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The Multiple Scattering of Waves*

I. General Theory of Isotropic Scattering by Randomly Distributed Scatterers

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While the problem of the multiple scattering of particles by a random distribution of scatterers has been treated classically through the use of the Boltzmann integro-differential equation, the corresponding problem of the multiple scattering of waves seems to have received scant attention. All previous treatments have considered the problem in the "geometrical optics" limit, where the rays are regarded as trajectories of particles and the treatment for particles is then applied, so that the interference phenomena in wave scattering are neglected. In this paper the problem of the multiple scattering of scalar waves by a random distribution of isotropic scatterers is considered in detail on the basis of a consistent wave treatment. The introduction of the concept of "randomness" requires averages to be taken over a statistical ensemble of scatterer configurations. Equations are derived for the average value of the wave function, the average value of the square of its absolute value, and the average flux carried by the wave. The second of these quantities satisfies an integral equation which has some similarities to the corresponding equation for particle scattering. The physical interpretation of the results is discussed in some detail and possible generalizations of the theory are outlined.

INTRODUCTION

THE problem of the multiple scattering of particles by a random distribution of scatterers has been treated in some detail in literature.[1] These treatments are all based essentially on the Boltzmann integro-differential equation of conservation in phase space and are thus classical rather than quantum-mechanical in foundation. On the other hand, a consistent treatment, based on the wave equation itself, of the multiple scattering of waves by a randomly distributed collection of scatterers, has not, to the writer's knowledge, yet appeared in literature. Those treatments which have appeared, for example, in discussions of the problem of the propagation of light through stellar atmospheres and through turbid media,[2] are all based on the approximation of considering light rays as the trajectories of particles (photons), which are then treated on the basis of the Boltzmann equation. This, in a sense, corresponds to treating light scattering in the "geometrical optics limit" where interference phenomena characteristic of waves is neglected. Such a treatment might be expected, *a priori*, to be valid if the wave-length of the waves is small compared to the mean free path of the associated particles in the scattering medium, but the only satisfactory justification for such a treatment lies in showing that the wave treatment leads to the same results.

The investigation of the multiple scattering of waves is not only important for the above reason but also for several others. The multiple scattering of sound waves by the water droplets of a fog or by other collections of small obstacles often occurs in cases where the interference effects may not be legitimately neglected, and thus this problem cannot be satisfactorily discussed except by a wave treatment. Also in the scattering of electrons and other fundamental particles by aggregates of nuclei, atoms, or molecules in gases, liquids, and amorphous solids, the use of the classical particle mechanics may be questionable, and the quantum-mechanical treatment of this problem necessitates the inves-

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[1] E. Fermi, Ricerca Scienta **VII–II**, 13 (1936); Yost and Dickinson, Phys. Rev. **50**, 128 (1936); G. C. Wick, Atti del Acad. Reale dei Lincei **23**, 775 (1936); L. S. Ornstein, Kon. Akad. van Wet. te Amst. Proc. **XXXIX**, No. 9 (1936); Halpern, Lueneburg, and Clark, Phys. Rev. **53**, 173 (1938); Schuchard and Uehling, Phys. Rev. **58**, 611 (1940).

[2] E. Hopf, "Problems of radiative equilibrium," Cambridge Tract No. 31 (1934); E. A. Milne, "Thermodynamics of stars," *Handbuch der Astrophys.*, Vol. III/I (Julius Springer, Berlin, 1930); G. Jaffe, Ann. d. Physik [5] **6**, 195 (1930); S. Q. Duntley, J. Opt. Soc. Am. **33**, 252 (1934); E. O. Hulbert, J. Opt. Soc. Am. **33**, 42 (1943).

tigation of the multiple scattering of waves. Finally, the connection between the so-called coherent scattering (leading to the macroscopic phenomena of refraction and specular reflection) and the incoherent scattering can only be appreciated through a consistent wave treatment of scattering from beginning to end.

In this paper we shall develop a wave theory of multiple scattering for the case of scalar waves which are isotropically scattered by a random distribution of scatterers, the "randomness" being defined explicitly in the next section. We deal with abstract waves of unspecified type so that the treatment is applicable to any type of wave satisfying the scalar wave equation. The concept of randomness being essentially statistical, it is necessary to concentrate one's interest in the average values of physical quantities, the averages being taken over a statistical ensemble of collections of scatterers. The quantities of greatest interest are the wave function itself, the square of its absolute magnitude, or its mean square value, and the flux of some quantity (energy, probability, etc.) carried by the wave.

The determination of the average value of the wave function is practically identical with the already treated theory of the index of refraction of material media (other than crystalline media) for light,[3] but is included in the present treatment since it forms a necessary introduction to the second problem of determining the average value of the square of the absolute value of the wave function. It is shown that the average value of the wave function satisfies the wave equation in a continuous medium in which no scatterers are present and in which the velocity of propagation is different from the velocity in the original medium in the absence of the scatterers and is, in general, a function of position and complex rather than real in value. In other words, the incident wave and the scattered waves combine on the average to form a wave which travels uniformly without scattering at a different velocity from the incident wave and with attenuation. This wave shows refraction and reflection phenomena and thus describes the coherent aspects of scattering, such as the specular reflection taking place at a boundary between a region where scatterers are present (in sufficiently great density) and a region where no scatterers are present. An example would be the reflection of light at a glass-vacuum interface, the atoms of the glass being the scatterers.

The principal contribution of this paper, however, lies in the treatment of the average of the square of the wave function, which is in general different from the square of the average value of the wave function. The difference represents what is usually referred to as the incoherent scattering, the term incoherent referring not to a lack of definite phase relationship between the incident wave and the wave scattered by a particular scatterer, but to the statistical superposition of these scattered waves when propagated to the point of observation because of the "random" distribution of the scatterers. The average of the square of the wave function will be shown to satisfy an integral equation which has some similarities to the form of the Boltzmann integro-differential equation governing the multiple scattering of particles. This integral equation is just as fundamental to the problem of the multiple scattering of waves as the Boltzmann integro-differential equation is to the problem of the multiple scattering of particles. The physical interpretation of the integral equation may be fairly readily recognized: The equation states that the average of the square of the wave function at any point is equal to the square of the average wave function at the point plus contributions representing scattering from all other portions of the medium; these contributions are proportional to the average of the square of the wave function at each point and to a scattering cross section per unit volume at each of these points. The value of the scattering cross section per unit volume at each point and the mode of propagation from each point to the point of interest are exactly defined in terms of the scattering properties of the individual scatterers and the properties of the statistical ensemble of scatterers studied. In fact the integral equation could be derived from its *a posteriori* physical interpretation, but in that case the value of the scattering cross section per unit volume and the mode of propagation of the scattered waves would have to be intro-

[3] M. Born, *Optik* (Julius Springer, Berlin, 1933), pp. 313 ff.

duced *ad hoc*, so that such a derivation could hardly be considered rigorous or fundamental.

