TABLE II. Constants referring to films of tridecylic, myristic, pentadecylic, palmitic acid, and to liquid decane.

Substance	C_{13}	C_{14}	C_{15}	C_{16}	Decane
Temperat.	14.5°C	14.5°C	14.5°C	12°C	23°C
	$\lambda=6$,			$\mu=3$	
$\delta f\sigma_0^{-6}\times 10^{13}$	11.2	12.6	14.2	17.4	
$\delta g\sigma_0^{-3}\times 10^{13}$	6.12	6.82	7.67	9.20	
B dynes/cm	279	321	355	447	
δ	17.2	18.6	20.0	23.1	
$\sigma_0\times 10^8$	7.5	7.8	8.1	8.7	
	$\lambda=9$,			$\mu=6$	
$\delta f\sigma_0^{-9}\times 10^{13}$	42.9	47.7	53.8	64.3	64.1
$\delta g\sigma_0^{-6}\times 10^{13}$	37.7	42.0	47.0	56.3	48.7
B dynes/cm	958	1067	1206	1450	
δ	26.5	28.1	30.8	34.1	34.3
$\sigma_0\times 10^8$	9.3	9.6	10.0	10.6	12.1

If the geometrical formula for σ is accepted, this amounts to setting $\delta=1$ in Eqs. (2.28) to (2.30). The theoretical values of the bulk coefficient remain the same, but the mean distances between nearest neighbors become, uniformly for all films and with both sets of exponents, $\sigma_0=1.8$A. This appears to be a little too small, and a more correct value of the function $p_1(r)$ would probably lead to a value intermediate between the two values given here.

On the whole it can be said that the agreement between empirical data and our theory is as good as might have been anticipated, given the preliminary nature of the potential of interaction.

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On Birkhoff's New Theory of Gravitation

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It is pointed out in the first place: (1) in Birkhoff's gravitational theory based on "flat" space-time, the "red shift" is accounted for by the energy change of the photon as it travels from the emitting body, whereas the photon plays no especial role in the Einstein theory; (2) the solution of the problem of two or more bodies is feasible in the new theory because of its simpler character. Four comments of H. Weyl concerning the Birkhoff theory are discussed, and it is concluded that these are to be taken with much reserve. In regard to the third of these comments it is pointed out that the "perfect fluid" used by Birkhoff as the ultimate carrier of mass and electric charge is to be characterized as the simplest fluid with disturbance velocity that of light (c). It is affirmed to be a glaring defect of earlier relativistic theories that the disturbance velocity in matter has been taken as arbitrary, although that of gravitation and of the electromagnetic field have been equal to c. The differential equations of the theory are then set up. An additional cosmological term in the gravitational potentials h_{ij} is suggested, namely,

$$\overset{*}{h}_{ij}=(K/8)(t^2-x^2-y^2-z^2)g_{ij},$$

where x, y, z, t are Lorentz coordinates and K is the (small) cosmological constant. The explicit formula for the rate of advance of periastron P of two bodies (mass points) of masses m and m_2 is given, as obtained from the solution of the two-body problem in the theory, and its possible application to double stars is referred to. The authors propose to give a detailed development of the theory and its applications in papers to be published shortly elsewhere.

I. PRELIMINARY OBSERVATIONS

IN a recent number of a journal of wide circulation,[1] Hermann Weyl has given expression to several critical remarks on G. D. Birkhoff's theory of gravitation of 1942.[2] In this note we intend first to analyze briefly the substance of Weyl's comments, secondly to consider the structure of the new theory from the physical point of view, and lastly to refer briefly to its physical applications.

Before doing this, however, we would like to make certain general observations.

The explanation of the "red shift" is fundamentally different in the gravitational theories of Einstein and Birkhoff. In the new theory the red shift has turned out to be accounted for by the energy change of the photon as it travels from the emitting body to the Earth; this explanation fully takes account of the role of

[1] H. Weyl, Math. Rev. 4, 285 (1943).
[2] G. D. Birkhoff, Proc. Nat. Acad. Sci. 29, 231 (1943).

the light signal (photon). In Einstein's interpretation, on the other hand, the frequency of the light wave emitted by an atom on a distant body is compared with the frequency of the light wave emitted by the same atom on the Earth, and any phenomena taking place while the light signal (photon) travels from the emitting body to the Earth play no part whatsoever.

Again, the difficulties accompanying the two-body problem and other questions in the general theory of relativity are too well known to require mention, while in Birkhoff's theory the solution of the problem of two or more bodies is quite within reach and will soon be available. This is due to the essentially simpler character of the latter theory.

