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Free Vibrations of Anisotropic Bodies

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Approximate solutions for free vibrations of a finite anisotropic body are derived by a perturbation method. As an example, some extensional modes of thin crystal plates are calculated. Calculated frequencies and deformation patterns are compared with observations.

1. SKETCH OF THE METHOD

ONLY very few problems of elastic vibration are susceptible of rigorous solution by the methods of the general theory of elasticity. During the last years stationary vibrations of crystalline bodies have become of practical importance because of the discovery of piezoelectric excitation of vibrations. Rigorous solutions for free vibrations of a crystalline body are known only for an infinitely extended plane parallel plate and an infinitely thin rod. It has become desirable to obtain even rough approximations, as rigorous solutions for most problems are inaccessible. Some attempts in this direction have been made by Mason[1] and Bechmann.[2]

It is attempted here to give a more systematic method of approach. The perturbation method has been used for a long time especially in quantum mechanics. To our knowledge, adaptation of this powerful method to the theory of elasticity has not been made.

We first make a resume of well-known equations of elastic vibrations. If U, V, W are the Cartesian components of the displacement and if we have stationary vibrations of angular frequency $\Omega=2\pi\nu$, they take the form[3]

$$\begin{aligned}\rho\Omega^2 U&=\partial X_x/\partial x+\partial X_y/\partial y+\partial X_z/\partial z,\\ \rho\Omega^2 V&=\partial X_y/\partial x+\partial Y_y/\partial y+\partial Y_z/\partial z,\\ \rho\Omega^2 W&=\partial X_z/\partial x+\partial Y_z/\partial y+\partial Z_z/\partial z.\end{aligned}\tag{1}$$

The strain components are defined by

$$\begin{aligned}x_x&=\partial U/\partial x, & x_y&=\partial U/\partial y+\partial V/\partial x,\\ y_y&=\partial V/\partial y, & y_z&=\partial V/\partial z+\partial W/\partial y,\\ z_z&=\partial W/\partial z, & x_z&=\partial U/\partial z+\partial W/\partial x.\end{aligned}\tag{2}$$

The general linear relation between stress components and strain components is

$$\begin{aligned}-X_x&=c_{11}x_x+c_{12}y_y+c_{13}z_z+c_{14}y_z+c_{15}x_z+c_{16}x_y,\\ &\cdots\cdots\cdots\\ -X_y&=c_{61}x_x+c_{62}y_y+c_{63}z_z+c_{64}y_z+c_{65}x_z+c_{66}x_y,\end{aligned}\tag{3}$$

with

$$c_{ik}=c_{ki};$$

and the inversion of (3) is

$$\begin{aligned}-x_x&=s_{11}X_x+s_{12}Y_y+\cdots+s_{16}X_y,\\ &\cdots\cdots\cdots\\ -x_y&=s_{61}X_x+s_{62}Y_y+\cdots+s_{66}X_y.\end{aligned}\tag{4}$$

If we eliminate the strain and stress components from these equations, we obtain three equations of the form

$$\begin{aligned}\rho\Omega^2 U+K(U,V,W)&=0,\\ \rho\Omega^2 V+L(U,V,W)&=0,\\ \rho\Omega^2 W+M(U,V,W)&=0,\end{aligned}$$

with

$$K(U,V,W)=-\left(\frac{\partial X_x}{\partial x}+\cdots\right),\qquad\cdots\cdots\tag{5}$$

where K, L, M are linear operators of second order which act on three components U, V, W.

The boundary conditions for the free body take the form:

$$\begin{aligned}X_x\cos(n,x)+X_y\cos(n,y)+X_z\cos(n,z)&=0,\\ X_y\cos(n,x)+Y_y\cos(n,y)+Y_z\cos(n,z)&=0,\\ X_z\cos(n,x)+Y_z\cos(n,y)+Z_z\cos(n,z)&=0,\end{aligned}\tag{6}$$

where n signifies the normal to the surface, directed inwardly to the body.

The fact is remembered that any two rigorous solutions of (5) and (6) are orthogonal to each other:[4]

$$\int(U_1U_2+V_1V_2+W_1W_2)dv=0,\tag{7}$$

where the integration is extended over the body.

[1] W. P. Mason, Bell Sys. Tech. J. **13**, 405 (1934).
[2] R. Bechmann, Zeits. f. Physik **117**, 180 (1941); **118**, 515 (1941); **120**, 107 (1942).
[3] W. Voigt, *Lehrbuch der Kristallphysik* (B. G. Teubner, Leipzig, 1910).
[4] A. E. H. Love, *The Mathematical Theory of Elasticity* (Cambridge University Press, 1934).

We shall try to represent the solution (U, V, W) approximately as a sum of zero-order functions (u_i, v_i, w_i) which are chosen so that they are rough approximations of the actual solution. We write:

$$U=\sum A_i u_i, \quad V=\sum A_i v_i, \quad W=\sum A_i w_i. \quad (8)$$

However, this expansion is possible only if the functions (u_i, v_i, w_i) satisfy the boundary conditions (6). In the following, we shall not suppose this to be true, so that (8) does not hold in general. We will construct a set of auxiliary functions (u_i', v_i', w_i') so that they are equal to the functions (u_i, v_i, w_i) everywhere but in a small domain close to the boundary. In this domain, (u_i', v_i', w_i') will behave so as to satisfy the conditions (6). We will write:

$$U=\sum A_i u_i', \quad V=\sum A_i v_i', \quad W=\sum A_i w_i'. \quad (9)$$

We now define strain and stress functions:

$$x_{xi}=\partial u_i/\partial x \cdots x_{yi}=\partial u_i/\partial y+\partial v_i/\partial x, \quad (10)$$

and

$$-X_{xi}=c_{11}x_{xi}+\cdots+c_{16}x_{yi}. \quad (11)$$

. . . .