STATISTICAL CONSIDERATIONS

Since there is no inherent limitation in the theory restricting it to the case where all scatterers have identical scattering properties, we shall assume that the scattering properties of the scatterers are determined by a *scattering parameter* denoted by s. In some applications s may be a continuously varying physical parameter such as the radius of water droplets in the problem of the scattering of sound by a fog, or it may be simply an index identifying particular types of scatterers such as various types of nuclei present in the problem of the scattering of neutrons by an amorphous solid or by a gas. In what follows we shall assume it to be a continuous variable, but the transformation of the equations in case it takes on only discrete values involves only an obvious replacement of integrals by summations.

If we have a collection of N scatterers and are given for each its position and the value of s which characterizes its scattering properties we shall say that we have a particular configuration of the scatterers. Now, in the problem of multiple scattering of waves by a collection of amorphously distributed scatterers, one rarely has sufficient information to establish the microscopic configuration of the collection and one is therefore not so much interested in the values of physical quantities for a particular configuration of the scatterers as in the average value of these quantities, the average being taken over the ensemble of possible configurations of the scatterers consistent with what information one has as to the macroscopic state of the collection. Thus if we specify the positions of the scatterers by their position vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N$, and their scattering properties by the values of the scattering parameters $s_1, s_2, \cdots s_N$, a particular configuration of the scatterers will be given when all the $\mathbf{r}_j$'s and s_j's are specified. We may then indicate the ensemble of configurations in which we are interested by a probability distribution function $P(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N, s_1, s_2, \cdots s_N)$ so that

$$P(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N, s_1, s_2, \cdots s_N) d\mathbf{r}_1 d\mathbf{r}_2 \cdots \times d\mathbf{r}_N ds_1 ds_2 \cdots ds_N, \quad (1)$$

represents the probability of finding the scatterers in a configuration in which the first scatterer lies in the element of volume $d\mathbf{r}_1$ about the point $\mathbf{r}_1$ and has a scattering parameter lying between s_1 and s_1+ds_1, the second scatterer lies in the element of volume $d\mathbf{r}_2$ about the point $\mathbf{r}_2$ and has a scattering parameter lying between s_2 and s_2+ds_2, etc. To find the average of a physical quantity over the ensemble of configurations we then multiply its value for the configuration $(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N, s_1, s_2, \cdots s_N)$ by (1) and integrate over all values of the $\mathbf{r}_j$'s and s_j's accessible to the scatterers.

In this paper we shall be interested only in probability distribution functions in which the probability that a particular scatterer is located in some volume element and has a value of s located within some range ds is independent of the locations and scattering parameters of the other scatterers and is the same for all scatterers. In this case we may write for the probability distribution function

$$P = (1/N)^N n(\mathbf{r}_1, s_1) n(\mathbf{r}_2, s_2) \cdots n(\mathbf{r}_N, s_N). \quad (2)$$

The average number of scatterers per unit volume in the neighborhood of the point $\mathbf{r}$ having scattering parameters lying between s and $s+ds$ is then

$$n(\mathbf{r}, s) ds,$$

and the total number of scatterers per unit volume in the neighborhood of the point $\mathbf{r}$ is

$$n(\mathbf{r}) = \int n(\mathbf{r}, s) ds,$$

the integration being taken over all values of s. Finally the normalization of the probability requires

$$\int_V \int n(\mathbf{r}, s) ds d\mathbf{r} = N$$

where V represents the total volume accessible to the scatterers.

We shall call the average of a physical quantity over the ensemble of configurations a *configurational average* and shall denote it by enclosing the quantity which is averaged in angular brackets. Thus if f is a function of the $\mathbf{r}_j$'s and s_j's, its

configurational average is

$$\langle f\rangle=\int_V\cdots\int_V\int_V\cdots\int f(\mathbf{r}_j, s_j)\times P(\mathbf{r}_j, s_j)ds_1ds_2\cdots ds_N d\mathbf{r}_1 d\mathbf{r}_2\cdots d\mathbf{r}_N. \quad (3)$$

If we wish to omit the integration over the positions and scattering powers of one or more of the scatterers we shall indicate this by subscribing the indices of these scatterers to the angular brackets; thus $\langle f\rangle_j$ indicates that the integration over $\mathbf{r}_j$ and s_j is to be omitted in (3).

STATEMENT OF THE PROBLEM

In this section we shall make a precise statement of the problem which we shall treat in this paper. Interest here will be restricted to the case of steady-state scattering of waves of a single frequency ω so that the value of the scalar wave function at the point $\mathbf{r}$ at time t can be represented as $\psi(\mathbf{r})e^{i\omega t}$. In the absence of scatterers from the medium, $\psi(\mathbf{r})$ will then satisfy the wave equation

$$\nabla^2\psi+k_0^2\psi=0, \quad (4)$$

where $k_0=\omega/c_0$ and c_0 is the wave velocity in the scatterer-free medium. Furthermore we shall consider here only isotropic point scatterers, that is, those which scatter spherically symmetrical waves (s-wave scattering only). In this case the wave function in the presence of the scatterers will have simple poles at the positions of the scatterers and in the neighborhood of the jth scatterer will behave like

$$\frac{A_j\exp[-ik_0|\mathbf{r}-\mathbf{r}_j|]}{|\mathbf{r}-\mathbf{r}_j|}.$$

We may then define the *external field* acting on the jth scatterer as

$$\psi^j(\mathbf{r})=\psi(\mathbf{r})-\frac{A_j\exp[-ik_0|\mathbf{r}-\mathbf{r}_j|]}{|\mathbf{r}-\mathbf{r}_j|}, \quad (5)$$

this field having no singularity at $\mathbf{r}=\mathbf{r}_j$. We then characterize the scattering properties of the scatterers by the relationship

$$A_j=g(s_j, \omega)\psi^j(\mathbf{r}_j), \quad (6)$$

making the strength of the scattered wave from a scatterer proportional to the external field acting on it. In the above $g(s_j, \omega)$ will be referred to as the *scattering coefficient* for the jth scatterer and will be abbreviated to g_j.

The problem which we shall consider in this paper is then the following: Given the function $g(s, \omega)$ and the distribution function $n(\mathbf{r}, s)$ for the scatterers, and given also the form of the wave function $\psi_0(\mathbf{r})$ which is present in the medium in the absence of the scatterers (the incident wave), to find the configurational average of the wave function, $\psi(\mathbf{r})$, and of its mean-squared value, $|\psi(\mathbf{r})|^2$, in the presence of the scatterers.

FUNDAMENTAL EQUATIONS

Consider a particular configuration of the scatterers. We then attempt to represent the value of the wave function at the point $\mathbf{r}$ in the form

$$\psi(\mathbf{r})=\psi_0(\mathbf{r})+\sum_j A_jE(\mathbf{r}, \mathbf{r}_j),$$

where we have introduced the abbreviation

$$E(\mathbf{r}, \mathbf{r}')=\frac{\exp[-ik_0|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}.$$

The equation represents the field as the sum of the incident wave and spherical waves diverging from each of the scatterers. The external field acting on the jth scatterer is then

$$\psi^j(\mathbf{r})=\psi_0(\mathbf{r})+\sum_{j'(\neq j)} A_{j'}E(\mathbf{r}, \mathbf{r}_{j'}).$$

We must then have, according to Eq. (6),

$$A_j=g_j\psi^j(\mathbf{r}_j),$$

so our equations become

$$\psi(\mathbf{r})=\psi_0(\mathbf{r})+\sum_j g_j\psi^j(\mathbf{r}_j)E(\mathbf{r}, \mathbf{r}_j), \quad (7)$$

$$\psi^j(\mathbf{r}_j)=\psi_0(\mathbf{r}_j)+\sum_{j'(\neq j)} g_{j'}\psi^{j'}(\mathbf{r}_{j'})E(\mathbf{r}_j, \mathbf{r}_{j'}). \quad (8)$$

These represent the fundamental equations of multiple scattering.