An objection which may be made to Birkhoff's theory is the introduction of an absolute reference system, which runs counter to Einstein's general relativity principle that matter determines space, and analogous philosophical ideas previously developed by Ernst Mach. But, in a sense, this objection to Birkhoff might also be urged against Einstein because in the latter's theory a *single* rotating body can still be supposed alone in the universe, which is absurd from Mach's point of view. It would be difficult indeed to set up any physical theory to which objections of this general nature could not be raised.

II. ANALYSIS OF WEYL'S COMMENTS

To begin with, Weyl contends that Birkhoff's theory is much the same as Einstein's theory of 1916, for the case of weak gravitational fields. In a note appearing elsewhere, Barajas[3] has shown that the factual consequences of Einstein's theory in this case differ from Birkhoff's; further, that the choice of the gravitational potentials suggested by Weyl is unsatisfactory; and, lastly, that the trajectories of a test particle in Birkhoff's theory are not geodesics in any four-dimensional space-time.

Weyl further raises four fundamental objections to Birkhoff's theory: (1) "The connection between metric and gravitation is dissolved." (2) The proportionality between inertial and gravitational mass "has again become as mysterious as it was before Einstein." (3) Birkhoff's perfect fluid "appears as a primitive irreducible physical entity." (4) "There seems to be no indication that the mechanical equations spring from a universal law of conservation of energy and momentum." We now propose to take up these points in detail.

(1) With regard to the first, it should be recalled that the connection between matter and geometry, as developed by Einstein, is purchased at the expense of giving up a fundamental reference system. This implies abandoning the description of nature in terms of four fundamental independent variables, essentially unique except for the arbitrariness involved in position and velocity of a single point (Lorentz group). In the abandonment of such coordinates may be discerned a sufficient reason for the early exhaustion of all observational tests of the general theory of relativity, and also for the fundamental difficulty of assigning actual physical meaning to the coordinates introduced in problems of this theory. Indeed, almost thirty years of intensive research have failed to provide another test besides the three classical ones, or to apply the theory to other fields of physics. In spite of a great deal of work, there does not exist any satisfactory solution of the two-body problem, and the n-body problem has not even been formulated. Even in the simple case of Schwarzschild's solution of the one-body problem, no clear-cut physical interpretation of the Schwarzschild coordinates seems to be available. This last difficulty is so serious that it led Milne to give up the use of general coordinates in relativity.

(2) Weyl's second objection seems to be based largely on a misinterpretation. In Birkhoff's theory, just as well as in Einstein's, the equality between inertial and gravitational mass comes out of the fact that in the gravitational equations the energy tensor is linear in the mass density. While nothing similar to Einstein's "equivalence principle" has been explicitly formulated in Birkhoff's theory, it must be stated that the exact significance of the principle has yet to be found. As far as can be seen at the present time it merely asserts that bodies moving freely in empty space near attracting matter behave, relatively to a hypothetical attached reference system, in the same way under all circumstances

[3] A. Barajas, Proc. Nat. Acad. Sci. **30**, 54 (1944).

—insofar as they so behave! From the point of view of differential geometry, it amounts to saying that in the infinitesimal neighborhood of any point in a Riemannian curved space there exists always a tangent flat space, and this is just as true in Einstein's theory as in any other.

According to Einstein, the proper language for the description of nature is that of tensors, together with the underlying group of general transformations; and the equivalence principle expresses the theorem just stated. According to Birkhoff, the proper language for this description is that of four vectors, and the underlying space-time is everywhere flat. In the latter case, the basic group involves only ten arbitrary constants, while in the former it involves four completely arbitrary functions of four completely arbitrary variables. Whichever choice is made, the equality between gravitational and inertial mass follows as a consequence both of Birkhoff's and Einstein's theories.

(3) Birkhoff has chosen the name "perfect fluid" for perhaps the simplest substance in which all disturbances are propagated with the velocity of light, on account of the similarity between the equations governing such a substance and those of a perfect fluid. The particular type of perfect fluid considered by Birkhoff is characterized by only one scalar (the mass density) and vector (the velocity). Fundamental difficulties arise if the disturbance velocity is different from that of light. From our point of view, the fact that in Einstein's theory no attention has been paid to this requirement constitutes a glaring defect, which by itself explains the possibility of Birkhoff's theory. Indeed, the postulation of a primordial substance in which all disturbances are propagated with the velocity of light is fundamentally a consequence of the assumption that the basic space-time is everywhere that of Minkowski. This is actually the only physical assumption made in Birkhoff's theory.

