$\Lambda=0$, we find Eq. (44) yields

$$8\pi\lambda=3k^2, \tag{45}$$

so that

$$8\pi T_4{}^4=8\pi T_1{}^1=3k^2, \tag{46}$$

whence the proper density of energy *and* negative pressure are connected in the same way as the proper density of energy in the case $T_1{}^1\neq T_4{}^4$.

The algebraic consequences of the theory could be carried out further.

CONCLUSION

On the basis of the field equation and the proposed supplement, it was found possible to construct a universal model having a pseudo-stationary property capable of explaining the red-shift as caused by velocity shifts. It is not maintained, however, that the supplemental equations are universal in scope although the possibility for this may exist. Nevertheless, in regards to the large scale behavior of matter, the equations may be the correct ones to use inasmuch as most of the facts of observational cosmology are consistent with the deductions, such as those which flow from the structure of the Robertson element itself, together with the rather attractive feature of a constant proper-density of energy and the aforementioned stationary character of the universe. Reality, however, is not always very attractive.

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Auxiliary Conditions and Electrostatic Interaction in Generalized Quantum Electrodynamics

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Paralleling a work of Fock we are able to eliminate the auxiliary conditions in our generalized quantum electrodynamics. As in the work of Fock this leads to a determination of both the electrostatic self-energy and electrostatic particle-particle interaction. Both turn out to be finite and in agreement with results obtained classically.

1. INTRODUCTION

IN an earlier paper[1] we have developed the quantum-mechanical formalism of a generalized electrodynamics. Here we shall derive further consequences, particularly those of the auxiliary conditions derived earlier.

Consider the wave equation

$$\left(\bar{H}_f+H_a-i\hbar\frac{\partial}{\partial T}\right)\Psi=0, \tag{1.1}$$

where $\bar{H}_f$ is the Hamiltonian of the radiation field, H_a the sum of the relativistic Hamiltonian of charged particles and their field-particle interactions, and T is the common time

$$H_a\equiv\sum_{s=1}^{n}(H_s+V_s)=\sum_{s=1}^{n}\{c\boldsymbol{\alpha}_s\cdot\mathbf{p}_s+c^2m_s\alpha_s{}^4+\epsilon_s[\phi(\mathbf{r}_s,T)-\boldsymbol{\alpha}_s\cdot\mathbf{A}(\mathbf{r}_s,T)]\}.$$

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[1] The two previous papers, B. Bodolsky, Phys. Rev. **62**, 68 (1942) and B. Podolsky and C. Kikuchi, **65**, 228 (1944) will be referred to as GE I and II, respectively.

Rosenfeld[2] has shown that, if the dynamical variables are transformed according to

$$F' = \exp(i\bar{H}_f T/\hbar) F \exp(-i\bar{H}_f T/\hbar),$$

Eq. (1.1) can be written as

$$\left[\sum_{s=1}^{n} (H_s + V_s') - i\hbar\partial/\partial T\right]\Psi' = 0, \tag{1.2}$$

where

$$\Psi' = \Psi'(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_n; J; T) = \exp(i\bar{H}_f T/\hbar)\Psi.$$

Now Dirac[3] has shown that the conditions under which this equation can be replaced by the set

$$(R_s - i\hbar\partial/\partial t_s)\Psi'(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_n; t, t_1, t_2, \cdots t_n; J) = 0, \tag{1.3}$$

where

$$\Psi'(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_n; J; T) \equiv [\Psi'(\mathbf{r}_1, \cdots \mathbf{r}_n; t, t_1, \cdots t_n; J)]_{t=t_1=\cdots=t_n=T}$$

and

$$R_s = c\boldsymbol{\alpha}_s \cdot \mathbf{p}_s + c^2 m_s \alpha_s^4 + \epsilon_s[\phi(\mathbf{r}_s, t_s) - \boldsymbol{\alpha}_s \cdot \mathbf{A}(\mathbf{r}_s, t_s)], \tag{1.4}$$

are

(A)
$$\frac{\partial\Psi'}{\partial T} = \left[\left(\frac{\partial}{\partial t} + \sum_{s=1}^{n} \frac{\partial}{\partial t_s}\right)\Psi'\right]_{t=t_1=\cdots=t_n=T} \tag{1.5}$$

and

(B)
$$[R_i, R_j] = 0, \tag{1.6}$$

for every pair i, j.

2. ELIMINATION OF AUXILIARY CONDITIONS

The quantity J includes, if the field variables are expressed in the Fourier components, the argument functions[4]

$$\phi(\mathbf{k}),\ A_1^*(\mathbf{k}),\ A_2^*(\mathbf{k}),\ A_3^*(\mathbf{k}), \quad \text{and} \quad \tilde{\phi}(\mathbf{k}),\ \tilde{A}_1^*(\mathbf{k}),\ \tilde{A}_2^*(\mathbf{k}),\ \tilde{A}_3^*(\mathbf{k}). \tag{2.1}$$

Fock has shown that, in Maxwellian electrodynamics, the scalar and the longitudinal component of the vector potential can be eliminated from the wave functional. It does not seem possible to carry through the same elimination from the set (2.1), because the longitudinal component of the extraordinary field does not vanish even in the absence of charged particles. Our problem, therefore, becomes that of finding "how much" of the longitudinal component can be eliminated.