The direct method of solving our problem would then consist of solving the set of simultaneous linear algebraic Eqs. (8) for the $\psi^j(\mathbf{r}_j)$ and substituting these in Eq. (7), thus giving $\psi(\mathbf{r})$ as a function of the positions and scattering

parameters of the scatterers. Taking the configurational average of this quantity and of the square of its absolute value according to Eq. (3) would then give us the desired results. Unfortunately, it does not seem possible to carry this procedure through because of the complexity of the necessary integrations, and it is necessary to resort to another procedure. This alternative method consists of attempting to find equations satisfied by $\langle|\psi(\mathbf{r})|\rangle$ and $\langle|\psi(\mathbf{r})|^2\rangle$ and then solving these equations for the desired averaged quantities. We shall now carry through, with some approximations, this latter procedure.

THE CONFIGURATIONAL AVERAGE OF THE WAVE FUNCTION

Let us take the configurational average of both sides of Eq. (7) formally to obtain

$$\langle\psi(\mathbf{r})\rangle=\psi_0(\mathbf{r})+\sum_j\int_V\int g_j\langle\psi^j(\mathbf{r}_j)\rangle_j \times E(\mathbf{r},\mathbf{r}_j)\frac{n(\mathbf{r}_j,s_j)}{N}ds_jd\mathbf{r}_j$$

$$=\psi_0(\mathbf{r})+\int_V G(\mathbf{r}_j)\langle\psi^j(\mathbf{r}_j)\rangle_j E(\mathbf{r},\mathbf{r}_j)d\mathbf{r}_j, \quad (9)$$

where

$$G(\mathbf{r})=\int g(s,\omega)n(\mathbf{r},s)ds. \quad (10)$$

The quantity $\langle\psi^j(\mathbf{r}_j)\rangle_j$ represents the external field acting on the jth scatterer averaged over all possible configurations of all of the other scatterers. The only rigorous way of evaluating it seems to be to solve the set of simultaneous linear algebraic Eqs. (8) and then carry out the necessary integrations to obtain the above configurational average. This again is not feasible. We therefore must resort to the approximation of replacing the external field acting on the jth scatterer averaged over all configurations of the other scatterers by the average field which would exist at the position of the jth scatterer when the scatterer is not present. This last average field would differ only by a term of the order $1/N$ from[4] $\langle\psi(\mathbf{r}_j)\rangle$, so that if N is large, and if the above approximation is valid, which it appears to be on physical grounds, we may substitute $\langle\psi(\mathbf{r}_j)\rangle$ for $\langle\psi^j(\mathbf{r}_j)\rangle_j$ under the integral sign in the above equation thus obtaining the integral equation

$$\langle\psi(\mathbf{r})\rangle=\psi_0(\mathbf{r})+\int_V G(\mathbf{r}')\langle\psi(\mathbf{r}')\rangle E(\mathbf{r},\mathbf{r}')d\mathbf{r}', \quad (11)$$

for the configurational average of the wave function.

To see the physical significance of this equation, let us apply the operator

$$\nabla^2+k_0^2 \quad (12)$$

to both sides of the integral equation, remembering that

$$(\nabla^2+k_0^2)E(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}'), \quad (13)$$

where $\delta(\mathbf{r}-\mathbf{r}')$ is the three-dimensional Dirac δ-function defined by the equation

$$\int_V f(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')d\mathbf{r}'=\begin{cases}f(\mathbf{r}) \text{ if } \mathbf{r} \text{ lies in } V\\ 0 \text{ if } \mathbf{r} \text{ lies outside of } V.\end{cases} \quad (14)$$

We then obtain

$$\nabla^2\langle\psi(\mathbf{r})\rangle+k_0^2\langle\psi(\mathbf{r})\rangle=-4\pi G(\mathbf{r})\langle\psi(\mathbf{r})\rangle. \quad (15)$$

If we write this as

$$\nabla^2\langle\psi(\mathbf{r})\rangle+k^2(r)\langle\psi(\mathbf{r})\rangle=0, \quad (16)$$

where

$$k^2(\mathbf{r})=k_0^2+4\pi G(\mathbf{r}),$$

we see that $\langle\psi(\mathbf{r})\rangle$ satisfies the wave equation in a "continuous medium" in which the velocity of propagation depends upon the scattering coefficients and density of distribution of the scatterers and is, in general, a function of position. Thus the problem of finding the average value of the wave function has been essentially reduced to solving a boundary value problem in the wave equation. The boundary conditions are implied in the integral equation itself and depend on the function $G(\mathbf{r})$. If $G(\mathbf{r})$ is everywhere continuous and approaches a constant value or zero at infinity, then the boundary conditions are that $\langle\psi(\mathbf{r})\rangle-\psi_0(\mathbf{r})$ be everywhere continuous and have a continuous gradient and at infinity represent outward travelling waves. In another important case, that in which $G(\mathbf{r})$ is sectionally continuous, the boundary conditions are that $\langle\psi(\mathbf{r})\rangle-\psi_0(\mathbf{r})$ be everywhere continuous and have

[4] $\psi(\mathbf{r}_j)=\lim_{\mathbf{r}\to\mathbf{r}_j}\psi(\mathbf{r})$ is singular, but $\langle\psi(\mathbf{r}_j)\rangle=\lim_{\mathbf{r}\to\mathbf{r}_j}\langle\psi(\mathbf{r})\rangle$ is regular.

a continuous normal derivative across a surface of discontinuity of $G(\mathbf{r})$ and at infinity represent outward travelling waves, provided again that $G(\mathbf{r})$ approaches a constant value or zero at infinity. In both cases, of course $\langle\psi(\mathbf{r})\rangle$ must approach $\psi_0(\mathbf{r})$ as $G(\mathbf{r})$ becomes zero everywhere.

This reduction of the determination of $\langle\psi(\mathbf{r})\rangle$ to the solution of a boundary value problem in the wave equation corresponds, of course, to the well-known theory of the refractive index for light of gases.

We note also that the integral equation (11) can in principle be solved directly. One method of solution is the Liouville-Neumann method of successive substitutions or iteration method.[5] It consists in repeatedly substituting the expression for $\langle\psi(\mathbf{r})\rangle$ as given by the right-hand side of the integral equation for $\langle\psi(\mathbf{r}')\rangle$ under the integral sign, thus yielding in our case the infinite series

$$\langle\psi(\mathbf{r})\rangle=\sum_{m=0}^{\infty}\psi_m(\mathbf{r}) \tag{17}$$

where

$$\psi_m(\mathbf{r})=\int_V G(\mathbf{r}')\psi_{m-1}(\mathbf{r}')E(\mathbf{r},\mathbf{r}')d\mathbf{r}' \quad (m\neq 0) \tag{18}$$

which, if it converges uniformly, is the desired solution.

