Theory of the Retraction of Stressed Rubber

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THE striking behavior of stressed rubber in free retraction reported by B. Adalbert Mrowca, S. Leonard Dart, and Eugene Guth[1] can be explained by the application of the theory of stress-wave propagation in elastic solids.

We consider a uniform elastic rod of length l, fixed at one end, with linear density ρ in the unstretched state. We assume that Hooke's law holds, with Young's modulus E, and that frictional forces are proportional to the rate of deformation. Let x be a measure of distance in the material of the unstretched rod, starting at the fixed end, and let $u(x)$ be the displacement of the material corresponding to the indicated value of x. In the absence of body forces, the behavior of the rod in extension and retraction is governed by the wave equation

$$\rho(\partial^2u/\partial t^2)=f(\partial^3u/\partial x^2\partial t)+E(\partial^2u/\partial x^2). \quad (1)$$

If the rod is initially uniformly stretched with fractional extension ϵ, and is released at time $t=0$, the boundary conditions are:

$$\begin{aligned} \partial u/\partial x=\epsilon,\quad \partial u/\partial t=0 &\quad \text{for all } x \text{ at } t=0,\\ u=0 &\quad \text{for all } t \text{ at } x=0, \quad (2)\\ f(\partial^2u/\partial x\partial t)+E(\partial u/\partial x)=0 &\quad \text{for } x=l \text{ when } t>0. \end{aligned}$$

In the limiting case of small friction, $f=0$, the solution becomes, with $v=(E/\rho)^{\frac{1}{2}}$,

$$\begin{aligned} u(x,t)&=\epsilon x & l-x&>vt,\\ u(x,t)&=\epsilon(l-vt) & l-x&<vt<l. \quad (3) \end{aligned}$$

According to this solution a pulse of acceleration arises at the free end at the moment it is released, and travels, with respect to the unstretched material, at a rate v; its rate of transmission in space is $v_p=(\epsilon+1)v$. The velocity of the tip of the bar is $v_t=-\partial u/\partial t=\epsilon v$. Each portion of the bar is stationary until it is reached by this pulse. As the pulse passes over it, the stress is completely relaxed, and the material reaches its final velocity ϵv. When the acceleration pulse reaches the fixed end of the material, after time $t_c=l/v$, the bar is unstressed and in uniform motion at the rate ϵv; the original potential energy is completely converted into kinetic energy. After this time a solution other than Eq. (3) begins to apply. The wave pulse is reflected at the fixed end and a pulse of deceleration begins to progress back down the bar. In the case of a rod in which transverse motion is not prevented, as in the case of a retracting rubber band, buckling begins at the moment of reflection of the pulse.

Deviations from Hooke's law cause broadening of the contraction pulse. Consideration of momentum and energy transfer makes it evident at once that an acceleration pulse of fixed form and limited extent cannot occur except for a Hooke's law bar.

In the case of high internal friction the inertia term in Eq. (1) may be neglected. The solution then becomes

$$u(x,t)\cong\epsilon x\exp[-(E/f)t]. \quad (4)$$

Equation (4) holds after the wave passed down the sample; there is a discontinuity at $t=0$.

For treatments of the intermediate case, forced vibrations, and the effect of a mass loading at the free end, reference must be made to a full treatment of the problem which is to follow.

[1] B. A. Mrowca, S. L. Dart, and E. Guth, Phys. Rev. 66, 30 (1944).