(4) Weyl's fourth objection also seems to spring from a misunderstanding. In Birkhoff's theory the law of conservation of energy and momentum at material velocities small compared with that of light is as much a consequence of the mechanical equations of motion (and conversely) as in Einstein's. It is true that in Birkhoff's case the conservation of energy and momentum is not connected with a fundamental geometrical theorem (Ricci's theorem), as in Einstein's theory. But, although Einstein obtains the formal result that the divergence of the energy-momentum tensor must vanish, this does not imply the conservation of energy and momentum in an exact sense because the four-dimensional integrals of a *covariant* partial derivative in curved space-time cannot be transformed into three-dimensional integrals. Consequently the conservation theorems of Birkhoff's theory are at least much more precise.

These remarks may serve to point out that Weyl's assertions are to be taken with much reserve and that additional research is required before the usefulness of Birkhoff's theory for physics can be adequately assessed.

III. STRUCTURE OF THE BIRKHOFF THEORY

In the light of the preceding remarks, it is possible to recapitulate as follows the considerations which lead to Birkhoff's theory. The fundamental concept of electromagnetic space-time, associated with the names of Fitzgerald, Larmor, Lorentz, Einstein, and Minkowski, has been of the first importance for physics and has played an ever increasing role in the developments of the last fifty years. The revolutionary change which this concept has brought about in physical theory is reflected in the mathematical apparatus employed. Four homogeneous variables of space and time replace the three homogeneous variables of space and the single disparate variable of time; the underlying Lorentz group replaces the Galilean group, and the language of 4 vectors replaces the language of 3 vectors characteristic of classical physics.

The early attempt of Nordström (1912) and others to incorporate gravitation in this framework failed to explain certain delicate gravitational phenomena which alone provide a crucial test. Thus, without an intensive study, electromagnetic space-time was abandoned for a curved or Riemannian space-time, latent in the ideas of Minkowski and realized in Einstein's brilliant gravitational theory of 1916.

It is exceedingly easy to exaggerate the significance of the three so-called critical confirmations of this theory, of which by far the most certain is afforded by the excessive perihelial advance of the planet Mercury. In fact, the building of theories from the aesthetical-mathematical point of view has shown that dimensional considerations enter which lead always to the same result, aside from simple numerical factors. More definitely, the three formulas for perihelial advance, bending of light, and red shift take always the respective forms

$$qm/a(1-e^2), \quad rm/p, \quad sm(1/r_1-1/r_2)$$

where q, r, s, are simple numerical constants.

Consequently, the basic test of any such theory really reduces to the single requirement that the first constant r has a value not very different from Einstein's value 6π! Thus the theory of Einstein, stripped of all mystical trappings, is seen in its proper perspective. It becomes obvious that the question "What is the simplest theory of gravitation and other physical phenomena, based on electromagnetic space-time, which explains the known facts?" deserves the most careful consideration. This question becomes all the more urgent since the Einstein theory, with its enormous mathematical complication and its lack of proper independent variables, seems to be essentially unworkable.

Table I. Table of disturbance velocities.

	Newton, Maxwell	Nordström, Einstein	Birkhoff
Matter	Arbitrary	Arbitrary	c
Gravitation	∞	c	c
Electromagnetic field	c	c	c

Birkhoff's theory of 1942 provided perhaps the first thoroughgoing attempt to answer this question. Its point of departure arises from the valid criticism of earlier theories that the forms of matter therein employed are inconsistent with the actual requirement that the disturbance velocity be that of light. This situation may be roughly set forth in the comparative Table I, in which the entries indicate disturbance velocity. The physical necessity for a disturbance velocity c of matter appears from the fact that, with any other velocity, the equations of motion break down at the collision of two atomic constituents. Here the matter referred to is not gross matter, but the ultimate carrier of mass and electric charge.

If this requirement is admitted, it appears that the first condition of any new theory based on electromagnetic space-time should be the condition that matter has a disturbance velocity equal to that of light under all circumstances. The theory of Birkhoff is based on one such type, the "perfect fluid," which seems to be the simplest conceivable from a conceptual point of view. But it appears almost certain that any other type of medium obeying this same fundamental requirement will lead to essentially the same gravitational theory. The "perfect fluid" may be approached in the following manner: The state of matter is regarded as characterized by a single scalar density ρ and vector velocity $u^i=dx^i/ds$ in the sense of local causation; the equations of motion are to be linear in the rates of change of these variables; free equilibrium is possible at a certain density ρ_0; the disturbance velocity is to be that of light c, which becomes 1 if the light-second is the unit of distance.