Reverting for a moment to the approximation $(\langle\psi^j(\mathbf{r}_j)\rangle_j\simeq\langle\psi(\mathbf{r}_j)\rangle)$ made at the beginning of this section, it should be mentioned that it would be desirable to demonstrate its validity mathematically rather than on intuitional grounds. Unfortunately it has not yet been found possible to do this. A similar approximation must be made in the next section to which this remark also applies.

THE CONFIGURATIONAL AVERAGE OF $|\psi(\mathbf{r})|^2$

To obtain the configurational average of $|\psi(\mathbf{r})|^2$ we follow an analogous procedure to that used for $\psi(\mathbf{r})$, but the analysis now becomes somewhat more involved. To begin we first multiply the expression for $\psi(\mathbf{r})$ as given by Eq. (7) by the corresponding expression for $\psi^*(\mathbf{r})$ with $\mathbf{r}$ set equal to $\mathbf{r}_0$, where the asterisk denotes the complex conjugate of the quantity to which it is attached. This yields

$$\begin{aligned}\psi(\mathbf{r})\psi^*(\mathbf{r}_0)&=\psi_0(\mathbf{r})\psi_0^*(\mathbf{r}_0)+\psi_0(\mathbf{r})[\psi^*(\mathbf{r}_0)-\psi_0^*(\mathbf{r}_0)]\\&+\psi_0^*(\mathbf{r}_0)[\psi(\mathbf{r})-\psi_0(\mathbf{r})]\\&+\sum_{j'}g_{j'}g_{j'}^*\psi^{j'}(\mathbf{r}_{j'})\psi^{j'*}(\mathbf{r}_{j'})E(\mathbf{r},\mathbf{r}_{j'})E^*(\mathbf{r}_0,\mathbf{r}_{j'})\\&+\sum_{j'j''}{}'g_{j'}g_{j''}^*\psi^{j'}(\mathbf{r}_{j'})\psi^{j''*}(\mathbf{r}_{j''})E(\mathbf{r},\mathbf{r}_{j'})E^*(\mathbf{r}_0,\mathbf{r}_{j''}).\end{aligned}$$

The prime on the last summation indicates that terms for which $j'=j''$ are to be omitted from the sum. If we now take the configurational average of both sides of this equation we obtain

$$\begin{aligned}\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle&=\psi_0(\mathbf{r})\psi_0^*(\mathbf{r}_0)\\&+\psi_0(\mathbf{r})[\langle\psi^*(\mathbf{r}_0)\rangle-\psi_0^*(\mathbf{r}_0)]\\&+\psi_0^*(\mathbf{r}_0)[\langle\psi(\mathbf{r})\rangle-\psi_0(\mathbf{r})]\\&+\int_V H(\mathbf{r}_{j'})\langle\psi^{j'}(\mathbf{r}_{j'})\psi^{j'*}(\mathbf{r}_{j'})\rangle_{j'}\\&\quad\times E(\mathbf{r},\mathbf{r}_{j'})E^*(\mathbf{r}_0,\mathbf{r}_{j'})d\mathbf{r}_{j'}\\&+\frac{N-1}{N}\int_V\int_V G(\mathbf{r}_{j'})G^*(\mathbf{r}_{j''})\langle\psi^{j'}(\mathbf{r}_{j'})\psi^{j''*}(\mathbf{r}_{j''})\rangle_{j'j''}\\&\qquad\times E(\mathbf{r},\mathbf{r}_{j'})E^*(\mathbf{r}_0,\mathbf{r}_{j''})d\mathbf{r}_{j'}d\mathbf{r}_{j''},\end{aligned} \tag{19}$$

where

$$H(\mathbf{r})=\int|g(s,\omega)|^2n(\mathbf{r},s)ds. \tag{20}$$

We now make a similar assumption to that made in the preceding section, although the physical meaning is not so clear in this case. We assume that it is valid to replace

$$\langle\psi^{j'}(\mathbf{r}_{j'})\psi^{j'*}(\mathbf{r}_{j'})\rangle_{j'} \text{ by } \langle\psi(\mathbf{r}_{j'})\psi^*(\mathbf{r}_{j'})\rangle$$

and

$$\langle\psi^{j'}(\mathbf{r}_{j'})\psi^{j''*}(\mathbf{r}_{j''})\rangle_{j'j''} \text{ by } \langle\psi(\mathbf{r}_{j'})\psi^*(\mathbf{r}_{j''})\rangle.$$

Making this substitution and replacing $(N-1)/N$ by unity since we are assuming N is large (compared to unity) we obtain the integral equation

$$\begin{aligned}\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle&=\psi_0(\mathbf{r})\psi_0^*(\mathbf{r}_0)\\&+\psi_0(\mathbf{r})[\langle\psi^*(\mathbf{r}_0)\rangle-\psi_0^*(\mathbf{r}_0)]\\&+\psi_0^*(\mathbf{r}_0)[\langle\psi(\mathbf{r})\rangle-\psi_0(\mathbf{r})]\\&+\int_V H(\mathbf{r}')\langle\psi(\mathbf{r}')\psi^*(\mathbf{r}')\rangle E(\mathbf{r},\mathbf{r}')E^*(\mathbf{r}_0,\mathbf{r}')d\mathbf{r}'\\&+\int_V\int_V G(\mathbf{r}')G^*(\mathbf{r}'')\langle\psi(\mathbf{r}')\psi^*(\mathbf{r}'')\rangle\\&\qquad\times E(\mathbf{r},\mathbf{r}')E^*(\mathbf{r}_0,\mathbf{r}'')d\mathbf{r}'d\mathbf{r}''.\end{aligned} \tag{21}$$

[5] Whittaker and Watson, *Modern Analysis* (Cambridge University Press, 1927), Chap. XI.

By substituting the right-hand side of Eq. (9) for $\langle\psi(\mathbf{r})\rangle$ and the complex conjugate for $\langle\psi^*(\mathbf{r}_0)\rangle$ in the above equation we obtain a simpler form of the equation:

$$\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle-\langle\psi(\mathbf{r})\rangle\langle\psi^*(\mathbf{r}_0)\rangle$$
$$=\int_V H(\mathbf{r}')\langle\psi(\mathbf{r}')\psi^*(\mathbf{r}')\rangle E(\mathbf{r},\mathbf{r}')E^*(\mathbf{r}_0,\mathbf{r}')d\mathbf{r}'$$
$$+\int_V\int_V G(\mathbf{r}')G^*(\mathbf{r}'')[\langle\psi(\mathbf{r}')\psi^*(\mathbf{r}'')\rangle$$
$$-\langle\psi(\mathbf{r}')\rangle\langle\psi^*(\mathbf{r}'')\rangle]E(\mathbf{r},\mathbf{r}')E^*(\mathbf{r}_0,\mathbf{r}'')d\mathbf{r}'d\mathbf{r}''. \quad (22)$$

