the isothermal core of the red-giant stars in which they must have been formed.

The results of more detailed calculations of the above "three-layer" stellar model, and also the discussion of possible astrophysical consequences will be published in due course.

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A Note on the Kepler Problem in a Space of Constant Negative Curvature

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SCHRÖDINGER[1] recently solved the "Kepler problem" in a spherical or Einstein universe and obtained the interesting result that the energy spectrum is discrete everywhere. It is instructive to compare this spectrum with that in an "open" universe of constant negative curvature, which is in fact Milne's universe.[2] As will be shown, the spectrum consists of a *finite* number of (mostly negative) energy levels in addition to a continuous spectrum. To our knowledge, this is the first quantum mechanical problem to exhibit a finite number of discrete energy values.

The line element of the hyperbolic 3-space may be written in the form:

$$ds^2=R^2d\alpha^2+R^2\sinh^2\alpha(d\theta^2+\sin^2\theta d\phi^2). \quad (1)$$

Letting $R\rightarrow\infty$, $\alpha\rightarrow 0$ such that $\alpha R=r$ remains finite, this line element reduces to that of Euclidean space, in which r, θ, ϕ are the usual polar coordinates. Schrödinger's equation is taken in the form

$$\Delta\psi+(2\mu/\hbar^2)(E-V)\psi=0, \quad (2)$$

with the proper Laplacian belonging to (1).

The potential energy $V(\alpha)$ of an electron in a central Coulomb field must satisfy Laplace's equation

$$\Delta V=g^{ij}V_{|ij}=0.$$

We find

$$V=-(Ze^2/R)(\coth\alpha-1), \quad (3)$$

if we demand that $V\rightarrow 0$ as $\alpha\rightarrow\infty$ and $V\rightarrow -Ze^2/r$ as $R\rightarrow\infty$, $\alpha R\rightarrow r$.

Putting $\psi=\sigma(\alpha)Y_l(\theta,\phi)$ in Eq. (2), where the Y_l are normalized spherical harmonics, we obtain the radial equation

$$(d/d\alpha)(\sinh^2\alpha d\sigma/d\alpha)+(\lambda+1-2\nu)\sinh^2\alpha\,\sigma +2\nu\sinh\alpha\cosh\alpha\,\sigma-l(l+1)\sigma=0; \quad (4)$$

where $\lambda=2\mu R^2E/\hbar^2-1$ and $\nu=Ze^2\mu R/\hbar^2$.

Introducing the "density function" $\rho=\sigma\sinh\alpha$, we can factorize Eq. (4):

$$\{(l+1)\coth\alpha-\nu/(l+1)+d/d\alpha\}\times\{(l+1)\coth\alpha-\nu/(l+1)-d/d\alpha\}\rho =\{\lambda-2\nu+(l+1)^2+\nu^2/(l+1)^2\}\rho; \quad (5)$$

$$\{l\coth\alpha-\nu/l-d/d\alpha\}\{l\coth\alpha-\nu/l+d/d\alpha\}\rho =\{\lambda-2\nu+l^2+\nu^2/l^2\}\rho. \quad (5')$$

The only possible discrete solutions are immediately obtained by the factorization method. We have, in the notation of Infeld's paper,[1]

$$\lambda_n=2\nu-(n+1)^2-\nu^2/(n+1)^2;\quad n=0,1,2\cdots, \quad (6)$$

$$\rho_n{}^n=c\sinh^{n+1}\alpha\exp(-\nu\alpha/(n+1)); \quad (7)$$

$$\rho_n{}^{l-1}=\{l^2+\nu^2/l^2-(n+1)^2-\nu^2/(n+1)^2\}^{-\frac{1}{2}}\times(l\coth\alpha-\nu/l+d/d\alpha)\rho_n{}^l;\quad l=0,1\cdots n, \quad (7')$$

since $\sinh\alpha\sim\frac{1}{2}e^\alpha$ for large α, Eq. (7) shows that if the normalization integral for $\rho_n{}^n$ is to exist, we must have

$$(n+1)^2<\nu;\quad n=0,1\cdots n_0, \quad (8)$$

where n_0 is the greatest integer satisfying the inequality.