The functions (u_i, v_i, w_i) will not satisfy Eq. (5), but will satisfy equations of the form:

$$\begin{aligned} -K(u_i, v_i, w_i) &= \partial X_{xi}/\partial x+\partial X_{yi}/\partial y \\ &\quad +\partial X_{zi}/\partial z=-f_i, \\ -L(u_i, v_i, w_i) &= \partial X_{yi}/\partial x+\partial Y_{yi}/\partial y \\ &\quad +\partial Y_{zi}/\partial z=-g_i, \\ -M(u_i, v_i, w_i) &= \partial X_{zi}/\partial x+\partial Y_{zi}/\partial y \\ &\quad +\partial Z_{zi}/\partial z=-h_i. \end{aligned} \quad (12)$$

Instead of the boundary conditions (6), the functions (u_i, v_i, w_i) will satisfy equations of the form:

$$\begin{aligned} X_{xi}\cos(n, x)+\cdots &= \varphi_i(s), \\ X_{yi}\cos(n, x)+\cdots &= \psi_i(s), \\ X_{zi}\cos(n, x)+\cdots &= \chi_i(s). \end{aligned} \quad (13)$$

Similarly, the corresponding functions derived from the auxiliary functions (u_i', v_i', w_i') will be denoted by

$$f_i', g_i', h_i' \quad \text{and} \quad \varphi_i', \psi_i', \chi_i'.$$

It will be noticed that $u_i \cdots$ can be considered as solution of the following problem: Given a volume force of components $-f_i/\rho$, $-g_i/\rho$, and $-h_i/\rho$, respectively, and surface tractions of components φ_i, ψ_i, χ_i, respectively, the equilibrium deformations of the body are to be determined. From Eqs. (12) and (13) it can be seen that the requirements for the solution of this problem are met by u_i, v_i, and w_i.

Let S' be a surface internal to the surface S of the body and parallel to S, having a small distance ϵ from S so that to each point s on S there corresponds a point s' on S'. The corresponding points are joined by the normal n which is directed toward the interior. Let the domain included between S and S' be called D, while the rest of the body volume is V. The tractions across S' which have the form of the first members of (13) will be found by integrating the functions $-f_i$, $-g_i$, $-h_i$ along the normal n:

$$\begin{aligned} \int\left(\frac{\partial X_{xi}}{\partial x}+\frac{\partial X_{yi}}{\partial y}+\frac{\partial X_{zi}}{\partial z}\right)dn &= |X_{xi}\cos(n, x)+X_{yi}\cos(n, y) \\ &\quad +X_{zi}\cos(n, z)|_{n=0}^{n=\epsilon}, \end{aligned} \quad (14)$$

$$\int\left(\frac{\partial X_{yi}}{\partial x}+\cdots\right)dn=|X_{yi}\cos(n, x)+\cdots|_{n=0}^{n=\epsilon},$$

$$\int\left(\frac{\partial X_{zi}}{\partial x}+\cdots\right)dn=|X_{zi}\cos(n, x)+\cdots|_{n=0}^{n=\epsilon}.$$

If we use (12) and (13) we obtain

$$\{X_{xi}\cos(n, x)+X_{yi}\cos(n, y) +X_{zi}\cos(n, z)\}_{n=\epsilon}=\varphi_i(s)-\int_0^\epsilon f_i dn, \quad (15)$$

. . . . ,

where the dots stand for the two analogous equations.

The same calculation for the functions $u_i' \cdots$ yields similarly:

$$\{X'_{xi}\cos(n, x)+X'_{yi}\cos(n, y) +X'_{zi}\cos(n, z)\}_{n=\epsilon}=\varphi_i'(s)-\int_0^\epsilon f_i' dn, \quad (16)$$

. . . .

Returning to the definition of the auxiliary functions $u_i' \cdots$, we remember that they should be equal to $u_i \cdots$ within V. Therefore, they must be solutions of a problem involving the same forces $-f_i/\rho\cdots$ within V and the tractions across S' given by (15). Thus, the first members of (15) and (16) are equal:

$$\varphi_i(s)-\int_0^\epsilon f_i dn=\varphi_i'(s)-\int_0^\epsilon f_i' dn. \quad (17)$$

By definition of the $u_i'\cdots$, the tractions $\varphi_i'\cdots$ must vanish:

$$\int_0^\epsilon f_i' dn=\int_0^\epsilon f_i dn-\varphi_i,$$
$$\int_0^\epsilon g_i' dn=\int_0^\epsilon g_i dn-\psi_i, \quad (18)$$
$$\int_0^\epsilon h_i' dn=\int_0^\epsilon h_i dn-\chi_i.$$

We substitute (9) into (5):

$$\rho\Omega^2\sum A_i u_i'+\sum A_i K(u_i', v_i', w_i')=0,$$
$$\rho\Omega^2\sum A_i v_i'+\sum A_i L(u_i', v_i', w_i')=0, \quad (19)$$
$$\rho\Omega^2\sum A_i w_i'+\sum A_i M(u_i', v_i', w_i')=0.$$

We multiply these equations by u_k', v_k', w_k', respectively, add them, and integrate over the crystal:

$$\sum_i A_i\left[\rho\Omega^2\int(u_i'u_k'+v_i'v_k'+w_i'w_k')dv+\int\{u_k'K(u_i', \cdots)+v_k'L(u_i', \cdots)+w_k'M(u_i', \cdots)\}dv\right]=0. \quad (20)$$

As $u_i'\cdots$ tends toward $u_i\cdots$, the first integral in (20) becomes:

$$\int(u_iu_k+v_iv_k+w_iw_k)dv=N_{ik}. \quad (21)$$

In view of (7) the non-orthogonality integrals N_{ik} would vanish for $i\neq k$ if the functions $(u_i\cdots)$ were exact solutions of the problem.