Now the above integral equation for $\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle$ is in too complicated a form to obtain practical solutions of it directly. However, we can simplify the equation further in such a way that reasonably rapidly convergent series solutions may be obtained in many cases. An examination of the integral equation (22) shows that by obtaining the Liouville-Neumann iteration solution of the equation, considered as an integral equation in $[\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle-\langle\psi(\mathbf{r})\rangle\langle\psi^*(\mathbf{r}_0)\rangle]$ with the first term on the right in Eq. (22) considered as the inhomogeneous term of the integral equation, $\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle$ satisfies the integral equation

$$\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle=\langle\psi(\mathbf{r})\rangle\langle\psi^*(\mathbf{r}_0)\rangle$$
$$+\int\langle\psi(\mathbf{r}')\psi^*(\mathbf{r}')\rangle L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')d\mathbf{r}' \quad (23)$$

where the kernel $L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$ is given by the iteration series:

$$L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=\sum_{p=0}^{\infty}M_p(\mathbf{r},\mathbf{r}_0;\mathbf{r}'), \quad (24)$$

$$M_p(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=\int_V\int_V G(\mathbf{r}'')G^*(\mathbf{r}''')$$
$$\times M_{p-1}(\mathbf{r}'',\mathbf{r}''';\mathbf{r}')E(\mathbf{r},\mathbf{r}'')$$
$$\times E^*(\mathbf{r}_0,\mathbf{r}''')d\mathbf{r}''d\mathbf{r}''', \quad (25)$$

$$M_0(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=H(r')E(\mathbf{r},\mathbf{r}')E^*(\mathbf{r}_0,\mathbf{r}'). \quad (26)$$

This solution has the disadvantage that the series solution for the kernel $L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$ will be very slowly convergent since the factors $E(\mathbf{r},\mathbf{r}'')$ and $E^*(\mathbf{r}_0,\mathbf{r}''')$ in the iteration integrals are imaginary exponentials and do not promote convergence of the series, in general. An improvement on the situation is possible by obtaining a more rapidly convergent series for the kernel.

To do this we first find the partial differential equation satisfied by $L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$. If we apply the operator $(\nabla_0^2+k_0^2)(\nabla^2+k_0^2)$ to both sides of Eq. (24), where ∇^2 is the Laplacian operator in the $\mathbf{r}$ coordinates and ∇_0^2 the Laplacian operator in the $\mathbf{r}_0$ coordinates, the result of this operator acting on M_0 will be $16\pi^2\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{r}_0-\mathbf{r}')$, while the result of the operator acting on the remainder of the series is readily found to be $16\pi^2G(\mathbf{r})\times G^*(\mathbf{r}_0)L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$. Thus the integral equation kernel satisfies the partial differential equation

$$[(\nabla_0^2+k_0^2)(\nabla^2+k_0^2)-16\pi^2G(\mathbf{r})G^*(\mathbf{r}_0)]L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$$
$$=16\pi^2\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{r}_0-\mathbf{r}'). \quad (27)$$

If we now add

$$\{4\pi G(\mathbf{r})[\nabla_0^2+k^{*2}(\mathbf{r}_0)]$$
$$+4\pi G^*(\mathbf{r}_0)[\nabla^2+k^2(\mathbf{r})]\}L(\mathbf{r},\mathbf{r}_0;\mathbf{r}') \quad (28)$$

to both sides of this equation, and remember that

$$k^2(\mathbf{r})=k_0^2+4\pi G(\mathbf{r}), \quad (16)$$

and

$$k^{*2}(\mathbf{r}_0)=k_0^2+4\pi G^*(\mathbf{r}_0), \quad (16')$$

the equation can be written

$$[(\nabla_0^2+k^{*2}(\mathbf{r}_0)][\nabla^2+k^2(\mathbf{r})]L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$$
$$=16\pi^2\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{r}_0-\mathbf{r}')$$
$$+\{4\pi G(\mathbf{r})[\nabla_0^2+k^{*2}(\mathbf{r}_0)]$$
$$+4\pi G^*(\mathbf{r}_0)[\nabla^2+k^2(\mathbf{r})]\}L(\mathbf{r},\mathbf{r}_0;\mathbf{r}'). \quad (29)$$

Let us now turn our attention to the function $K(\mathbf{r},\mathbf{r}')$ defined as the solution of the equation

$$\nabla^2K(\mathbf{r},\mathbf{r}')+k^2(\mathbf{r})K(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}'), \quad (30)$$

which represents outward travelling waves at infinity. It will be noted that $K(\mathbf{r},\mathbf{r}')$ bears the same relationship to Eq. (15) as $E(\mathbf{r},\mathbf{r}')$ does to Eq. (4). $K(\mathbf{r},\mathbf{r}')$ is a Green's function for Eq. (15) and can be interpreted as the wave field produced by a point source of unit strength placed at the point $\mathbf{r}'$ in a medium in which the propagation constant at the point $\mathbf{r}$ is given by $k(\mathbf{r})$. Then $K(\mathbf{r},\mathbf{r}')K^*(\mathbf{r}_0,\mathbf{r}')$ satisfies the equa-

tion

$$[\nabla_0^2+k^{*2}(\mathbf{r}_0)][\nabla^2+k^2(\mathbf{r})]K(\mathbf{r},\mathbf{r}')K^*(\mathbf{r}_0,\mathbf{r}')$$
$$=16\pi^2\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{r}_0-\mathbf{r}'). \quad (31)$$

From this one readily finds that $L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$ satisfies the integral equation

$$L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=K(\mathbf{r},\mathbf{r}')K^*(\mathbf{r}_0,\mathbf{r}')$$
$$+\frac{1}{4\pi}\int_V\int_V[\{G(\mathbf{r}'')[\nabla'''^2+k^{*2}(\mathbf{r}''')]$$
$$+G^*(\mathbf{r}''')[\nabla''^2+k^2(\mathbf{r}'')]\}L(\mathbf{r}'',\mathbf{r}''';\mathbf{r}')]$$
$$\times K(\mathbf{r},\mathbf{r}'')K^*(\mathbf{r}_0,\mathbf{r}''')d\mathbf{r}''d\mathbf{r}''', \quad (32)$$

as may be verified by substitution in Eq. (29). The Liouville-Neumann iteration procedure applied to this integral equation gives us a new series solution for the kernel of our integral equation:

$$L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=\sum_{p=0}^{\infty}L_p(\mathbf{r},\mathbf{r}_0;\mathbf{r}'), \quad (33)$$

$$L_p(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=\frac{1}{4\pi}\int_V\int_V[\{G(\mathbf{r}'')[\nabla'''^2+k^{*2}(\mathbf{r}''')]$$
$$+G^*(\mathbf{r}''')[\nabla''^2+k^2(\mathbf{r}'')]\}$$
$$\times L_{p-1}(\mathbf{r}'',\mathbf{r}''';\mathbf{r}')]K(\mathbf{r},\mathbf{r}'')$$
$$\times K^*(\mathbf{r}_0,\mathbf{r}''')d\mathbf{r}''d\mathbf{r}''', \quad (34)$$

$$L_0(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=K(\mathbf{r},\mathbf{r}')K^*(\mathbf{r}_0,\mathbf{r}'). \quad (35)$$