We have established the mathematical theorem that the equations of motion of the perfect fluid may be written in the 4-vector form

$$\text{div } T=\partial T^{i\alpha}/\partial x^\alpha=0, \quad T^{ij}=\rho(u^iu^j-\tfrac{1}{2}g^{ij}),$$

where $g_{ii}=1, -1, -1, -1$ for $i=1, 2, 3, 4$, respectively, and $g_{ij}=0$ for $i\neq j$, with appropriate choice of the scalar density ρ, unique up to a choice of a unit. Furthermore, the perfect fluid so obtained satisfies the conservation principle that the integral $\int\int\int\sqrt{\rho}dv$ is conserved (dv, element of volume referred to a rest system).

The equation of motion under arbitrary force densities f^i may be written correctly

$$\partial T^{i\alpha}/\partial x^\alpha=f^i.$$

There is then the essential further requirement to be imposed on the forces f^i that if the particles of the fluid return to an initial state of position and velocities, the density ρ must also return. For this, it seems necessary and is certainly sufficient that the condition of orthogonality

$$f_\alpha u^\alpha=0$$

be identically satisfied. Thus the force vectors are always required to be identically orthogonal to the velocity vectors. A primary force vector of this type is evidently the acceleration vector itself.

Now all force vectors which have been used in previous relativistic theories are rational and integral in the components of the velocity vector u^i. It is therefore natural to set:

$$f^i = A^i + \overset{i}{B}_\alpha u^\alpha + \overset{i}{C}_{\alpha\beta} u^\alpha u^\beta + \cdots,$$

where the coefficients are functions of position alone, But the electromagnetic force is known to be linear and homogeneous in the velocities and also in the first partial derivatives of the electromagnetic potential φ_i, which itself is of dimension 0 in the unit of length and time. Similarly, in the theory of Einstein, the gravitational forces are homogeneous and quadratic in the velocities, and linear and homogeneous in the first partial derivatives of the gravitational potentials g_{ij} which again are of dimension 0. It is therefore natural to assume that B and C pertain to electromagnetic and gravitational forces, respectively. Birkhoff imposes the analogous requirements upon the vector A and thus arrives at the first (covariant) form for the forces[4]

$$f_i = \rho \frac{\partial \psi}{\partial x^i} + \sigma \left(\frac{\partial \varphi_i}{\partial x^\alpha} - \frac{\partial \varphi_\alpha}{\partial x^i} \right) u^\alpha + \rho \left(\frac{\partial h_{i\alpha}}{\partial x^\beta} - \frac{\partial h_{\alpha\beta}}{\partial x^i} \right) u^\alpha u^\beta,$$

where ψ is his "atomic potential," constant along the world-lines of the fluid, φ_i is the electromagnetic potential satisfying the Maxwell-Lorentz equations (σ, density of electricity) .

$$\frac{\partial}{\partial x^\alpha} \left(\frac{\partial \varphi_i}{\partial x^\alpha} - \frac{\partial \varphi_\alpha}{\partial x^i} \right) = -4\pi\sigma u_i$$

and where h_{ij} is his gravitational potential satisfying

$$\Box h_{ij} = \frac{\partial^2 h_{ij}}{\partial t^2} - \frac{\partial^2 h_{ij}}{\partial x^2} - \frac{d^2 h_{ij}}{\partial y^2} - \frac{\partial^2 h_{ij}}{\partial z^2} = 8\pi T_{ij},$$

which is the generalized form of Poisson's equation of the theory.

Hence the gravitational theory involved may be singled out as follows: If the equations of motion for a small total amount of matter (absence of gravitational force) are written

$$\partial T^{i\alpha}/\partial x^\alpha = f^i;$$

then for the general case, these equations are

$$\frac{\partial T^{i\alpha}}{\partial x^\alpha} = f^i + f^{*i}, \quad f_i^* = \rho \left(\frac{\partial h_{i\alpha}}{\partial x^\beta} - \frac{\partial h_{\alpha\beta}}{\partial x^i} \right) u^\alpha u^\beta,$$

where f^{*i} is the gravitational force and where the gravitational potentials h_{ij} are defined by

$$\Box h_{ij} = 8\pi T_{ij}.$$

Such is the structure of the Birkhoff theory as hitherto formulated. However, to account for the phenomena of nebular recession on this basis, it is necessary to suppose that at some time in the past there was a nuclear distribution of matter, with a wide range of velocities. But from the physical point of view it appears to be more natural to suppose an initial nuclear distribution at low relative velocities. In our recent investigations, we have been led to the following extension of the original theory involving the introduction of a cosmological constant.