We have found that if the operators

$$Q(\mathbf{k}) = \frac{\mathbf{k}\cdot\mathbf{A}(\mathbf{k})}{k}, \quad \tilde{Q}(\mathbf{k}) = \frac{a\mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k}) - \tilde{B}_0(\mathbf{k})}{a\tilde{k}} \tag{2.2}$$

are introduced, the calculation can be carried through in a manner very similar to Fock's. As in GE I and GE II, $k = |\mathbf{k}|$ and $\tilde{k} = (1+a^2k^2)^{\frac{1}{2}}/a$. The operator $\tilde{B}_0(\mathbf{k})$ is related to $\tilde{B}(\mathbf{k})$, defined in GE II, Eq. (5.2), through the following equation:

$$\tilde{B}_0(\mathbf{k}) = i\tilde{B}(\mathbf{k}).$$

It is to be noted that although $Q(\mathbf{k})$ is the total longitudinal component of the ordinary vector potential, the quantity $\mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k})/\tilde{k}$ is not the total longitudinal part of $\tilde{\mathbf{A}}(\mathbf{k})$. Using the commutation rules for $A_i(\mathbf{k})$, $\tilde{A}_i(\mathbf{k})$, and $\tilde{B}_0(\mathbf{k})$, it is easy to show that

$$[Q(\mathbf{k}), Q^*(\mathbf{k}')] = \frac{c\hbar}{2k}\delta(\mathbf{k}-\mathbf{k}'), \quad [\tilde{Q}(\mathbf{k}), \tilde{Q}^*(\mathbf{k}')] = -\frac{c\hbar}{2\tilde{k}}\delta(\mathbf{k}-\mathbf{k}'). \tag{2.3}$$

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[2] L. Rosenfeld, Zeits. f. Physik **76**, 729 (1932).

[3] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, 1935), p. 286.

[4] This section closely parallels the work of V. Fock, Physik. Zeits. Sowjetunion **6**, 449–460 (1934). Actually we ought to distinguish the operators $A_1^*(\mathbf{k})$, $\tilde{A}_2^*(\mathbf{k})$, etc. from their argument functions $\tilde{A}_1(\mathbf{k})$, $\tilde{A}(\mathbf{k})$, etc., as was done by Fock, but we prefer not to do this because the notation would become very cumbersome. The context will make it clear whether we mean the operator or the corresponding argument function.

Fock has further shown, for Maxwellian electrodynamics, how the field potential operators can be represented by functional derivatives. For the appropriate representation of the extraordinary field potentials, we note that, if Ω is a functional of some functions, say $\tilde{b}(\mathbf{k})$, the following general relation will be satisfied:

$$\frac{\delta}{\delta\tilde{b}(\mathbf{k}')}\tilde{b}(\mathbf{k})\Omega-\tilde{b}(\mathbf{k})\frac{\delta\Omega}{\delta\tilde{b}(\mathbf{k}')}=\delta(\mathbf{k}-\mathbf{k}')\Omega.$$

Comparing this with the commutation rules developed earlier, we see that it is possible to make the following associations:[5]

$$\begin{aligned}
\phi^*(\mathbf{k})&\rightarrow\frac{c\hbar}{2k}\frac{\delta}{\delta\phi(\mathbf{k})}, &\quad \tilde{\phi}^*(\mathbf{k})&\rightarrow-\frac{c\hbar}{2\tilde{k}}\frac{\delta}{\delta\tilde{\phi}(\mathbf{k})},\\
A_l(\mathbf{k})&\rightarrow\frac{c\hbar}{2k}\frac{\delta}{\delta A_l^*(\mathbf{k})}, &\quad \tilde{A}_l(\mathbf{k})&\rightarrow-\frac{c\hbar}{2\tilde{k}}\frac{\delta}{\delta\tilde{A}_e^*(\mathbf{k})},\\
Q(\mathbf{k})&\rightarrow\frac{c\hbar}{2k}\frac{\delta}{\delta Q^*(\mathbf{k})}, &\quad \tilde{Q}(\mathbf{k})&\rightarrow-\frac{c\hbar}{2\tilde{k}}\frac{\delta}{\delta\tilde{Q}^*(\mathbf{k})},\\
&&\tilde{B}_0(\mathbf{k})&\rightarrow-\frac{c\hbar}{2\tilde{k}}\frac{\delta}{\delta\tilde{B}_0^*(\mathbf{k})}.
\end{aligned}\tag{2.4}$$