The integrals N_{ii} will be determined arbitrarily by normalization:

$$\int(u_i^2+v_i^2+w_i^2)dv=1=N_{ii}. \quad (22)$$

The volume V inside of S' contributes to the second integral of (20) the term:

$$\int[u_kK(u_i, \cdots)+v_kL(u_i, \cdots)+w_kM(u_i, \cdots)]dv.$$

The contribution of D to this integral is

$$\int_D[u_k'K(u_i', \cdots)+\cdots]dv=\int_D(u_k'f_i'+\cdots)dv$$
$$=\int_S ds\left[u_k\int_0^\epsilon f_i' dn+\cdots\right], \quad (23)$$

where u_k' is put equal to u_k and considered as a constant along the normal n. Substituting (18) into (23) we find

$$\int_D[u_k'K(u_i', \cdots)+]dv=\int_S ds\left[u_k\left\{\int_0^\epsilon f_i dn-\varphi_i\right\}+v_k\left\{\int_0^\epsilon g_i dn-\psi_i\right\}+w_k\left\{\int_0^\epsilon h_i dn-\chi_i\right\}\right].$$

When ϵ tends toward zero, the integrals $\int_0^\epsilon f_i dn\cdots$ vanish and we have

$$\int_D[u_k'K(u_i', \cdots)+\cdots]dv=-\int_S ds(u_k\varphi_i+v_k\psi_i+w_k\chi_i).$$

Equation (20) now takes the form:

$$\sum_i A_i\left[\rho\Omega^2N_{ik}+\int\{u_kK(u_i, \cdots)+v_kL(v_i, \cdots)+w_kM(u_i, \cdots)\}dv-\int(\varphi_iu_k+\psi_iv_k+\chi_iw_k)ds\right]=0,$$

or

$$\sum_i A_i(\rho\Omega^2N_{ik}-H_{ik})=0, \quad (24)$$

where

$$H_{ik}=-\int(f_iu_k+g_iv_k+h_iw_k)dv+\int(\varphi_iu_k+\psi_iv_k+\chi_iw_k)ds. \quad (25)$$

If we consider, as above, two sets of functions u_i, v_i, w_i and u_k, v_k, w_k as solutions of two different equilibrium problems involving the given volume forces and surface tractions,

$$-f_i/\rho,\ -g_i/\rho,\ -h_i/\rho;\quad \varphi_i,\ \psi_i,\ \chi_i$$

and

$$-f_k/\rho,\ -g_k/\rho,\ -h_k/\rho;\quad \varphi_k,\ \psi_k,\ \chi_k,$$

respectively, then a theorem due to Betti[3] states that:

$$-\int(f_iu_k+g_iv_k+h_iw_k)dv+\int(\varphi_iu_k+\psi_iv_k+\chi_iw_k)ds=-\int(f_ku_i+g_kv_i+h_kw_i)dv+\int(\varphi_ku_i+\psi_ku_i+\chi_kw_i)ds, \quad (26)$$

or, according to Eq. (25):

$$H_{ik}=H_{ki}. \quad (27)$$

In other words, the matrix H_{ik} is symmetric.

If the Eqs. (24) are to be compatible, the determinant

$$\begin{vmatrix} \rho\Omega^2-H_{11} & \rho\Omega^2N_{12}-H_{12} & \cdots \\ \rho\Omega^2N_{12}-H_{12} & \rho\Omega^2-H_{22} & \cdots \\ \vdots & \vdots & \end{vmatrix}=0 \quad (28)$$

must vanish. This will be possible only when Ω^2 assumes certain characteristic values which correspond to the approximate characteristic frequencies of the free vibrations. With a particular value Ω_i^2 we can solve Eqs. (24) for the coefficients A_i and by substitution into (9) we can obtain the approximate solutions U, V, W. For all practical purposes, the functions $u_i\cdots$ can then be substituted for the u_i'; in other words, Eq. (8) can be used instead of (9).

2. APPLICATION: EXTENSIONAL VIBRATIONS OF THIN CRYSTAL PLATES

Let a crystal plate have its two parallel larger surfaces normal to the Z axis of the reference system. The origin is situated in a plane midway between the layer surfaces. On the boundaries, for a free crystal

$$X_z=Y_z=Z_z=0. \quad (29)$$

For a thin crystal, we may assume that the variation of the stress is linear, in a first approximation. As (29) is true for both surfaces, it must then hold everywhere. Substituting (29) into (4), we obtain

$$\begin{aligned} -x_x&=s_{11}X_x+s_{12}Y_y+s_{16}X_y,\\ -y_y&=s_{12}X_x+s_{22}Y_y+s_{26}X_y,\\ -x_y&=s_{16}X_x+s_{26}Y_y+s_{66}X_y. \end{aligned} \quad (30)$$

And if we solve (30) for the stresses:

$$\begin{aligned} -X_x&=\gamma_{11}x_x+\gamma_{12}y_y+\gamma_{16}x_y,\\ -Y_y&=\gamma_{12}x_x+\gamma_{22}y_y+\gamma_{26}x_y,\\ -X_y&=\gamma_{16}x_x+\gamma_{26}y_y+\gamma_{66}x_y. \end{aligned} \quad (31)$$

In this equation, X_x, Y_y, X_y mean the average value of the stresses taken over the thickness of the plate. Equation (31) is the equation given by W. Voigt.[3] Explicit formulas for the γ_{ik} are given in the same book.

As explained by Voigt, Eq. (31) applies really only to points of the plane $z=0$. For these points, $W=0$ in the case of extensional vibrations, and the strains (more exactly, the average values of the strains) are given similarly to Eq. (2):

$$x_x=\frac{\partial U}{\partial x},\quad y_y=\frac{\partial V}{\partial y},\quad x_y=\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}. \quad (32)$$

The dynamical equation (1) reduces to:

$$\rho\Omega^2U=\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y},\quad \rho\Omega^2V=\frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}. \quad (33)$$

This problem has been solved for the special case of a circular isotropic plate by Love.[4] For a rectangular plate, no general solution is known. V. Petrzilka[5] has given an exact solution for a mode of an isotropic rectangular plate, but the solution exists only for certain dimensions of the rectangle.

For this two-dimensional case, Eqs. (10)–(13), (21), (22), and (25) are simplified:

$$x_{xi}=\frac{\partial u_i}{\partial x},\quad y_{yi}=\frac{\partial v_i}{\partial y},\quad x_{yi}=\frac{\partial u_i}{\partial y}+\frac{\partial v_i}{\partial x}, \quad (10a)$$

$$\begin{aligned} -X_{xi}&=\gamma_{11}x_{xi}+\gamma_{12}y_{yi}+\gamma_{16}x_{yi},\\ -Y_{yi}&=\gamma_{12}x_{xi}+\gamma_{22}y_{yi}+\gamma_{26}x_{yi},\\ -X_{yi}&=\gamma_{16}x_{xi}+\gamma_{26}y_{yi}+\gamma_{66}x_{yi}, \end{aligned} \quad (11a)$$

$$\frac{\partial X_{xi}}{\partial x}+\frac{\partial X_{yi}}{\partial y}=-f_i,\quad \frac{\partial X_{yi}}{\partial x}+\frac{\partial Y_{yi}}{\partial y}=-g_i, \quad (12a)$$

$$\begin{aligned} X_{xi}\cos(n,x)+X_{yi}\cos(n,y)&=\varphi_i(s),\\ X_{yi}\cos(n,x)+Y_{yi}\cos(n,y)&=\psi_i(s), \end{aligned} \quad (13a)$$

