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On the Binding Energy of Deuteron and the Neutron-Proton Scattering by a New Potential

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The binding energy of deuteron and scattering cross section of proton by fast neutron are calculated by using new forms of nuclear potential suggested by Wang. The results obtained are found to be in good agreement with the experimental values, when "zero cut-off" of the potential is employed.

IT is pointed out by Wang[1] that the force between two nuclear particles may be related to the gravitational force. He takes two alternative forms of the nuclear potential,

$$V=-Ae^{K/r}, \tag{1a}$$

and

$$V=-(B/r)e^{K/r}, \tag{1b}$$

with

$$K=\hbar/mc=3.84\times 10^{-11}\text{ cm},$$

where the constants A and B, as determined by the gravitational constant, are 4.78×10^{-45} and 1.84×10^{-55}, respectively.

The purpose of the present work is to determine whether the potential (1a) and (1b) can give correct results about the binding energy of a deuteron and the scattering cross section of the neutron by the proton. The calculations follow closely those of Bethe and Bacher.[2]

I. THE BINDING ENERGY OF THE DEUTERON

The wave equation for the relative motion of the two nuclear particles is

$$\Delta\psi+(M/\hbar^2)(E-V)\psi=0. \tag{2}$$

The potential is spherically symmetric; (2) can thus be separated in polar coordinates r, θ, ϕ, by putting

$$\psi(r,\theta,\phi)=(1/r)u_l P_{lm}(\theta)e^{im\phi},$$

where P_{lm} is a spherical harmonic. The wave equation for u_l is then

$$\frac{\hbar^2}{M}\left(\frac{d^2u_l}{dr^2}-\frac{l(l+1)}{r^2}u_l\right)+(E-V)u_l=0.$$

[1] K. C. Wang and H. L. Tsao, Phys. Rev. **66**, 155 (1944).
[2] H. A. Bethe and R. E. Bacher, Rev. Mod. Phys. **8**, 82 (1936).

For the ground state $l=0$, we have

$$\frac{d^2u_0}{dr^2}+\frac{M}{\hbar^2}(E-V)u_0=0. \tag{3}$$

To solve Eq. (3), two ways of cut-off of the potential given by (1a) and (1b) are employed; *viz.*,

(a) zero cut-off:

$V=0$, for $r<a$; $V=V(r)$, for $r>a$.

(b) straight cut-off:

$V=V(a)$, for $r<a$; $V=V(r)$, for $r>a$.

(a) Zero Cut-Off

For $r<a$, $V=0$; Eq. (3) takes the form,

$$\frac{d^2u_0}{dr^2}-\frac{M\epsilon}{\hbar^2}u_0=0,$$

where $\epsilon=-E$, the binding energy of the deuteron. Its solution is

$$u_0=D(e^{\beta r}-e^{-\beta r}),$$

where

$$\beta=(M\epsilon/\hbar^2)^{\frac{1}{2}},$$

and

$$du_0/dr=D\beta(e^{\beta r}+e^{-\beta r}).$$

Therefore

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a}=\beta\coth\beta a. \tag{4}$$

For $r>a$, $V=V(r)$; the asymptotic solution of (3) is

$$u_0=ce^{-\beta r}.$$

Now we treat c as a slowly varying quantity whose second derivative with respect to r may be set equal to zero. In so doing, we have

$$du_0/dr=c'e^{-\beta r}-\beta ce^{-\beta r},$$

$$d^2u_0/dr^2=-2\beta c'e^{-\beta r}+\beta^2ce^{-\beta r}.$$

TABLE I. Numerical values for Eq. (7) when $V=-Ae^{K/r}$ for $r>a$ and $V=0$ for $r<a$.

ϵ, in 10^{-6} erg	coth $[1.24\times10^{15}a(V_0-\epsilon)^{\frac{1}{2}}]$	$\frac{V_0}{2\epsilon}-1$
3.50	1.33	1.38
3.60	1.32	1.32
3.70	1.31	1.26

TABLE II. Numerical values for Eq. (7) when $V=-(B/r)e^{K/r}$ for $r>a$ and $V=0$ for $r<a$.