The auxiliary conditions, GE II, Eq. (5.8), can then be written in the form[6]

$$\begin{aligned}
&\frac{c\hbar}{2k}\frac{\delta\Psi}{\delta Q^*(\mathbf{k})}-\phi(\mathbf{k})\Psi=0, &\quad &\frac{c\hbar}{2\tilde{k}}\frac{\delta\Psi}{\delta\tilde{Q}^*(\mathbf{k})}+\tilde{\phi}(\mathbf{k})\Psi=0,\\
&\frac{c\hbar}{2k}\frac{\delta\Psi}{\delta\phi(\mathbf{k})}-Q^*(\mathbf{k})\Psi=0, &\quad &\frac{c\hbar}{2\tilde{k}}\frac{\delta\Psi}{\delta\tilde{\phi}(\mathbf{k})}+\tilde{Q}^*(\mathbf{k})\Psi=0.
\end{aligned}\tag{2.5}$$

The solution of these equations is

$$\Psi=\exp(\chi_0-\tilde{\chi}_0)\Omega_0,\tag{2.6}$$

where Ω_0 is a functional not containing $\phi(\mathbf{k})$, $Q^*(\mathbf{k})$, $\tilde{\phi}(\mathbf{k})$, and $\tilde{Q}^*(\mathbf{k})$; and

$$\chi_0=\frac{2}{c\hbar}\int Q^*(\mathbf{k})\phi(\mathbf{k})kd\mathbf{k},\quad \tilde{\chi}_0=\frac{2}{c\hbar}\int\tilde{Q}^*(\mathbf{k})\tilde{\phi}(\mathbf{k})\tilde{k}d\mathbf{k},\tag{2.7}$$

where $d\mathbf{k}$ stands for $dk_xdk_ydk_z$, the volume element in $\mathbf{k}$ space.

If there are charged particles in the field, the modified auxiliary conditions must be used.[7] If we put

$$\begin{aligned}
f(\mathbf{r}_s,t_s)&=(2\pi)^{-\frac{3}{2}}\sum_{s=1}^{n}\epsilon_s\exp i\varphi_s, &\quad \varphi_s&\equiv ckt_s-\mathbf{k}\cdot\mathbf{r}_s;\\
\tilde{f}(\mathbf{r}_s,t_s)&=(2\pi)^{-\frac{3}{2}}\sum_{s=1}^{n}\epsilon_s\exp i\tilde{\varphi}_s, &\quad \tilde{\varphi}_s&\equiv c\tilde{k}t_s-\mathbf{k}\cdot\mathbf{r}_s;
\end{aligned}\tag{2.8}$$

[5] For another representation, see C. Kikuchi, Thesis, University of Washington (1944).

[6] The wave functional Ψ used here is identical with Ψ' of Eq. (1.3). The prime has been omitted to simplify the notation. Similarly it is understood that all field operators are those of Eq. (1.2).

[7] Note that the f's defined here differ by a factor from those given earlier in GE II, Eq. (6.3).

Eqs. (6.4) GE II, become

$$\left[Q(\mathbf{k})-\phi(\mathbf{k})+\frac{1}{2k^2}f\right]\Psi=0, \qquad \left[\tilde{Q}(\mathbf{k})-\tilde{\phi}(\mathbf{k})-\frac{1}{2\tilde{k}^2}\tilde{f}\right]\Psi=0,$$
$$\left[Q^*(\mathbf{k})-\phi^*(\mathbf{k})+\frac{1}{2k^2}f^*\right]\Psi=0, \quad \left[\tilde{Q}^*(\mathbf{k})-\tilde{\phi}^*(\mathbf{k})-\frac{1}{2\tilde{k}^2}\tilde{f}^*\right]\Psi=0. \tag{2.9}$$

The wave functional satisfying these equations can be written

$$\Psi=\exp\,(\chi-\tilde{\chi})\Omega, \tag{2.10}$$

where Ω is again a functional independent of the functions occurring explicitly in Eq. (2.9);

$$\chi=\frac{2}{c\hbar}\int Q^*(\mathbf{k})\phi(\mathbf{k})kdk+\frac{1}{c\hbar}\int\phi(\mathbf{k})\frac{f^*}{k}d\mathbf{k}-\frac{1}{c\hbar}\int Q^*(\mathbf{k})\frac{f}{k}d\mathbf{k}+\chi',$$
$$\tilde{\chi}=\frac{2}{c\hbar}\int\tilde{Q}^*(\mathbf{k})\tilde{\phi}(\mathbf{k})\tilde{k}d\mathbf{k}-\frac{1}{c\hbar}\int\tilde{\phi}(\mathbf{k})\frac{\tilde{f}^*}{\tilde{k}}d\mathbf{k}+\frac{1}{c\hbar}\int\tilde{Q}^*(\mathbf{k})\frac{\tilde{f}}{\tilde{k}}d\mathbf{k}+\tilde{\chi}', \tag{2.11}$$

and χ' and $\tilde{\chi}'$ being arbitrary functions of the space-time coordinates of the particles.