[5] V. Petrzilka, Zeits. f. Physik 97, 436 (1935).

$$\int (u_i u_k + v_i v_k) dx dy = N_{ik}, \quad (21a)$$

$$\int (u_i^2 + v_i^2) dx dy = 1, \quad (22a)$$

$$H_{ik} = -\int (f_i u_k + g_i v_k) dx dy + \int (\varphi_i u_k + \psi_i v_k) ds. \quad (25a)$$

(a) Low Frequency Longitudinal Modes

A group of extensional modes has been called longitudinal because the strain components x_x and y_y are large in comparison to the shear x_y.

These vibrations have been studied experimentally and theoretically by several authors.[1,2,6,7] Their analysis is, however, unsatisfactory in several respects, mainly because it fails to account for the fact that *three* low frequency longitudinal modes are observed, usually with nearby frequencies.

We shall obtain the zero-order solutions from consideration of a simplified problem. In (31), we put

$$\gamma_{11} = \gamma_{22}, \quad \gamma_{66} = \tfrac{1}{2}\gamma_{11}, \quad \gamma_{12} = \gamma_{16} = \gamma_{26} = 0. \quad (34)$$

These are the equations for an isotropic plate with vanishing Poisson's ratio. Furthermore, we suppose that the plate is square, the edge having the length a. The directions X and Y shall be parallel to the edges, and the origin is in one corner of the plate. One exact solution is the one given by Petrzilka:

$$u_1 = C_1 \cos kx \sin ky, \quad v_1 = -C_1 \sin kx \cos ky, \quad \text{where } k = \pi/a. \quad (35)$$

The constant C_1 is arbitrary, but will later be defined by the normalization (22a).

Two other solutions are the longitudinal plane waves in the directions X and Y, respectively:

$$u_2 = C_2 \cos kx, \quad v_2 = 0, \quad (36)$$

and

$$u_3 = 0, \quad v_3 = C_3 \cos ky. \quad (37)$$

[6] A. Lissütin, Zeits. f. Physik **59**, 265 (1930).
[7] V. Petrzilka, reference 5, and Hoch: tech. u. Elek: akus. **50**, 1 (1937).

These three modes are degenerate, i.e., they have the same frequency:

$$\rho\omega^2 = k^2 \gamma_{11}. \quad (38)$$

We now abandon the simplifying assumptions (34). We then will find the familiar phenomenon of splitting of a degenerate eigenvalue by a perturbation.

For the more general case, we let two unequal edges of lengths a and b be parallel to X and Y, respectively. The zero-order functions can be written:

$$u_1 = C_1 \cos k_1 x \sin k_2 y, \quad v_1 = -C_1 \sin k_1 x \cos k_2 y, \quad (39)$$

$$u_2 = C_2 \cos k_1 x, \quad v_2 = 0, \quad (40)$$

$$u_3 = 0, \quad v_3 = C_3 \cos k_2 y, \quad (41)$$

where

$$k_1 = \pi/a, \quad k_2 = \pi/b.$$

The normalization (22a) requires that:

$$C_1^2 = C_2^2 = C_3 = 2/ab. \quad (42)$$

The non-orthogonality integrals N_{ik} are, according to (21a):

$$N_{12} = 2/\pi, \quad N_{13} = -2/\pi, \quad N_{23} = 0. \quad (43)$$

The equations (10a), (11a), (12a), (13a), and (25a) yield after some elementary calculations:

$$\begin{aligned} H_{11} &= \tfrac{1}{2}[k_1^2\gamma_{11} + k_2^2\gamma_{22} - 2k_1k_2\gamma_{12} + (k_2 - k_1)^2\gamma_{66}], \\ H_{12} &= H_{21} = (2/\pi)[k_1^2\gamma_{11} - k_1k_2\gamma_{12}], \\ H_{22} &= k_1^2\gamma_{11} \\ H_{13} &= H_{31} = (2/\pi)[k_1k_2\gamma_{12} - k_2^2\gamma_{22}], \\ H_{33} &= k_2^2\gamma_{22}, \\ H_{23} &= H_{32} = (8/\pi^2)k_1k_2\gamma_{12}. \end{aligned} \quad (44)$$

These are the elements of the secular determinant (28).

We first discuss the case where the major face is normal to a crystallographic threefold axis. In this case, according to Voigt, the plate behaves as if it were isotropic, i.e.,

$$\gamma_{11} = \gamma_{22}; \quad \gamma_{16} = \gamma_{26} = 0; \quad \gamma_{66} = \tfrac{1}{2}(\gamma_{11} - \gamma_{12}). \quad (45)$$

If the plate is also square ($k_1 = k_2 = k$) the expressions (44) reduce to:

$$\begin{aligned} H_{11} &= k^2(\gamma_{11} - \gamma_{12}), & H_{12} &= (2/\pi)k^2(\gamma_{11} - \gamma_{12}), \\ H_{22} &= k^2\gamma_{11}, & H_{13} &= -(2/\pi)k^2(\gamma_{11} - \gamma_{12}), \\ H_{33} &= k^2\gamma_{11}, & H_{23} &= k^2(8/\pi^2)\gamma_{12}. \end{aligned} \quad (46)$$

The secular determinant (28) has the form:

$$\begin{vmatrix} \rho\Omega^2-k^2(\gamma_{11}-\gamma_{12}) & \frac{2}{\pi}\rho\Omega^2-\frac{2}{\pi}k^2(\gamma_{11}-\gamma_{12}) & -\frac{2}{\pi}\rho\Omega^2+\frac{2}{\pi}k^2(\gamma_{11}-\gamma_{12}) \\ \frac{2}{\pi}\rho\Omega^2-\frac{2}{\pi}k^2(\gamma_{11}-\gamma_{12}) & \rho\Omega^2-k^2\gamma_{11} & -k^2\frac{8}{\pi^2}\gamma_{12} \\ -\frac{2}{\pi}\rho\Omega^2+\frac{2}{\pi}k^2(\gamma_{11}-\gamma_{12}) & -k^2\frac{8}{\pi^2}\gamma_{12} & \rho\Omega^2-k^2\gamma_{11} \end{vmatrix}=0. \quad (47)$$