This, in general, will be a quite rapidly converging series since $K(\mathbf{r},\mathbf{r}')$ contains a real negative exponential. The successive series terms will contain increasingly larger numbers of integrations involving the negative exponential and will thus die out rapidly. Thus, for example, the second term in the series is

$$L_1(\mathbf{r},\mathbf{r}_0;\mathbf{r}')=-\int_V G(\mathbf{r}'')K(\mathbf{r}'',\mathbf{r}')$$
$$\times K(\mathbf{r},\mathbf{r}'')K^*(\mathbf{r}_0,\mathbf{r}')d\mathbf{r}''$$
$$-\int_V G^*(\mathbf{r}'')K^*(\mathbf{r}'',\mathbf{r}')K^*(\mathbf{r}_0,\mathbf{r}'')K(\mathbf{r},\mathbf{r}')d\mathbf{r}''. \quad (36)$$

The solution to our problem has thus been reduced to solving the integral equation

$$\langle|\psi(\mathbf{r})|^2\rangle=|\langle\psi(\mathbf{r})\rangle|^2$$
$$+\int_V H(\mathbf{r}')\langle|\psi(\mathbf{r}')|^2\rangle L(\mathbf{r},\mathbf{r};\mathbf{r}')d\mathbf{r}'. \quad (37)$$

In case the kernel $L(\mathbf{r},\mathbf{r};\mathbf{r}')$ can be accurately represented by the first term in the series (33), the integral equation becomes

$$\langle|\psi(\mathbf{r})|^2\rangle=|\langle\psi(\mathbf{r})\rangle|^2$$
$$+\int_V H(\mathbf{r}')\langle|\psi(\mathbf{r}')|^2\rangle|K(\mathbf{r},\mathbf{r}')|^2d\mathbf{r}'. \quad (38)$$

This latter equation is probably the most important practical result of our investigation.

AUXILIARY PHYSICAL QUANTITIES

There are several other quantities of importance in wave propagation theory and we shall now obtain expressions for the configurational average of these. The first of these is the gradient of the wave function. It is readily observed that the operation of taking the gradient and the integrations involved in obtaining the configurational average commute so that the average of the gradient of the wave function is simply the gradient of the configurational average of the wave function:

$$\langle\nabla\psi(\mathbf{r})\rangle=\nabla\langle\psi(\mathbf{r})\rangle. \quad (39)$$

The other quantity of importance is the flux which is proportional to the quantity

$$\mathfrak{F}=\psi^*\nabla\psi-\psi\nabla\psi^*. \quad (40)$$

We may obtain the configurational average of this quantity by taking the gradient of both sides of Eq. (23) with respect to $\mathbf{r}$ and subtracting from this the gradient of both sides of the same equation with respect to $\mathbf{r}_0$. On subsequently setting $\mathbf{r}_0=\mathbf{r}$, we obtain

$$\langle\mathfrak{F}(\mathbf{r})\rangle=\langle\psi^*(\mathbf{r})\nabla\psi(\mathbf{r})-\psi(\mathbf{r})\nabla\psi^*(\mathbf{r})\rangle$$
$$=\langle\psi^*(\mathbf{r})\rangle\nabla\langle\psi(\mathbf{r})\rangle-\langle\psi(\mathbf{r})\rangle\nabla\langle\psi^*(\mathbf{r})\rangle$$
$$+\int_V H(\mathbf{r}')\langle|\psi(\mathbf{r}')|^2\rangle[\nabla L(\mathbf{r},\mathbf{r}_0;\mathbf{r}')$$
$$-\nabla_0 L^*(\mathbf{r},\mathbf{r}_0;\mathbf{r}')]_{\mathbf{r}_0=\mathbf{r}}d\mathbf{r}'. \quad (41)$$

Again if $L(\mathbf{r}, \mathbf{r}_0; \mathbf{r}')$ can be replaced by the first term in its iteration series, this result can be written

$$\langle \mathfrak{F}(\mathbf{r}) \rangle = \langle \psi^*(\mathbf{r}) \rangle \nabla \langle \psi(\mathbf{r}) \rangle - \langle \psi(\mathbf{r}) \rangle \nabla \langle \psi^*(\mathbf{r}) \rangle + \int H(\mathbf{r}') \langle |\psi(\mathbf{r}')|^2 \rangle [K^*(\mathbf{r}, \mathbf{r}') \nabla K(\mathbf{r}, \mathbf{r}') - K(\mathbf{r}, \mathbf{r}') \nabla K^*(\mathbf{r}, \mathbf{r}')] d\mathbf{r}'. \quad (42)$$

Thus $\langle \mathfrak{F} \rangle$ can be calculated when $\langle |\psi(\mathbf{r})|^2 \rangle$ is known.

PHYSICAL INTERPRETATION

In order to interpret the results which we have obtained, it is useful first to consider the scattering of waves by a single scatterer. In this case the wave function becomes simply

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + g_1 \psi_0(\mathbf{r}_1) E(\mathbf{r}, \mathbf{r}_1), \quad (43)$$

which is the sum of the incident wave and the spherical wave scattered from the scatterer. If we consider a plane incident wave

$$\psi_0(\mathbf{r}) = A \exp [-i\mathbf{k}_0 \cdot \mathbf{r}], \quad (44)$$

and integrate the flux per unit area in the scattered wave, $\mathfrak{F} = \psi_s^* \nabla \psi_s - \psi_s \nabla \psi_s^*$, where ψ_s represents the second term on the right in Eq. (43), over a sphere whose center is at the position of the scatterer, we readily find that this total flux is

$$4\pi i k_0 |g_1|^2 A^2. \quad (45)$$

If we divide by the flux per unit area normal to $\mathbf{k}_0$ in the incident wave, $ik_0 A^2$, we obtain what is usually referred to as the *scattering cross section*, σ_s:

$$\sigma_s = 4\pi |g_1|^2. \quad (46)$$

It represents an area which when multiplied by the incident flux per unit area, gives the total flux in the scattered wave.

Now if one integrates the flux per unit area caused by both incident and scattered waves [that is $\psi^* \nabla \psi - \psi \nabla \psi^*$, where ψ is given by (43)] over the same sphere, one obtains

$$4\pi i k_0 |g_1|^2 A^2 + 4\pi (g_1 - g_1^*) A^2. \quad (47)$$

The negative of this quantity represents the net inward flux through the sphere. Dividing the latter by the flux per unit area in the incident wave gives the so-called *absorption* or *capture cross section*, σ_c, since it represents the area which when multiplied by the incident flux per unit area normal to $\mathbf{k}_0$, gives the flux which is effectively lost from the wave field. Thus

$$\sigma_c = -4\pi |g_1|^2 - \frac{4\pi Im(g_1)}{k_0} \equiv -\sigma_s + \sigma_e, \quad (48)$$

where

$$\sigma_e = -4\pi Im(g_1)/k_0 \quad (49)$$

is usually referred to as the *extinction cross section*, being the sum of the capture and scattering cross sections. In practical problems, the imaginary part of g_1 is usually negative so that the extinction cross section is positive.[6] Expressed in terms of the more familiar cross sections, the scattering coefficient g_1 becomes

$$g_1 = \left(\frac{\sigma_s}{4\pi} - \frac{k_0^2 \sigma_e^2}{16\pi^2} \right)^{\frac{1}{2}} - i \frac{k_0 \sigma_e}{4\pi}. \quad (50)$$