As we wish to obtain a wave equation for Ω corresponding to Eq. (1.3) for Ψ we must replace each operator F by

$$\exp\,(-\chi+\tilde{\chi})F\exp\,(\chi-\tilde{\chi}).$$

The quantities $\phi(\mathbf{k})$, $Q^*(\mathbf{k})$, $\tilde{\phi}(\mathbf{k})$, $\tilde{Q}^*(\mathbf{k})$, and $\tilde{B}_0^*(\mathbf{k})$ are invariant with respect to this transformation, but:

$$\exp\,(-\chi+\tilde{\chi})Q(\mathbf{k})\exp\,(\chi-\tilde{\chi})=Q(\mathbf{k})+\phi(\mathbf{k})-\frac{1}{2k^2}f,$$
$$\exp\,(-\chi+\tilde{\chi})\phi^*(k)\exp\,(\chi-\tilde{\chi})=\phi^*(\mathbf{k})+Q^*(\mathbf{k})+\frac{1}{2k^2}f^*,$$
$$\exp\,(-\chi+\tilde{\chi})\tilde{Q}(k)\exp\,(\chi-\tilde{\chi})=\tilde{Q}(\mathbf{k})+\tilde{\phi}(\mathbf{k})+\frac{1}{2\tilde{k}^2}\tilde{f},$$
$$\exp\,(-\chi+\tilde{\chi})\tilde{\phi}^*(\mathbf{k})\exp\,(\chi-\tilde{\chi})=\tilde{\phi}^*(\mathbf{k})+\tilde{Q}^*(\mathbf{k})-\frac{1}{2\tilde{k}^2}\tilde{f}^* \tag{2.12}$$

and

$$\exp\,(-\chi+\tilde{\chi})\tilde{B}_0(\mathbf{k})\exp\,(\chi-\tilde{\chi})=\tilde{B}_0(\mathbf{k})-\frac{1}{a\tilde{k}}\tilde{\phi}(\mathbf{k})-\frac{1}{2a\tilde{k}^3}\tilde{f}. \tag{2.13}$$

To illustrate how these are calculated, consider the transform of $\tilde{B}_0(\mathbf{k})$. From Eq. (2.4), if Λ is a functional of $\tilde{B}_0^*(\mathbf{k})$,

$$\exp\,(-\chi+\tilde{\chi})\tilde{B}_0(\mathbf{k})\exp\,(\chi-\tilde{\chi})\Lambda=-\exp\,(-\chi+\tilde{\chi})\left(\frac{c\hbar}{2\tilde{k}}\frac{\delta}{\delta\tilde{B}_0^*(\mathbf{k})}\right)(\exp\,(\chi-\tilde{\chi})\Lambda)$$
$$=\frac{-c\hbar}{2\tilde{k}}\frac{\delta\Lambda}{\delta\tilde{B}_0^*(\mathbf{k})}+\frac{c\hbar}{2\tilde{k}}\frac{\delta\tilde{\chi}}{\delta\tilde{B}_0^*(\mathbf{k})}=\left[\tilde{B}_0(\mathbf{k})+\frac{c\hbar}{2\tilde{k}}\frac{\delta\tilde{\chi}}{\delta\tilde{B}_0^*(\mathbf{k})}\right]\Lambda.$$

Since from Eq. (2.11),

$$\frac{\delta\tilde{\chi}}{\delta\tilde{B}_0^*(\mathbf{k})}=-\frac{2\tilde{\phi}(\mathbf{k})}{ac\hbar}-\frac{\tilde{f}}{ac\hbar\tilde{k}^2},$$

Eq. (2.13) follows.

Next we shall show that the vector potentials $\mathbf{A}(\mathbf{k})$ and $\tilde{\mathbf{A}}(\mathbf{k})$ transform as follows:

$$\begin{aligned}\exp(-\chi+\tilde{\chi})\mathbf{A}(\mathbf{k})\exp(\chi-\tilde{\chi})&=\mathbf{A}(\mathbf{k})+\frac{\mathbf{k}}{k}\phi(\mathbf{k})-\frac{\mathbf{k}f}{2k^3},\\ \exp(-\chi+\tilde{\chi})\tilde{\mathbf{A}}(\mathbf{k})\exp(\chi-\tilde{\chi})&=\tilde{\mathbf{A}}(\mathbf{k})+\frac{\mathbf{k}}{\tilde{k}}\tilde{\phi}(\mathbf{k})+\frac{\mathbf{k}\tilde{f}}{2\tilde{k}^3}.\end{aligned}\qquad(2.14)$$

Since χ and $\tilde{\chi}$ contain only the longitudinal component of the vector potentials, we have

$$\begin{aligned}\exp(-\chi+\tilde{\chi})\tilde{\mathbf{A}}(\mathbf{k})\exp(\chi-\tilde{\chi})&=\exp(-\chi+\tilde{\chi})(\tilde{\mathbf{A}}_T(\mathbf{k})+\tilde{\mathbf{A}}_L(k))\exp(\chi-\tilde{\chi})\\ &=\tilde{\mathbf{A}}_T(\mathbf{k})+\exp(-\chi+\tilde{\chi})\tilde{\mathbf{A}}_L(\mathbf{k})\exp(\chi-\tilde{\chi}),\end{aligned}\qquad(2.15)$$

where the subscripts L and T indicate the longitudinal and transverse parts of $\tilde{\mathbf{A}}(\mathbf{k})$. From Eqs. (2.2) and (2.11) we further obtain

$$\begin{aligned}\exp(-\chi+\tilde{\chi})\tilde{\mathbf{A}}_L(\mathbf{k})\exp(\chi-\tilde{\chi})&\equiv\exp(-\chi+\tilde{\chi})\left[\mathbf{k}\frac{\mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k})}{k^2}\right]\exp(\chi-\tilde{\chi})\\ &=\frac{\mathbf{k}\mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k})}{k^2}+\frac{\mathbf{k}}{\tilde{k}}\tilde{\phi}(\mathbf{k})+\frac{\mathbf{k}}{2\tilde{k}^3}\tilde{f}.\end{aligned}$$

Substitution of this into Eq. (2.15) yields the desired result.