Thus we have a finite number of discrete energy levels:

$$E_n=Ze^2/R-\hbar^2n(n+2)/2\mu R^2-Z^2e^4\mu/2\hbar^2(n+1)^2;\quad n=0,1\cdots n_0. \quad (9)$$

It is interesting to note that the number of discrete levels $n_0+1\sim\nu^{\frac{1}{2}}=(R/a)^{\frac{1}{2}}$, where a is the radius of the first Bohr orbit of the hydrogen-like atom. Taking R to be of the order of 10^{28} cm (the usual magnitude of the radius of the universe considered in relativistic cosmology), we find that n_0 is a large number of order 10^{18}. The highest discrete energy level E_{n_0} lies between $-3\hbar^2/2\mu R^2$ and $\hbar^2/2\mu R^2$ and may thus be positive or negative; all others are negative.

The factorization method breaks down for the continuous spectrum. Putting $x=\coth\alpha$, Eq. (4) transforms into

$$\frac{d^2\sigma}{dx^2}+\left[\frac{\lambda+1-2\nu+2\nu x}{(x^2-1)^2}-\frac{l(l+1)}{x^2-1}\right]\sigma=0. \quad (10)$$

The range of x (for $0\leqslant\alpha<\infty$) is from ∞ to 1. The only solution of this Riemann P-equation, bounded for large x, is[3]

$$\sigma=(x+1)^{-l-\frac{1}{2}(1+\sqrt{-\lambda})}(x-1)^{\frac{1}{2}(1+\sqrt{-\lambda})}\times F(1+l+\tfrac{1}{2}[-\lambda^{\frac{1}{2}}-(4\nu-\lambda)^{\frac{1}{2}}],\ 1+l+\tfrac{1}{2}[-\lambda^{\frac{1}{2}}+(4\nu-\lambda)^{\frac{1}{2}}];\ 2(l+1);\ 2/(x+1)). \quad (11)$$

This solution is continuous everywhere in our interval (including ∞), except possibly at $x=1$. For negative λ we regain the discrete spectrum discussed above.

If λ is positive, $-\lambda^{\frac{1}{2}}=i\lambda^{\frac{1}{2}}$ is imaginary. Introducing α for x in (11) and replacing σ by the "density function" ρ, we have

$$\rho=\sinh^{l+1}\alpha\exp(-(1+i\lambda^{\frac{1}{2}}+l)\alpha)\times F(1+l+\tfrac{1}{2}[i\lambda^{\frac{1}{2}}-(4\nu-\lambda)^{\frac{1}{2}}],\ 1+l+\tfrac{1}{2}[i\lambda^{\frac{1}{2}}+(4-\nu-\lambda)^{\frac{1}{2}}];\ 2(l+1);z);\quad z=2e^{-\alpha}\sinh\alpha. \quad (12)$$

We demand that ρ be continuous and bounded for $0\leqslant\alpha<\infty$. We have seen that ρ is continuous, and it is clear that the factor multiplying the hypergeometric function in (12) remains bounded for large α. It remains to be shown that the hypergeometric function is also bounded as $\alpha\rightarrow\infty$, i.e., as $z\rightarrow 1$.

Putting $z=1$, we find the ratio of successive terms of the hypergeometric series is of the form[4]

$$a_n/a_{n+1}=1+(1-i\lambda^{\frac{1}{2}})/n+0(n^{-2}).$$

It follows[5] that the infinite hypergeometric series oscillates finitely. Abel's theorem[6] then shows that the hypergeometric function in (12) remains bounded as $z \to 1$ or $\alpha \to \infty$.

This establishes the existence of a continuous eigenvalue spectrum of all positive λ or

$$E > \hbar^2/2\mu R^2. \tag{13}$$

In order to obtain an indication of how a relativistic treatment might affect this problem, the authors repeated the calculations by use of the Gordon-Klein equation. The general character of the solution was found to be unchanged.

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Classical Theory of the Point Electron

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THE field produced by a point charge can be split in two parts:

(1) The attached field, $F^{\mu\nu}_{\text{at}}$;

(2) The radiated field, $F^{\mu\nu}_{\text{rad}}$.

We assume the total field to be the sum of the attached, the radiated and the external fields and that the sum of the attached and radiated fields gives the usual retarded field $F^{\mu\nu}_{\text{ret}}$ of the particle.