From the formal point of view the following more general type of Poisson equation needs to be especially examined:

$$\Box h_{ij} + a T_{ij} + b g_{ij}.$$

With proper choice of a unit of density, it is then possible to specialize further the above equation to the form

$$\Box h_{ij} = 8\pi T_{ij} + K g_{ij},$$

where K is the "cosmological constant," which has the dimensions of a density and is supposed to be very small.

Now, with such an extended Poisson equation, it is no longer possible to demand that the gravitational potentials h_{ij} approach 0 regularly at ∞, nor even that these functions are linearly infinite. However, the condition may be imposed that h_{ij} are regularly infinite to at most the second order, with a boundary distribution at infinity which is spherically symmetric in a spatial sense. Obviously, the most natural possibility from the electromagnetic point of view is to take the cosmological term in the gravitational potentials to be

$$h_{ij}^* = \frac{K}{8}(t^2 - x^2 - y^2 - z^2) g_{ij},$$

where x, y, z, t are any Lorentz coordinates and $x=y=z=t=0$ is the origin in space-time.

[4] Birkhoff has changed the form of A_i from $\partial\psi/\partial x^i$ to $\rho\partial\psi/\partial x^i$ because of the dimensionality requirement.

It is our purpose to publish shortly detailed studies of the theory outlined above, its cosmological extension, and its applications.

IV. APPLICATIONS AND TESTS OF THE THEORY

It has not yet been possible for us to make an intensive and thoroughgoing study of the Birkhoff theory in relation to known gravitational phenomena; and still less has it been possible for us to consider other possible applications.

However, the basic two-body problem has already been solved by us to the requisite order of approximation. Thus, for instance, the formula for rate of advance of periastron P of two bodies (mass points) of masses m_1 and m_2 has been found to be (in absolute units)

$$P=\frac{3m_1^2+7m_1m_2+3m_2^2}{m_1+m_2}\frac{2\pi}{a(1-e^2)}$$

which in case of an infinitesimal mass ($m_1=0$, $m_2=m$) reduces to that made familiar by the Einstein theory, $6\pi m/a(1-e^2)$. Here, of course, the application to the rotation of the line of the apsides for double stars is at once suggested. But such a comparison with observation must await the further analysis of the observations themselves, so that appropriate data may be available in cases where the rotation of the line of the apsides due to tidal forces is relatively small.

We here naturally hoped also that the new theory would account for some of the hitherto unexplained features of lunar motion. However, this does not seem to be a likely outcome although it may turn out that the solution of the three-body problem (Earth, Sun, and Moon) will yield an explanation of the desired type. Instead, at the present writing, we are inclined to think that these supposed features are attributable to the large mass ratio (1/83) of Moon to Earth, which makes the convergency of the mathematical calculations involved slow and uncertain.

It seems not unlikely that the astronomical consequences of this theory for the interpretation of the red shift from extragalactic sources will be of special interest.

As yet no means appear to be available for a decisive experimental test between the gravitational theory of Birkhoff and that of Einstein. This theoretical uncertainty is likely to continue for some time, especially as it appears to be very difficult to carry the Einstein theory to specific conclusions other than those found by him at the outset.

There remains for later consideration the study of atomic phenomena on the basis of the electromagnetic and atomic potentials. It is to be hoped that, by proper assignment of the atomic potential ψ to the constituents of the atom, the explanation of atomic phenomena may be advanced and possibly the fundamental Schroedinger wave equation may be obtained in a conceptual manner.[5] Here the proton and electron constituting the hydrogen atom, for instance, are conceived of as freely interpenetrable and superimposed in case of equilibrium and as oscillating under disturbances.

In conclusion we would like to point out that, for the physicist, all mathematics is fundamentally a form of abstract model building, of more or less general aspects of nature; and that no experiments which the physicist may perform in his laboratory can advance very far without free access to a variety of abstract models which are not to be thought of as final. The new theory of matter, electricity, and gravitation in flat space-time proposed by Birkhoff, would seem to afford a model of unusually fundamental, simple, and complete type.

[5] See G. D. Birkhoff's two notes in the Proc. Nat. Acad. Sci. **13**, 160, 165 (1927).