We may then define a scattering cross section per unit volume, $S_s(\mathbf{r})$, by

$$S_s(\mathbf{r}) = \int \sigma_s(s) n(\mathbf{r}, s) ds = 4\pi \int |g(s)|^2 n(\mathbf{r}, s) ds = 4\pi H(\mathbf{r}), \quad (51)$$

and an extinction cross section per unit volume, $S_e(\mathbf{r})$, by

$$S_e(\mathbf{r}) = \int \sigma_e(s) n(\mathbf{r}, s) ds = -\frac{4\pi}{k_0} \int Im[g(s)] n(\mathbf{r}, s) ds = -\frac{4\pi}{k_0} Im[G(\mathbf{r})]. \quad (52)$$

Let us now examine Eq. (15) which governs the propagation of the average value of the wave function in the case where $G(\mathbf{r})$ is a constant independent of position. In this case the plane wave solution of the differential equation will be

[6] If there is no "true" capture, σ_c must be zero. For a particular field, this can be shown from the field equations and, where appropriate, the boundary conditions at the scatterer. However, g will still be complex because of "radiation resistance."

of the form

$$\langle\psi(\mathbf{r})\rangle = A \exp[-i\mathbf{k}\cdot\mathbf{r}], \tag{53}$$

where $|\mathbf{k}| = k$ and the direction of $\mathbf{k}$ is the direction of propagation of the wave. Now since k is in general complex the wave represented in Eq. (53) is attenuated with distance. Now we have

$$k^2 = k_0^2 + 4\pi G = k_0^2 + 4\pi Re(G) + 4\pi i Im(G), \tag{54}$$

so that when the number of scatterers per unit volume is sufficiently small we may expand the square root for k to obtain

$$k \simeq k_0 + \frac{2\pi}{k_0} Re(G) + \frac{2\pi i}{k_0} Im(G). \tag{55}$$

Thus the flux in the wave (53) is reduced by the factor

$$\left[\exp \frac{2\pi}{k_0} Im(G)\right]^2 = e^{-S_e} \tag{56}$$

per unit distance. Hence in this case the medium acts as if it removes in a unit distance a fraction equal to the extinction cross section per unit volume of the flux carried by the wave. Alternatively, we may say that the medium acts as if each scatterer removes from the wave an amount of flux equal to its extinction cross section times the flux per unit area incident on it in the average wave. This flux is partially absorbed (if the capture cross section is not zero) and partially scattered out of the average wave propagating through the medium. It should be noted that this simple picture is applicable only when the number of scatterers per unit volume is sufficiently small so that the above expansion of the square root is valid. Otherwise there is interference between the scatterers and they no longer can be considered to absorb and scatter independently.

In addition the phase velocity $c(\mathbf{r})$ of the average wave is different from the phase velocity in the scatterer-free medium:

$$\frac{1}{c^2} = \frac{1}{c_0^2} + \frac{4\pi G(\mathbf{r})}{\omega^2}. \tag{57}$$

This may be attributed to the fact that there is a difference in phase between the wave incident on a scatterer and the scattered wave from the scatterer. When the scattered waves are combined with the incident wave there is a progressive advancement or retardation of the phase of the disturbance and the result appears as a change in the phase velocity.[7]

Let us now turn our attention to the physical interpretation of our integral equation for the configurational average of the square of the wave function written in terms of the scattering cross section per unit volume [see Eqs. (51), (37)]:

$$\langle|\psi(\mathbf{r})|^2\rangle = |\langle\psi(\mathbf{r})\rangle|^2 + \frac{1}{4\pi}\int S_s(\mathbf{r}')\langle|\psi(\mathbf{r}')|^2\rangle L(\mathbf{r}, \mathbf{r}; \mathbf{r}')d\mathbf{r}'. \tag{58}$$

Assume first that we may replace $L(\mathbf{r}, \mathbf{r}; \mathbf{r}')$ by the first term in the series (33), $|K(\mathbf{r}, \mathbf{r}')|^2$. Now we have already pointed out that $K(\mathbf{r}, \mathbf{r}')$ can be interpreted as the value of the wave function at the point $\mathbf{r}$ caused by a unit point source at the point $\mathbf{r}'$, in the continuous medium [characterized by the propagation constant $k(\mathbf{r})$] in which $\langle\psi(\mathbf{r})\rangle$ is propagated. Thus in this case the integral equation above simply states that the value of $\langle|\psi(\mathbf{r})|^2\rangle$ at any point $\mathbf{r}$ is equal to the mean-square value of the average wave at the point $\mathbf{r}$, $|\langle\psi(\mathbf{r})\rangle|^2$, plus contributions owing to scattered waves from all other points of the medium, the strength of the sources per unit volume at these other points being proportional to the value of $\langle|\psi(\mathbf{r})|^2\rangle$ at these points and to the scattering cross section per unit volume at these points. The various contributions, it will be noticed, combine as if the waves to which they are owing are *incoherent*, that is, bear a random phase relationship to one another. If we obtain the series solution of this integral equation by the Liouville-Neumann method, this solution, too, has a simple physical interpretation. The solution can be written

$$\langle|\psi(\mathbf{r})|^2\rangle = \sum_{m=0}^{\infty} \Psi_m(\mathbf{r}), \tag{59}$$

$$\Psi_m(\mathbf{r}) = \frac{1}{4\pi}\int S_s(\mathbf{r}')\Psi_{m-1}(\mathbf{r}')|K(\mathbf{r}, \mathbf{r}')|^2 d\mathbf{r}', \tag{60}$$

$$\Psi_0(\mathbf{r}) = |\langle\psi(\mathbf{r})\rangle|^2. \tag{61}$$

[7] The condition for no dispersion in the presence of the scatterers is obviously: $g(\omega, s) = \omega^2\gamma(s)$, where γ is independent of ω.

Thus the first term in the series represents the contribution caused by the average wave, the second term, the contributions caused by a single scattering of the average wave from each point in the medium to the point in question, the third term, the contributions caused by scattering twice of the average wave, etc. Hence, when $L(\mathbf{r}, \mathbf{r}; \mathbf{r}')$ can be replaced by the first term in its series solution, we have a very simple and logical physical interpretation of the integral equation which we have derived.

When the full series solution for $L(\mathbf{r}, \mathbf{r}; \mathbf{r}')$ is substituted in the integral equation, however, the meaning of the additional terms is not so clear. An examination of them indicates that they are some sort of residual interference terms which are not eliminated in the averaging process. This is apparent from the fact that the imaginary exponential terms are still present [Eqs. (34), (36)]. The physical interpretation of each term is quite simple: Thus, the terms corresponding to the next term in the series for L, that is L_1 [Eq. (36)], may be interpreted as the resultant at the point $\mathbf{r}$ of a wave scattered from the point $\mathbf{r}'$ to $\mathbf{r}$ by two paths, a direct path from $\mathbf{r}'$ to $\mathbf{r}$ and also by propagation from $\mathbf{r}'$ to another point $\mathbf{r}''$ from which the wave is rescattered *coherently* to the point $\mathbf{r}$, the result being summed over all intermediate points $\mathbf{r}''$. Similarly the higher terms of the series correspond to larger numbers of intermediate coherent scatterings. It is difficult to see, however, why these additional terms should be present in the equation. A rough calculation of the order of magnitude of these terms shows that in many practical cases, they are not very significant. Thus the ratio of the terms caused by L_1 to the terms caused by $L_0 = |K(\mathbf{r}, \mathbf{r}')|^2$ seems to be of the order of magnitude of $(k^2 - k_0^2)/k_0^2$, so that if the change in the velocity of the waves in the medium produced by the introduction of the scatterers is small, these higher terms are of little importance.