The radiated field behaves as an external field and reacts on the emitting charge. The attached field does not act on the generating charge but acts on other charges. Therefore, each charge is under the action of the retarded fields of the others, but only under the radiated part of its own field.

The particle of charge e and mass m has two kinds of momentum and energy: (1) the kinetic four vector

$$m\frac{dx^\mu}{ds} = G^\mu_{\text{kin}},$$

and (2) the four vector of the acceleration energy and momentum

$$G^\mu_{\text{ac}} = -\tfrac{2}{3}e^2\frac{d^2x^\mu}{ds^2}.$$

The four vector G^μ_{ac} arises from the interaction of the charge and its radiated field. Since the particle is under the action of the external field and its own radiated field, the equations of motion ought to be:

$$m\frac{d^2x^\mu}{ds^2} = \frac{e}{c}\frac{dx^\nu}{ds}(F^\mu_{\text{ext},\nu} + F^\mu_{\text{rad},\nu}). \tag{1}$$

If we apply the laws of conservation of energy and momentum we get:

$$\frac{d}{ds}(G^\mu_{\text{kin}} + G^\mu_{\text{ac}}) - \tfrac{2}{3}e^2\frac{dx^\mu}{ds}\frac{d^2x^\rho}{ds^2}\frac{d^2x_\rho}{ds^2} = \frac{e}{c}F^\mu_{\text{ext},\nu}\frac{dx^\nu}{ds}, \tag{2}$$

because the rate of loss of energy and momentum is

$$\tfrac{2}{3}e^2\frac{dx^\mu}{ds}\frac{d^2x^\rho}{ds^2}\frac{d^2x_\rho}{ds^2}$$

according to the well-known Larmor formulae. Comparing Eqs. (1) and (2) we see that:

$$\frac{e}{c}F^\mu_{\text{rad},\nu}\frac{dx^\nu}{ds} = -\frac{dG^\mu_{\text{ac}}}{ds} + \tfrac{2}{3}e^2\frac{dx^\mu}{ds}\frac{d^2x^\rho}{ds^2}\frac{d^2x_\rho}{ds^2}. \tag{3}$$

Equation (3) is satisfied by taking:

$$F^{\mu\nu}_{\text{rad}} = \tfrac{1}{2}(F^{\mu\nu}_{\text{ret}} - F^{\mu\nu}_{\text{adv}}). \tag{4}$$

From Eq. (4) it results that:

$$F^{\mu\nu}_{\text{at}} = \tfrac{1}{2}(F^{\mu\nu}_{\text{ret}} + F^{\mu\nu}_{\text{adv}}). \tag{5}$$

We define the stress-tensor $T^{\mu\nu}$ of the field in which there are n point charges in the following way:

$$4\pi T_\nu{}^{\mu\prime} = F^{\mu\rho}F_{\rho\nu} + \tfrac{1}{4}\delta_\nu{}^\mu(F^{\rho\sigma}F_{\rho\sigma}) - \Sigma_i F^{\mu\rho}_{i,\text{at}}F_{i,\text{at};\rho\nu} - \tfrac{1}{4}\delta_\nu{}^\mu(\Sigma_i F^{\rho\sigma}_{i,\text{at}}F_{i,\text{at};\rho\sigma}). \tag{6}$$

Equation (6) ensures the conservation of the energy and momentum of the system of the field and particles, and leads to a finite field energy which is not always positive. The energy and momentum of the field derived from the stress tensor $T^{\mu\nu}$ are the components of a four vector. A detailed exposition of the theory will be given elsewhere.

Our theory leads to the Lorentz-Dirac classical equations of motion and to a finite field energy without introducing any energies and momenta of non-electromagnetic nature besides the kinetic ones. There are no subtractions of infinite quantities.

The Radiation Field of a Point Electron

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THE radiation field of the electron has been defined in different ways. The most usual is to take it as the part of the retarded field of the charge that depends on the retarded acceleration and varies inversely with the retarded distance to the charge. Though this definition presents many obvious inconveniencies, it has the advantage of giving the right amount of the radiation emission. Dirac[1] has proposed the following definition of the radiation field of a point charge:

$$F^{\mu\nu}_{\text{rad},D} = F^{\mu\nu}_{\text{ret}} - F^{\mu\nu}_{\text{adv}}, \tag{1}$$