One root of (47) is obvious from inspection:

$$\rho\Omega_1{}^2=k^2(\gamma_{11}-\gamma_{12}). \quad (48)$$

With this value, the ratio of the coefficients A_i in (24) is:

$$A_1{}^{(1)}:A_2{}^{(1)}:A_3{}^{(1)}=1:0:0. \quad (49)$$

The first solution is, according to (8) and (39):

$$U_1=C_1\cos kx\sin ky,\quad V_1=-C_1\sin kx\cos ky; \quad (39)$$

i.e., the result is the unchanged function (39). This was to be expected, for (39) is known to be a rigorous solution for the isotropic case with $a=b$.

To find the two other roots of (48), only a quadratic equation has to be solved. The result is:

$$\rho\Omega_2{}^2=k^2\gamma_{11},\quad \rho\Omega_3{}^2=k^2(\gamma_{11}+(8/\pi^2)\gamma_{12}), \quad (50)$$

and with these values

$$A_1{}^{(2)}:A_2{}^{(2)}:A_3{}^{(2)}=1:-\frac{\pi}{4}:+\frac{\pi}{4},\quad A_1{}^{(3)}:A_2{}^{(3)}:A_3{}^{(3)}=0:1:1. \quad (51)$$

Table I. Frequencies of square Z-cut quartz plates with edge $a=1$ mm (in kilocycles).

Mode	$N_{\text{calc.}}$	$N_{\text{obs.}}$
1	2.537×10^3 2.564×10^3	2.551×10^3
2	2.721×10^3 2.734×10^3	2.724×10^3
3	2.861×10^3 2.863×10^3	2.8962×10^3

Table II. Frequencies of a square Z-cut tourmaline plate with edge $a=1.836$ cm (in kilocycles).

Mode	Calc.	Obs.
1	218	215.1
2	253	247.6
3	278	291.0

The two equations (51) refer to the roots $\Omega_2{}^2$ and $\Omega_3{}^2$, respectively. The solutions U_2, V_2 and U_3, V_3 are then:

$$U_2=C(\sin ky-(\pi/4))\cos kx,\quad V_2=C(-\sin kx+(\pi/4))\cos ky \quad (52)$$

and

$$U_3=C\cos kx,\quad V_3=C\cos ky.$$

R. Bechmann[8] has observed some resonant frequencies of square quartz plates of the Z-cut type. He finds three frequencies for longitudinal modes and expresses them in terms of the frequency constant N, i.e., the frequency (in kilocycles) of a plate with $a=1$ mm so that $N=10^{-2}\nu a$. Table I gives a comparison between these measurements and the constants N as computed by Eqs. (48) and (50).

In this case, the parameters γ_{ik} are given by:

$$\gamma_{11}=\frac{s_{11}}{s_{11}{}^2-s_{12}{}^2},\quad \gamma_{12}=-\frac{s_{12}}{s_{11}{}^2-s_{12}{}^2}.$$

For s_{11} and s_{12} the values

	s_{11}	s_{12}
	12.948×10^{-13}	-1.690×10^{-13}
and	12.79×10^{-13}	-1.535×10^{-13}

are given by Bechmann[8] (according to Voigt's measurements) and W. P. Mason,[9] respectively.

The calculated values have been obtained with these two sets of elastic constants of Voigt (upper figure) and by W. P. Mason (lower figure), respectively. Two of the observed frequencies agree numerically with the formulas given by Bechmann although the expression in terms of the s_{ik} is quite different.

V. Petrzilka[7] has measured the frequencies of a square tourmaline plate of edge $a=1.836$ cm.

[8] R. Bechmann, Zeits. f. Physik **118**, 515 (1941).
[9] W. P. Mason, Bell Sys. Tech. J. **22**, 178 (1943).

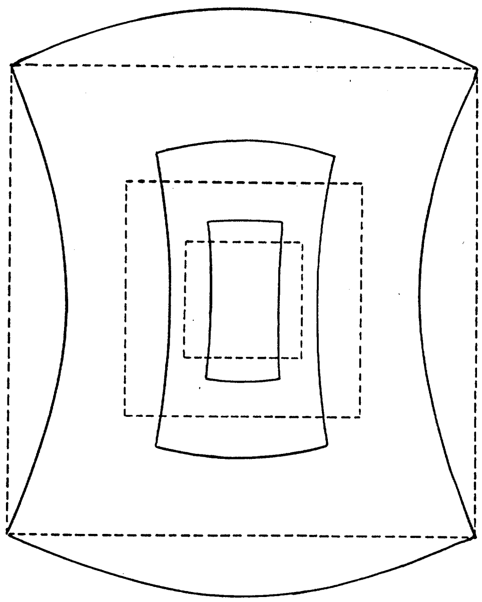

FIG. 1. Longitudinal vibration of a thin square plate. Mode 1: $U_1 = C\cos(\pi/a)x\sin(\pi/a)y$, $V_1 = C\sin(\pi/a)x\cos(\pi/a)y$. Dotted lines—crystal at rest. Full lines—crystal vibrating.

The frequencies calculated by Eqs. (48) and (50) with Voigt's values for the elastic moduli of tourmaline are shown in Table II.

Two of these calculated frequencies are equal to those given by Petrzilka. The first frequency is the rigorous value and is, of course, in agreement with our value.

The vibrational modes calculated by Eqs. (39) and (52) are represented by Figs. 1–3. It can be seen that none of them has nodal lines, but only nodal points. Petrzilka attributed to the frequency (2) a mode equal to our zero-approximation modes (40) or (41), but did not find the expected nodal line by lycopodium powder observations.

The calculated mode (3), Fig. 3, involves a nearly radial motion. It can clearly be identified with the picture of the lycopodium powder pattern published by Petrzilka. In this mode, according to the author, the powder is displaced radially and stays at rest only at the center. This is also what the theoretical picture predicts.