ϵ, in 10^{-6} erg	coth $[1.24\times10^{15}a(V_0-\epsilon)^{\frac{1}{2}}]$	$\frac{V_0}{2\epsilon}-1$
3.50	1.29	1.38
3.60	1.29^-	1.32
3.70	1.27	1.26
3.80	1.26^+	1.19

Substituting this last expression in (3), we get the differential equation for c,

$$dc/dr=c'=-MVc/2\beta\hbar^2,$$

the solution of which is

$$c=F\exp\left(-\int\frac{MV}{2\beta\hbar^2}dr\right).$$

Thus

$$u_0=F\exp\left(-\beta r-\int\frac{MV}{2\beta\hbar^2}dr\right),$$

and

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a}=\frac{M|V(a)|}{2\beta\hbar^2}-\beta. \quad (5)$$

In order that the wave function u_0 could be joined at $r=a$ smoothly, the expressions (4) and (5) must be equal; i.e.,

$$\frac{M|V(a)|}{2\beta\hbar^2}-\beta=\beta\coth\beta a. \quad (6)$$

(b) Straight Cut-Off

For $r<a$, $V=V(a)$; Eq. (3) takes the form

$$\frac{d^2u_0}{dr^2}+\frac{M}{\hbar^2}(|V(a)|-\epsilon)u_0=0.$$

Its solution is

$$u_0=G\sin\gamma r,$$

with

$$\gamma=\left[\frac{M}{\hbar^2}(|V(a)|-\epsilon)\right]^{\frac{1}{2}}.$$

Thus

$$\left(\frac{1}{u_0}\frac{du_0}{dr}\right)_{r=a}=\gamma\cot\gamma a.$$

For $r>a$, $(1/u_0)(du_0/dr)$ is same as given by (5). Therefore, instead of (6), we have

$$\frac{M|V(a)|}{2\beta\hbar^2}-\beta=\gamma\cot\gamma a. \quad (6a)$$

Equations (6) and (6a) serve to determine the binding energy of the deuteron ϵ if we use the experimental value of $-V(a)=V_0$, the depth of the potential well, and calculate a, the range of the nuclear force, from (1a) or (1b). Conversely, if we use the experimental value of ϵ, we can calculate the value of V_0, and a by the aid of Eq. (1a) or (1b).

(c) Numerical Calculations

(1) $V=0$, for $r<a$; $V=-Ae^{K/r}$, for $r>a$. We take the recent experimental value $V_0=10.5$ Mev; then a calculated from (1a) is equal to 4.21×10^{-13} cm. Equation (6) becomes numerically,

$$\coth[1.24\times10^{15}a(V_0-\epsilon)^{\frac{1}{2}}]=V_0/2\epsilon-1. \quad (7)$$

From Table I we see that the root of (7), the binding energy of deuteron, is 3.60×10^{-6} erg or 2.26 Mev.

(2) $V=0$, for $r<a$; $V=-(B/r)e^{K/r}$, for $r>a$. Also take $V_0=10.5$ Mev; then a is equal to 4.42×10^{-13} cm from (1b).

TABLE III. Numerical values for Eq. (7) when $V=V(a)$ for $r<a$ and $V=-Ae^{K/r}$ for $r>a$.

ϵ, in 10^{-6} erg	cot $[1.24\times10^{15}a(V_0-\epsilon)^{\frac{1}{2}}]$	$\frac{V_0/2\sqrt{\epsilon}-\sqrt{\epsilon}}{(V_0-\epsilon)^{\frac{1}{2}}}$
7.80	0.018	0.067
8.00	0.038	0.037
8.20	0.061	0.027

TABLE IV. Numerical values for Eq. (7) when $V=V(a)$ for $r<a$ and $V=-(B/r)e^{K/r}$ for $r>a$.

ϵ, in 10^{-6} erg	cot $[1.24\times10^{15}a(V_0-\epsilon)^{\frac{1}{2}}]$	$\frac{V_0/2\sqrt{\epsilon}-\sqrt{\epsilon}}{(V_0-\epsilon)^{\frac{1}{2}}}$
8.20	−0.020	0.206
8.30	−0.013	0.005
8.40	−0.004	−0.083

The binding energy of deuteron from Table II is 3.66×10^{-6} erg or 2.30 Mev. The results given by (1) and (2) are in good agreement with the experimental value 2.17 Mev.

(3) $V=V(a)$, for $r<a$; $V=-Ae^{K/r}$, for $r>a$. Also take $V_0=10.5$ Mev, then a is 4.21×10^{-13} cm from (1a). Equation (6a) turns out numerically to be

$$\cot\left[1.24\times10^{15}a(V_0-\epsilon)^{\frac{1}{2}}\right]=\left[(V_0/2\sqrt{\epsilon})-\sqrt{\epsilon}\right]/(V_0-\epsilon)^{\frac{1}{2}}.$$

(4) $V=V(a)$, for $r<a$; $V=-(B/r)e^{K/r}$, for $r>a$.