3. ELECTROSTATIC SELF-ENERGY

Let us transform back into coordinate space by means of the formula

$$F(\mathbf{r},t)=(2\pi)^{-\frac{3}{2}}\int\{F(\mathbf{k})\exp(-i\varphi)+F^*(\mathbf{k})\exp(i\varphi)+\tilde{F}(\mathbf{k})\exp(-i\tilde{\varphi})+\tilde{F}^*(\mathbf{k})\exp(i\tilde{\varphi})\}d\mathbf{k},\quad(3.1)$$

where

$$\varphi=ckt-\mathbf{k}\cdot\mathbf{r},\quad\tilde{\varphi}=c\tilde{k}t-\mathbf{k}\cdot\mathbf{r}.$$

We then obtain

$$\begin{aligned}\exp(-\chi+\tilde{\chi})\mathbf{A}(\mathbf{r}_s,t_s)\exp(\chi-\tilde{\chi})&=\mathbf{A}(\mathbf{r}_s,t_s)+(2\pi)^{-\frac{3}{2}}\int\frac{\mathbf{k}}{k}\phi(\mathbf{k})\exp(-i\varphi_s)d\mathbf{k}\\ &+(2\pi)^{-\frac{3}{2}}\int\frac{\mathbf{k}}{\tilde{k}}\tilde{\phi}(\mathbf{k})\exp(-i\tilde{\varphi}_s)d\mathbf{k}-\tfrac{1}{2}(2\pi)^{-\frac{3}{2}}\int\frac{\mathbf{k}}{k^3}f\exp(-i\varphi_s)d\mathbf{k}+\tfrac{1}{2}(2\pi)^{-\frac{3}{2}}\int\frac{\mathbf{k}}{\tilde{k}^3}\tilde{f}\exp(-i\tilde{\varphi}_s)d\mathbf{k},\end{aligned}\quad(3.2)$$

and

$$\begin{aligned}\exp(-\chi+\tilde{\chi})\phi(\mathbf{r}_s,t_s)\exp(\chi-\tilde{\chi})&=\phi(\mathbf{r}_s,t_s)+(2\pi)^{-\frac{3}{2}}\int Q^*(\mathbf{k})\exp(i\varphi_s)d\mathbf{k}+(2\pi)^{-\frac{3}{2}}\int\tilde{Q}^*(\mathbf{k})\exp(i\tilde{\varphi}_s)d\mathbf{k}\\ &+\tfrac{1}{2}(2\pi)^{-\frac{3}{2}}\int\frac{f^*}{k^2}\exp(i\varphi_s)d\mathbf{k}-\tfrac{1}{2}(2\pi)^{-\frac{3}{2}}\int\frac{\tilde{f}^*}{\tilde{k}^2}\exp(i\tilde{\varphi}_s)d\mathbf{k}.\end{aligned}\quad(3.3)$$

Furthermore

$$\exp(-\chi+\tilde{\chi})p_x^{(s)}\exp(\chi-\tilde{\chi})=p_x^{(s)}-i\hbar\frac{\partial}{\partial x_s}(\chi-\tilde{\chi}),$$

and

$$\exp(-\chi+\tilde{\chi})(i\hbar\partial/\partial t_s)\exp(\chi-\tilde{\chi})=i\hbar\partial/\partial t_s+i\hbar\frac{\partial}{\partial t_s}(\chi-\tilde{\chi}). \tag{3.4}$$

We are now ready to derive the transformation properties of

$$P_x^{(s)}\equiv -i\hbar\partial/\partial x_s-(\epsilon_s/c)A_x(\mathbf{r}_s,t_s)$$

and

$$T^{(s)}\equiv i\hbar\partial/\partial t_s-\epsilon_s\phi(\mathbf{r}_s,t_s) \tag{3.5}$$

occurring in Eq. (1.3).