In conclusion, it may be helpful to point out that in most measurements on waves it is the mean-square value of the wave function rather than the wave function itself that is measured; therefore $\langle|\psi(\mathbf{r})|^2\rangle$ will usually be of more interest than $\langle\psi(\mathbf{r})\rangle$. Now the difference between $\langle|\psi(\mathbf{r})|^2\rangle$ and $|\langle\psi(\mathbf{r})\rangle|^2$ can be interpreted from the view of our statistical ensemble of configurations as effected by the statistical fluctuations of $\psi(\mathbf{r})$ about $\langle\psi(\mathbf{r})\rangle$ in different members of the ensemble. Another method of viewing this difference may be helpful in bringing out the physical interpretation. Consider instead of a statistical ensemble of configurations of scatterers, a collection of scatterers which are moving about slowly (so that Doppler effects may be neglected) in such a way that over a period of time the collection passes through all the configurations of our original ensemble. In this case $\langle|\psi(\mathbf{r})|^2\rangle$ can be interpreted as the long time average of $|\psi(\mathbf{r}, t)|^2$, $\psi(\mathbf{r}, t)$ being the value of the wave function at the point $\mathbf{r}$ at the time t. For such a collection of scatterers $\langle\psi(\mathbf{r})\rangle$ has little physical significance unless

$$\langle|\psi(\mathbf{r})|^2\rangle - |\langle\psi(\mathbf{r})\rangle|^2 \ll \langle|\psi(\mathbf{r})|^2\rangle. \quad (62)$$

To define the meaning of $\langle\psi(\mathbf{r})\rangle$ precisely, one may say that if $\psi(\mathbf{r}, t)$ represents the value of the wave function at the point $\mathbf{r}$ at time t, then $\langle\psi(\mathbf{r})\rangle$ is the function which makes the integral

$$\int |\psi(\mathbf{r}, t) - \langle\psi(\mathbf{r})\rangle e^{i\omega t}|^2 dt \quad (63)$$

a minimum, the integration being taken over a length of time sufficient for the collection of scatterers to have passed through all the configurations of the ensemble.

On the other hand, we have still to indicate why in scattering by a fixed configuration of scatterers, "incoherent scattering" is observed. The answer to this question lies in the fact that the regularly propagated wave or "coherent scattering" is precisely defined only as an average over the ensemble of configurations. For a fixed configuration, all of the scattering is strictly coherent but is artificially divided into a part which is estimated to propagate uniformly and a remainder which is referred to as "incoherently scattered." The basis of this distinction lies in the fact that for a collection of a large number of scatterers a particular "unprepared" collection will have particular physical properties which do not deviate greatly from the average physical properties of a properly defined statistical ensemble of collections because of the lack of "correlation" implied in the word "unprepared" as

to the positions of the individual scatterers. This allows one to estimate the properties of the average over an ensemble from an experiment on a particular configuration with a very high probability that the estimate is correct, although the possibility exists of a wide deviation if the selected ensemble should be particularly strongly ordered. In experiments usually several samples are used so as to guard against this remote possibility. We cannot enter further into this discussion of the relationship between statistical concepts and experiment, but refer the reader to almost any text on statistical mechanics for further elucidation.

DISCUSSION

It is of interest to indicate the directions in which the present theory may be generalized to embrace a wider range of phenomena:

(1) We may attempt to lift the restriction to isotropic (*s*-) scattering. This could be accomplished by including higher order scattering terms. Thus, in the next approximation we would include terms corresponding to dipole (*p*-) scattering which are of the form

$$\mathbf{B}\cdot\nabla E(\mathbf{r}, \mathbf{r}_j), \tag{64}$$

where $\mathbf{B}$ is some vector function of the external field acting on the scatterer, such as its gradient. By adding further terms corresponding to quadripole scattering, etc., one could, in principle, adequately treat any type of scattered wave. The difficulties in this extension of the theory appear to be simply ones of increasing complexity of the expressions handled.

(2) One might attempt to consider more general forms of probability distribution functions than those included in Eq. (1). Such more general functions would involve correlations between the positions of various scatterers so that the probability of finding a particular scatterer in the neighborhood of some point would be a function of the positions of one or more of the other scatterers. This type of function would allow a more accurate representation of the state of affairs in "amorphous" solids, liquids, and imperfect gases where such correlations do exist. Tempting as such a generalization might be, the well-known difficulties which arise in averaging over correlative probability distribution functions, are discouraging of elegant attempts at a rigorous theory in this direction. Some promise might appear, however, in an approximate treatment of this problem by simplifying the correlation between scatterers to that between the position of each scatterer and the average distribution of the remainder such as is done in the well-known approximate treatments of cooperative phenomena in statistical mechanics, or by the use of such methods as are employed in the theory of x-ray (single) scattering in liquids.

(3) A further generalization may be made in the extension of the above treatment to vector waves. If the vector wave function satisfies the vector wave equation and if the wave scattered by a scatterer is proportional directly to the external field acting on the scatterer, then the generalization follows immediately. However, in the case of greatest practical importance, electromagnetic waves, the above is not true. The wave scattered by a scatterer is to first order proportional, not to the external field acting on the scatterer, but to the curlcurl of this field.[8] This introduces singularities into the integrands of the integral equations which must be omitted from the integration by enclosing them by small spheres. These singularities lead in the treatment of the average value of the wave function to the well-known Lorentz-Lorenz effect,[3] but the derivation along rigorous lines is long and involved. These singularities appear to cause still more difficulty in the treatment of the average value of the square of the wave function (electric or magnetic field strength) but there seems no reason why the analysis cannot be carried through. The problem is of sufficient practical importance as well as theoretical importance to warrant additional study.

(5) Another generalization could be attempted in the direction of considering changes of frequency of scattered waves produced through the Doppler effect by moving scatterers. This is closely related to the problem of inelastic scat-

[8] The vector wave function is the Hertz vector $\mathbf{Z}$ in this case; the scattered waves are of the form $\mathbf{p}E(\mathbf{r}, \mathbf{r}_i)$, where $\mathbf{p}$, the dipole moment of the scatterer is proportional to the external electric field acting on the scatterer, which is in turn proportional to the curlcurl of the external $\mathbf{Z}$ field acting on the scatterer.

tering of light and fundamental particles by free or bound atoms in the quantum theory, since a change of frequency of the wave function corresponds in quantum mechanics to a change in energy of the associated particle. Such a theory would be developed along lines in some ways similar to the treatment of diffuse spots in the theory of x-ray diffraction which are also the effect of inelastic scattering of photons by the crystal lattice. Such a theory might have considerable value in investigating experimentally the dynamics of thermal agitation in amorphous solids, liquids, and gases.

(6) Finally in a similar manner, the theory could be extended to include non-steady state scattering, in which the incident wave function does not have a harmonic dependence on the time.

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