We now apply Eqs. (44) to a square Y-cut quartz plate, i.e., a plate whose large face is normal to the crystallographic Y axis, while the edges are parallel to the optic and electric axis, respectively. Using Voigt's values for the elastic moduli, we obtain:

$$\gamma_{11} = 78.67\times10^{10},\quad \gamma_{22} = 103.08\times10^{10},\quad \gamma_{16} = \gamma_{26} = 0,\quad \gamma_{12} = 12.180\times10^{10}. \tag{53}$$

The secular equation takes the form:

$$\begin{vmatrix} \rho\Omega^2 - \frac{1}{2}k^2[\gamma_{11}+\gamma_{22}-2\gamma_{12}] & \frac{2}{\pi}\rho\Omega^2 - \frac{2}{\pi}k^2[\gamma_{11}-\gamma_{12}] & -\frac{2}{\pi}\rho\Omega^2 + \frac{2}{\pi}k^2[\gamma_{22}-\gamma_{12}] \\ \frac{2}{\pi}\rho\Omega^2 - \frac{2}{\pi}k^2[\gamma_{11}-\gamma_{12}] & \rho\Omega^2 - k^2\gamma_{11} & -k^2\frac{8}{\pi^2}\gamma_{12} \\ -\frac{2}{\pi}\rho\Omega^2 + \frac{2}{\pi}k^2[\gamma_{22}-\gamma_{12}] & -k^2\frac{8}{\pi^2}\gamma_{12} & \rho\Omega^2 - k^2\gamma_{22} \end{vmatrix} = 0. \tag{54}$$

The three eigenvalues of (55) are: $\rho\Omega^2 = k^2\times73.9\times10^{10}$; $k^2\times89.6\times10^{10}$; $k^2\times106.8\times10^{10}$.

The corresponding frequencies expressed in terms of the frequency constant $N = 10^{-2}\nu a$ are shown in Table III. As a comparison the values observed by Bechmann[8] and Wright and Stuart[10] are given. The second calculated frequency was not observed by these authors.

Wright and Stuart studied the mode No. 1 ($N = 2.67\times10^3$) by means of powder patterns. While the experiments with powder on the crystal surface are difficult to interpret, the pattern created by the air currents from the periph-

TABLE III. Frequencies of square Y-cut quartz plates edge = 1 mm, in kilocycles.

Mode	$N_{calc.}$	Bechmann	Wright and Stuart
1	2.64×10^3	2.68×10^3	2.675×10^3
2	2.90×10^3		
3	3.17×10^3	3.19×10^3	

[10] R. B. Wright and D. M. Stuart, Bur. Stand. J. Research 7, 519 (1931).

ery of the crystal should give a simple idea of the motion of the boundary.

In attempting to calculate the ratio $A_1:A_2:A_3$ corresponding to the mode No. 1, one meets a difficulty. While the discrepancy between the observed and calculated values of the frequency (2.64 and 2.68) is as small as can be expected from the uncertainty of the elastic moduli, the ratio $A_1:A_2:A_3$ is very sensitive to small variations of the root, so that the mode cannot be described definitely. As a first approximation, we can expect it to be similar to the mode No. 2 described for the Z-cut plate (Fig. 2). In this mode, the corners of the plate move in a tangential direction and create thereby a wind such as is shown in Wright and Stuart's Fig. 13a. The agreement between this picture and the air currents to be expected from the theoretical figure is quite satisfactory.

(b) Shear Modes

A type of modes where the shearing strain x_y predominates is usually called "shear mode."

The zero-order modes will be found by considering two extreme cases: If the plate is infinitely extended along the x direction ($a\rightarrow\infty$) standing shear waves are exact solutions:

$$u_{py}=C\cos pk_2y,\quad v_{py}=0,\qquad p=1,2,3\cdots. \tag{55}$$

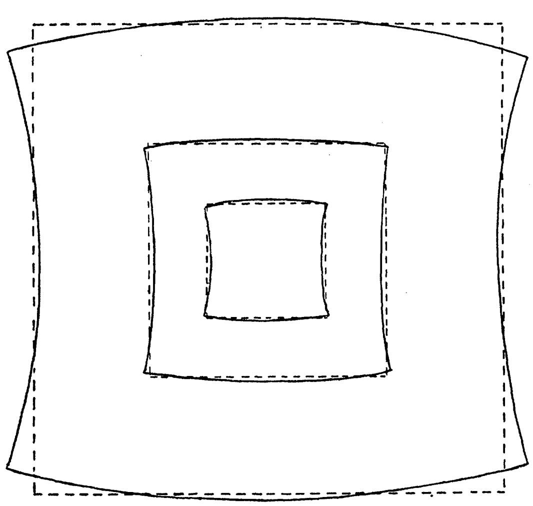

FIG. 2. Longitudinal vibration of a thin square plate. Mode 2: $U_2=C[\sin(\pi/a)y-(\pi/4)]\cos(\pi/a)x$, $V_2=C[-\sin(\pi/a)x+(\pi/4)]\cos(\pi/a)y$. Dotted lines—crystal at rest. Full lines—crystal vibrating.

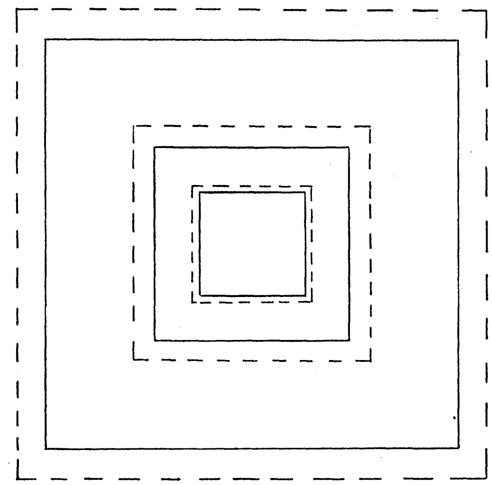

FIG. 3. Longitudinal vibration of a thin square plate. Mode 3: $U_3=C\cos kx$, $V_3=C\cos ky$. Dotted lines—crystal at rest. Full lines—crystal vibrating.

If the dimension b is very long, standing shear waves along the y direction are solutions:

$$u_{px}=0,\quad v_{px}=C\cos pk_1x,\qquad p=1,2,3\cdots, \tag{56}$$

where again

$$k_1=\pi/a,\quad k_2=\pi/b.$$

The modes (55) and (56) will be taken as zero-order modes for the study of almost square plates.