For convenience, we shall introduce the quantities

$$\mathbf{D}(\mathbf{k})=\mathbf{A}(\mathbf{k})-\mathbf{k}\mathbf{k}\cdot\mathbf{A}(\mathbf{k})/k^2$$

and

$$\tilde{\mathbf{D}}(\mathbf{k})=\tilde{\mathbf{A}}(\mathbf{k})-\mathbf{k}\mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k})/\tilde{k}^2+\mathbf{k}\tilde{B}_0(\mathbf{k})/a\tilde{k}^2 \tag{3.6}$$

which satisfy the commutation rules

$$[D_i(\mathbf{k}),D_j^*(\mathbf{k}')]=\frac{c\hbar}{2k}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\delta(\mathbf{k}-\mathbf{k}'),$$

and

$$[\tilde{D}_i(\mathbf{k}),\tilde{D}_j^*(\mathbf{k}')]=-\frac{c\hbar}{2\tilde{k}}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\delta(\mathbf{k}-\mathbf{k}'). \tag{3.7}$$

We also need the relations:

$$\frac{\partial\chi}{\partial x_s}=\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int\frac{k_x}{k}\phi(\mathbf{k})\exp(-i\varphi_s)d\mathbf{k}+\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int\frac{k_x}{k}Q^*(\mathbf{k})\exp(i\varphi_s)d\mathbf{k}+\partial\chi'/\partial x_s,$$

$$-\frac{\partial\tilde{\chi}}{\partial x_s}=\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int\frac{k_x}{\tilde{k}}\tilde{\phi}(\mathbf{k})\exp(-i\tilde{\varphi}_s)d\mathbf{k}+\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int\frac{k_x}{\tilde{k}}\tilde{Q}^*(\mathbf{k})\exp(i\tilde{\varphi}_s)d\mathbf{k}-\partial\tilde{\chi}'/\partial x_s,$$

$$\frac{\partial\chi}{\partial t_s}=-\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int\phi(\mathbf{k})\exp(-i\varphi_s)d\mathbf{k}-\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int Q^*(\mathbf{k})\exp(i\varphi_s)d\mathbf{k}+\partial\chi'/\partial t_s,$$

and

$$-\frac{\partial\tilde{\chi}}{\partial t_s}=-\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int\tilde{\phi}(\mathbf{k})\exp(-i\tilde{\varphi}_s)d\mathbf{k}-\frac{i\epsilon_s}{c\hbar}(2\pi)^{-\frac{3}{2}}\int\tilde{Q}^*(\mathbf{k})\exp(i\tilde{\varphi}_s)d\mathbf{k}-\partial\tilde{\chi}'/\partial t_s. \tag{3.8}$$

Therefore, the transforms of Eq. (3.5) turn out to be

$$P_x'^{(s)}\equiv\exp(-\chi+\tilde{\chi})P_x^{(s)}\exp(\chi-\tilde{\chi})$$

$$=p_x^{(s)}-\frac{\epsilon_s}{c}D_x(\mathbf{r}_s,t_s)-i\hbar\frac{\partial}{\partial x_s}(\chi'-\tilde{\chi}')+\frac{\epsilon_s}{2c}(2\pi)^{-\frac{3}{2}}\int\left[\frac{k_xf}{k^3}\exp(-i\varphi_s)-\frac{k_x\tilde{f}}{\tilde{k}^3}\exp(-i\tilde{\varphi}_s)\right]d\mathbf{k} \tag{3.9}$$

and similarly,

$$T'^{(s)}=i\hbar\frac{\partial}{\partial t_s}+i\hbar\frac{\partial}{\partial t_s}(\chi'-\tilde{\chi}')-\frac{\epsilon_s}{2}(2\pi)^{-\frac{3}{2}}\int\left[\frac{f^*}{k^2}\exp(i\varphi_s)-\frac{\tilde{f}^*}{\tilde{k}^2}\exp(i\tilde{\varphi}_s)\right]d\mathbf{k}. \tag{3.10}$$

In both Eqs. (3.9) and (3.10) the terms involving $Q(\mathbf{k})$, $\tilde{Q}(\mathbf{k})$, $\phi^*(\mathbf{k})$, $\tilde{\phi}^*(\mathbf{k})$ drop out because their argument functions have been eliminated from the wave functional; i.e.,

$$Q(\mathbf{k})\Omega=\tilde{Q}(\mathbf{k})\Omega=\phi^*(\mathbf{k})\Omega=\tilde{\phi}^*(\mathbf{k})\Omega=0.$$

Since χ' and $\tilde{\chi}'$ are arbitrary functions of the space-time coordinates, they can be chosen in such a way that their derivatives will just cancel the imaginary terms in Eqs. (3.9) and (3.10). The suitable choices are

$$\chi'=-\frac{1}{4c\hbar}\int\frac{f^*f}{k^3}d\mathbf{k},\quad \tilde{\chi}'=-\frac{1}{4c\hbar}\int\frac{\tilde{f}^*\tilde{f}}{\tilde{k}^3}d\mathbf{k}. \tag{3.11}$$

The real parts of these functions are:

$$-\frac{1}{4c\hbar(2\pi)^3}\sum_{u,v}\epsilon_u\epsilon_v F(\varphi_u-\varphi_v)\quad\text{and}\quad -\frac{1}{4c\hbar(2\pi)^3}\sum_{u,v}\epsilon_u\epsilon_v\tilde{F}(\tilde{\varphi}_u-\tilde{\varphi}_v), \tag{3.12}$$

where

$$F(\varphi_u-\varphi_v)=\int\frac{\cos\{ck(t_u-t_v)-\mathbf{k}\cdot(\mathbf{r}_u-\mathbf{r}_v)\}}{k^3}d\mathbf{k}$$

and (3.13)

$$\tilde{F}(\tilde{\varphi}_u-\tilde{\varphi}_v)=\int\frac{\cos\{c\tilde{k}(t_u-t_v)-\mathbf{k}\cdot(\mathbf{r}_u-\mathbf{r}_v)\}}{\tilde{k}^3}d\mathbf{k}.$$

Hence the operators $P_x'^{(s)}$ and $T'^{(s)}$ become

$$P_x'^{(s)}=p_x^{(s)}-\frac{\epsilon_s}{c}D_x(\mathbf{r}_s,t_s)+\frac{\epsilon_s}{2c(2\pi)^3}{\sum_u}'\epsilon_u\int\left[\frac{k_x}{k^3}\cos(\varphi_u-\varphi_s)-\frac{k_x}{\tilde{k}^3}\cos(\tilde{\varphi}_u-\tilde{\varphi}_s)\right]d\mathbf{k}$$
$$+\frac{\epsilon_s^2}{2c(2\pi)^3}\int\left[\frac{k_x}{k^3}-\frac{k_x}{\tilde{k}^3}\right]d\mathbf{k}, \tag{3.14}$$

and

$$T'^{(s)}=i\hbar\frac{\partial}{\partial t_s}-\frac{\epsilon_s}{2(2\pi)^3}{\sum_u}'\epsilon_u\int\left[\frac{\cos(\varphi_u-\varphi_s)}{k^2}-\frac{\cos(\tilde{\varphi}_u-\tilde{\varphi}_s)}{\tilde{k}^3}\right]d\mathbf{k}-\frac{\epsilon_s^2}{2(2\pi)^3}\int\left[\frac{1}{k^2}-\frac{1}{\tilde{k}^2}\right]d\mathbf{k}. \tag{3.15}$$

The primes over the sigmas indicate that the terms $u=s$ are to be omitted in the summation. The last integral in Eq. (3.14) vanishes because the integrand is odd; the analogous term in Eq. (3.15) gives

$$\frac{1}{4\pi}\frac{\epsilon_s^2}{2a},$$

which can be interpreted as the electrostatic self-energy, and agrees with the earlier result calculated classically.[8]

[8] GE I, Eq. (5.4). There ϵ_s is in electrostatic units.

4. ELECTROSTATIC INTERACTION

It will be observed that Eqs. (3.14) and (3.15) can be somewhat simplified by writing

$$U_s=\sum_u{}' \frac{\epsilon_u}{(2\pi)^3}\int\left[\frac{\sin(\varphi_s-\varphi_u)}{k^3}-\frac{\sin(\tilde{\varphi}_s-\tilde{\varphi}_u)}{\tilde{k}^3}\right]d\mathbf{k}. \tag{4.1}$$

Accordingly,

$$P_x{}'^{(s)}=p_x{}^{(s)}-\frac{\epsilon_s}{c}D_x(\mathbf{r}_s,t_s)-\frac{\epsilon_s}{2c}\frac{\partial U_s}{\partial x_s}, \tag{4.2}$$

and

$$T'^{(s)}=i\hbar\frac{\partial}{\partial t_s}-\frac{\epsilon_s}{2c}\frac{\partial U_s}{\partial t_s}-\frac{1}{4\pi}\frac{\epsilon_s{}^2}{2a}. \tag{4.3}$$

It is possible to eliminate the last term in Eq. (4.2) by means of canonical transformation, i.e.,

$$P_x{}''^{(s)}\equiv\exp(-i\epsilon_s U_s/2c\hbar)P_x{}'^{(s)}\exp(i\epsilon U_s/2c\hbar)=p_x{}^{(s)}-\frac{\epsilon_s}{c}D_x(\mathbf{r}_s,t_s), \tag{4.4}$$

$$T''^{(s)}\equiv\exp(-i\epsilon_s U_s/2c\hbar)T'^{(s)}\exp(i\epsilon_s U_s/2c\hbar)=i\hbar\frac{\partial}{\partial t_s}-\frac{\epsilon_s}{c}\frac{\partial U_s}{\partial t_s}-\frac{\epsilon_s{}^2}{8\pi a}. \tag{4.5}$$

Substituting the above results into Eq. (1.3), we obtain the following for the relativistic wave equation of the sth particle:

$$[c\boldsymbol{\alpha}_s\cdot\mathbf{p}^{(s)}-\epsilon_s\boldsymbol{\alpha}_s\cdot\mathbf{D}(\mathbf{r}_s,t_s)+m_sc^2\alpha_s{}^4]\Omega=\left[i\hbar\frac{\partial}{\partial t_s}-\frac{\epsilon_s}{c}\frac{\partial U_s}{\partial t_s}-\frac{\epsilon_s{}^2}{8\pi a}\right]\Omega. \tag{4.6}$$

We note that the above equation differs from the usual wave equation in several respects. In the first place, it contains the self-energy term. Secondly, the term $\mathbf{D}(\mathbf{r}_s, t_s)$, which represents the interaction of the sth particle with the field, is not solenoidal but satisfies the more general equation

$$(1-a^2\square)\operatorname{div}\mathbf{D}=0. \tag{4.7}$$

Finally, we shall show that $\partial U_s/\partial t_s$ gives the particle-particle electrostatic interaction function derived classically in GE I, Eq. (2.6).