The normalization (22a) requires that

$$C^2=2/ab \tag{57}$$

for all these modes.

The definition (25a) yields, after elementary calculations, the elements of the matrix H:

$$\begin{aligned} &H_{py,\,qy}=H_{px,\,qx}=0,\quad (p\neq q)\\ &H_{py,\,py}=\gamma_{66}p^2k_2^2,\\ &H_{px,\,px}=\gamma_{66}p^2k_1^2,\\ &H_{px,\,qy}=\begin{cases}0 \text{ if } p \text{ or } q \text{ is even}\\ (8/ab)\gamma_{66} \text{ if } p \text{ and } q \text{ are odd,}\end{cases}\\ &N_{ik}=0 \text{ for any two different modes.}\end{aligned} \tag{58}$$

There is no interaction between the modes with p or q even and the odd modes. Therefore, the even modes can be dropped, and in the following, p and q will be only odd integers.

The secular determinant reads:

$$\begin{vmatrix} \rho\Omega^2-\gamma_{66}k_1^2 & -\frac{8\gamma_{66}}{ab} & 0 & -\frac{8\gamma_{66}}{ab} & \cdots \\ -\frac{8\gamma_{66}}{ab} & \rho\Omega^2-\gamma_{66}k_2^2 & -\frac{8\gamma_{66}}{ab} & 0 & \cdots \\ 0 & -\frac{8\gamma_{66}}{ab} & \rho\Omega^2-9\gamma_{66}k_1^2 & -\frac{8\gamma_{66}}{ab} & \cdots \\ -\frac{8\gamma_{66}}{ab} & 0 & -\frac{8\gamma_{66}}{ab} & \rho\Omega^2-9\gamma_{66}k_2^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} = 0. \tag{59}$$

As $k_1 \cong k_2$, each diagonal element of H is quasi-degenerate with one neighboring diagonal element. The first step in the solution of (59) is to find new linear combinations of the zero-order functions so that this degeneracy is removed. Considering the case of the square plate ($k_1=k_2$) and disregarding the coupling between non-degenerate functions, we find Eq. (24) to be:

$$\begin{aligned} A_{px}(\rho\Omega^2-\gamma_{66}p^2k^2)-A_{py}(8\gamma_{66}/a^2)=0, \\ -A_{px}(8\gamma_{66}/a^2)+A_{py}(\rho\Omega^2-\gamma_{66}p^2k^2)=0. \end{aligned} \tag{60}$$

These two equations are compatible if

$$A_{px}=A_{py} \quad \text{or} \quad A_{px}=-A_{py}. \tag{61}$$

Consequently, the sought linear combinations are:

$$\begin{aligned} &u_{p+}=C\cos pk_2y, \quad u_{p-}=-C\cos pk_2y, \\ &\qquad\qquad \text{where } p=1,3,5\cdots \\ &v_{p+}=C\cos pk_1x, \quad v_{p-}=C\cos pk_1x. \end{aligned} \tag{62}$$

With these new zero-order functions the matrix elements of H and N are found to be:

$N_{ik}=0$ for any two functions (62),

$$\begin{aligned} &H_{p+,p+}=\gamma_{66}\left(p^2\frac{k_1^2+k_2^2}{2}-\frac{8}{ab}\right), \\ &H_{p+,q+}=\frac{8}{ab}\gamma_{66}, \quad (p\neq q) \\ &H_{p+,q-}=\begin{cases} 0 & (p\neq q) \\ \gamma_{66}p^2\frac{k_1^2-k_2^2}{2} & (p=q). \end{cases} \end{aligned} \tag{63}$$

The interaction between the modes $p+$ and $p-$ is so small for our case that it can be neglected.

Putting

$$b=a(1-\epsilon), \tag{64}$$

where ϵ is a small number, we find that the matrix elements referring to the modes $p+$ become

$$\begin{aligned} H_{p+,p+}&=(1/a^2)(1+\epsilon)(p^2\pi^2+8)\gamma_{66}, \\ H_{p+,q+}&=(1/a^2)(1+\epsilon)8\gamma_{66} \quad (p\neq q). \end{aligned} \tag{65}$$

Now, the secular determinant has the form:

$$\begin{vmatrix} \rho\Omega^2-\frac{\gamma_{66}}{a^2}(1+\epsilon)(\pi^2+8) & -\frac{8}{a^2}\gamma_{66}(1+\epsilon) & -\frac{8}{a^2}\gamma_{66}(1+\epsilon) & \cdots \\ -\frac{8}{a^2}\gamma_{66}(1+\epsilon) & \rho\Omega^2-\frac{\gamma_{66}}{a^2}(1+\epsilon)(9\pi^2+8) & -\frac{8}{a^2}\gamma_{66}(1+\epsilon) & \cdots \\ -\frac{8}{a^2}\gamma_{66}(1+\epsilon) & -\frac{8}{a^2}\gamma_{66}(1+\epsilon) & \rho\Omega^2-\frac{\gamma_{66}}{a^2}(1+\epsilon)(25\pi^2+8) & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}. \tag{66}$$

If we put

$$r=\rho\Omega^2a^2/8\gamma_{66}(1+\epsilon) \tag{67}$$

and divide each row of (66) by $8\gamma_{66}(1+\epsilon)/a^2$, the determinant is simplified:

$$\begin{vmatrix} r-\left(1+\frac{\pi^2}{8}\right) & -1 & -1 & \cdots \\ -1 & r-\left(1+\frac{9\pi^2}{8}\right) & -1 & \cdots \\ -1 & -1 & r-\left(1+\frac{25\pi^2}{8}\right) & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}=0. \tag{68}$$

The roots of (68) are approximated by the Schrödinger perturbation formula:

$$r_p=1+\frac{p^2\pi^2}{8}-\frac{8}{\pi^2}\sum_{q=1}^{\infty}{}'\frac{1}{q^2-p^2}. \tag{69}$$