Let

$$V_s=\frac{1}{c}\frac{\partial U_s}{\partial t_s}=\sum_u{}' V_{su},\quad V_{su}=V'_{su}-\tilde{V}'_{su}, \tag{4.8}$$

where

$$V'_{su}=\frac{\epsilon_u}{(2\pi)^3}\int\frac{\cos(\varphi_u-\varphi_s)}{k^2}d\mathbf{k}\quad\text{and}\quad\tilde{V}'_{su}=\frac{\epsilon_u}{(2\pi)^3}\int\frac{\cos(\tilde{\varphi}_u-\tilde{\varphi}_s)}{\tilde{k}^2}d\mathbf{k}. \tag{4.9}$$

The first integral can be evaluated very readily. Putting $T=t_u-t_s$, $R=|\mathbf{r}_u-\mathbf{r}_s|$, we get

$$\frac{\epsilon_u}{(2\pi)^3}\int\frac{\cos\{ckT-kR\cos\theta\}}{k^2}d\mathbf{k}=\begin{cases}\epsilon_u/4\pi R, & cT<R\\ 0, & cT>R.\end{cases} \tag{4.10}$$

For the integration in (4.9) we make use of the formula[9]

$$\int_0^\infty J_{\frac{1}{2}}(bk)\frac{J_{\frac{1}{2}}(a\tilde{k})}{\tilde{k}^{\frac{1}{2}}}k^{\frac{1}{2}}dk=\begin{cases}0, & \text{for } a<b\\ -\left(\dfrac{b}{a}\right)^{\frac{1}{2}}\dfrac{\lambda}{(a^2-b^2)^{\frac{1}{2}}}J_1[\lambda(a^2-b^2)^{\frac{1}{2}}], & \text{for } a>b\end{cases}$$

where $\tilde{k}=(k^2+\lambda^2)^{\frac{1}{2}}$. Multiply both sides by $a^{\frac{1}{2}}$ and integrate. Then, since

$$\int_0^a a^{\frac{1}{2}}J_{\frac{1}{2}}(a\tilde{k})da=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\tilde{k}^{-\frac{3}{2}}-\frac{a^{\frac{1}{2}}}{\tilde{k}}J_{-\frac{1}{2}}(a\tilde{k}),$$

we obtain, upon substituting the trigonometric equivalents of the half-integral Bessel functions, and finally replacing λ by $1/a$, the result

$$\int_0^\infty \frac{ka^2}{a^2k^2+1}\sin(kR)\cos\left[\frac{cT}{a}(a^2k^2+1)^{\frac{1}{2}}\right]dk$$

$$=\begin{cases}\dfrac{1}{8\pi R}\exp(-R/a) & \text{for } c|T|<R\\ \dfrac{1}{8\pi}\left[\exp(-R/a)/R+\dfrac{1}{a}\displaystyle\int_1^{CT/R}\frac{J_1[(R/a)(\xi^2-1)^{\frac{1}{2}}]}{(\xi^2-1)^{\frac{1}{2}}}d\xi\right] & \text{for } c|T|>R;\end{cases} \quad (4.11)$$

with the help of Eqs. (4.10) and (4.11), we finally obtain

$$V_{su}=\begin{cases}\dfrac{\epsilon_u}{4\pi R}[1-\exp(-R/a)] & \text{for } c|T|<R\\ -\dfrac{\epsilon_u}{4\pi}\left[\exp(-R/a)/R+\dfrac{1}{a}\displaystyle\int_1^{CT/R}\frac{J_1[R(\xi^2-1)^{\frac{1}{2}}/a]}{(\xi^2-1)^{\frac{1}{2}}}d\xi\right] & \text{for } c|T|>R.\end{cases} \quad (4.12)$$

But from the commutation rule, GE II Eqs. (1.7) to (4.10) we see that condition (B), discussed in the introduction, will not be satisfied unless $c|T|<R$. Therefore we shall have to discard that part of Eq. (4.12) for which $c|T|>R$. This does not constitute any limitation, since we are always interested in interactions for particles at the same common time. This would make $T=0<R$.

Our relativistic wave equation thus becomes

$$[c\boldsymbol{\alpha}_s\cdot\mathbf{p}^{(s)}-\epsilon_s\boldsymbol{\alpha}_s\cdot\mathbf{D}(\mathbf{r}_s,t_s)+m_sc^2\alpha_s{}^4]\Omega=\left[i\hbar\frac{\partial}{\partial t_s}-\frac{1}{4\pi}\sum_u{}'\frac{\epsilon_u\epsilon_s}{|\mathbf{r}_s-\mathbf{r}_u|}[1-\exp(-|\mathbf{r}_s-\mathbf{r}_u|/a)]-\frac{\epsilon_s{}^2}{8\pi a}\right]\Omega. \quad (4.13)$$

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[9] G. N. Watson, *Theory of Bessel Function* (Cambridge University Press, 1922)— p. 415, Eq. (1), with $\mu=\nu=\frac{1}{2}$.