The sum in this expression can be readily calculated. As p and q are odd numbers, the substitutions $p=2n+1$ and $q=2m+1$ give:

$$\sum_{m=0}^{\infty}{}'\frac{1}{(2m+1)^2-(2n+1)^2}$$
$$=\frac{1}{4(2n+1)}\sum{}'\left(\frac{1}{m-n}-\frac{1}{m+n+1}\right),$$

where m takes all integer values except $m=n$. Inspection of this expression shows that the sum has the value

$$\frac{1}{4(2n+1)}\times\frac{1}{2n+1}.$$

Thus, (69) yields:

$$r_p=1+\frac{p^2\pi^2}{8}-\frac{2}{\pi^2p^2}. \tag{70}$$

In view of (67), the frequencies are:

$$\nu_p=\frac{(1+\epsilon)^{\frac{1}{2}}}{2a}\left(\frac{\gamma_{66}}{\rho}\right)^{\frac{1}{2}}\left(\frac{8}{\pi^2}r_p\right)^{\frac{1}{2}},$$

or, in view of (64):

$$\nu_p=\frac{1}{a+b}\left(\frac{\gamma_{66}}{\rho}\right)^{\frac{1}{2}}\left(p^2+\frac{8}{\pi^2}-\frac{16}{\pi^4p^2}\right)^{\frac{1}{2}}. \tag{71}$$

Systematic observations were made on the lowest shear mode of nearly square quartz plates by Hight and Willard.[11] They studied quartz plates whose edge was parallel to the crystallographic X axis. In this case

$$s_{16}=s_{26}=\gamma_{16}=\gamma_{26}=0;\quad \gamma_{66}=1/s_{66}.$$

Equation (71) gives, for $p=1$,

$$\nu_1=\frac{1}{a+b}\left(\frac{1}{s_{66}\rho}\right)^{\frac{1}{2}}\times 1.28,$$

while Hight and Willard's observations give

$$\nu_1=\frac{1}{a+b}\left(\frac{1}{s_{66}\rho}\right)^{\frac{1}{2}}\times 1.25,$$

so that the difference between theory and experiment is 2.5 percent. Later observations of Bechmann[8] agree within 1 percent with those of Hight and Willard.

For the resultant modes, the Rayleigh-Schrödinger perturbation formula yields from Eq. (68):

$$U_p=u_p+\frac{8}{\pi^2}\sum_{q\neq p}\frac{u_{q+}}{q^2-p^2},$$
$$V_p=v_p+\frac{8}{\pi_2}\sum_{q\neq p}\frac{v_{p+}}{q^2-p^2}. \tag{72}$$

The lowest mode is

$$U_1=\cos k_2y+\frac{8}{\pi^2}\sum_{q\neq 1}^{\infty}\frac{\cos qk_2y}{q^2-1},$$
$$V_1=\cos k_1x+\frac{8}{\pi^2}\sum_{q\neq 1}\frac{\cos qk_1x}{q^2-1}. \tag{73}$$

[11] S. C. Hight and G. W. Willard, Proc. I.R.E. **25**, 549 (1937).

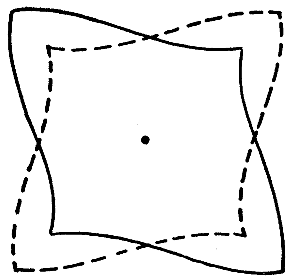

FIG. 4. Shear vibration of a thin square plate. Observations by R. A. Sykes. Full and dotted lines represent the contour deformation in two opposite phases.

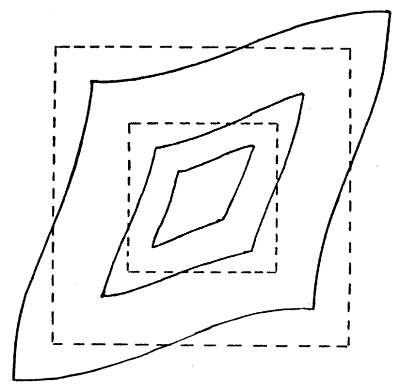

FIG. 5. Shear vibration of a thin square plate (calculated). $U=\cos(\pi/a)y$, $V=\cos(\pi/a)x$. Dotted line—crystal at rest. Full line—crystal vibrating.

This result can be compared with observations on the mode of a square quartz plate vibrating in the lowest shear mode. Figure 4 shows the distorted edges in opposite phases, one a solid curve and the other a dotted curve, according to microscopic observations by Sykes.[12] Figure 5 shows one phase (full line) of the motion calculated by (73). Only the first terms of (73) have been used for the drawing because the supplementary terms are small. The agreement between observation and theory is satisfactory.

Theoretical considerations of the shear mode by Mason[1] and Bechmann[13] led to a mode in which the corners of the plate should be at rest. Their solution apparently bears no resemblance to the experimental picture.

All comparisons with experiment for the longitudinal as well as shear modes were made in cases where the parameters γ_{16} and γ_{26} vanish. Where these parameters are not negligibly small, the longitudinal and shear modes would be coupled and could not be treated separately as is done here.

The author is indebted to Dr. F. Beer for advice and help. He wishes to thank Mr. A. H. Schlesinger for his help in numerical calculation.

[12] R. A. Sykes, Bell Sys. Tech. J. **23**, 52 (1944).
[13] R. Bechmann, Zeits. f. Physik **117** and **118** (1941).

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Electromagnetic Waves in Metal Tubes Filled Longitudinally with Two Dielectrics

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The propagation of e-m waves is investigated: (1) in a rectangular metal tube half-filled longitudinally with a dielectric; (2) in a cylindrical dielectric guide of radius a, surrounded by a coaxial metal tube of radius b. In (1) the waves are of the longitudinal-section type; for high frequencies they are confined to the medium of higher dielectric constant. The problem is discussed also with the model of the criss-cross component waves. In (2) the waves are linear combinations of E and H waves. The case of axial symmetry is considered, in which there are simple E and H waves. For E_0 a critical wave-length exists, depending only on the dimensions of the inner dielectric, and on the dielectric constants, below which the system behaves more or less as a dielectric guide in free space, and above which as an ordinary hollow tube. For H_0 this critical wave-length depends also on the ratio b/a. The case in which the external medium has a higher dielectric constant is also briefly investigated.

1

THE purpose of this note is to investigate from a purely theoretical point of view the propagation of electromagnetic waves in hollow metal tubes, the interior of which is filled with two different dielectrics, the distribution in all