From Tables III and IV, the binding energy of the deuteron is 8.0×10^{-6} and 8.3×10^{-6} erg, respectively. These values are too large in comparison with the experimental value. The correct value will be obtained if we take $V_0=5.93$ Mev.

II. THE SCATTERING OF NEUTRON BY PROTON

Let us denote E the kinetic energy of a proton and a neutron in a coordinate system in which the center of gravity of the two particles is at rest, which is equal to one-half of the kinetic energy of the incident neutron in a system at rest. The wave function u_l will satisfy the equation,

$$\frac{\hbar^2}{M}\left(\frac{d^2u_l}{dr^2}-\frac{l(l+1)}{r^2}u_l\right)+(E-V)u_l=0. \quad (8)$$

Asymptotically for large r, the solution of (8) is

$$u_l=c\sin(Kr-\tfrac{1}{2}l\pi+\delta_l), \quad (9)$$

with

$$K=(ME/\hbar^2)^{\frac{1}{2}}.$$

Then the cross section $d\sigma$, the number of the neutrons scattered per unit time through an angle θ and $\theta+d\theta$, if there is one neutron crossing unit area per unit time in the incident beam, is given by the well-known formula,

$$d\sigma=(\pi/2K^2)\times\left|\textstyle\sum_l(2l+1)P_l(\theta)[\exp(2i\delta_l)-1]\right|^2\sin\theta d\theta. \quad (10)$$

It has been shown[3] that if $1/K\gg a$, a being the range of the force, all phases δ_l will be small except δ_0. Then

$$d\sigma=2\pi K^{-2}\sin^2\delta_0\sin\theta d\theta,$$

[3] Cf. reference 2, p. 115.

TABLE V. Comparison of values of $\sigma\times10^{24}$.

E, Mev	σ_0	σ	$\sigma_0(B)$	$\sigma(B)$	σ(obs.)
2.15	0.85	1.43	1.2	1.8	(0.5–0.8)
1.05	1.03	1.86	1.6	2.4	(1.1–1.5)

and

$$\sigma=\int d\sigma=4\pi K^{-2}\sin^2\delta_0. \quad (11)$$

For the ground state of deuteron, we have already shown that

$$\left(\frac{1}{u_0^-}\frac{du_0^-}{dr}\right)_{r=a}=\left(\frac{MV_0}{2\beta\hbar^2}-\beta\right)\equiv\alpha. \quad (12)$$

Now, in the present case E is positive; we should have[3]

$$\left(\frac{1}{u_0^+}\frac{du_0^+}{dr}\right)_{r=a}=\alpha-\frac{M(E+\epsilon)}{\hbar^2u_0^+(a)u_0^-(a)}\int_0^a u_0^+u_0^-dr$$
$$=\alpha-\frac{\eta aM(E+\epsilon)}{\hbar^2}\equiv A, \quad (13)$$

where u_0^+ and u_0^- are the u_0 functions for $r<a$ corresponding to the positive and negative values of E, respectively; i.e.,

$$u_0^+=B\sin Kr, \quad u_0^-=D(e^{\beta r}-e^{-\beta r});$$

therefore

$$\eta a=\frac{1}{u_0^+(a)u_0^-(a)}\int_0^a u_0^+u_0^-dr$$
$$=\frac{\beta\coth\beta a-K\cot Ka}{\beta^2+K^2}. \quad (14)$$

For large r,

$$\left(\frac{1}{u_0^+}\frac{du_0^+}{dr}\right)_{r=a}=K\cot(Ka+\delta_0). \quad (15)$$

In order to join the wave function at $r=a$ smoothly, the expressions given by (13) and (15) must be equal, thus

$$K\cot(Ka+\delta_0)=A. \quad (16)$$

Under the assumption that Ka is small, we have

$$\cot\delta_0=A/K, \quad (17)$$

and by (11),

$$\sigma=4\pi/K^2+A^2. \quad (18)$$

If η is small $A\approx\alpha$, and we have

$$\sigma_0=4\pi/K^2+\alpha^2. \quad (19)$$

The numerical results are given in Table V in taking $V=-Ae^{K/r}$, and the values of α are obtained from Table I at the point $r=a$. Corresponding to (18) and (19), Bethe and Bacher have derived[3] the formulae: $\sigma_0(B)=4\pi\hbar^2/\epsilon+E$, $\sigma(B)=3\sigma_0/2$, and the values calculated from these are given for comparison in the fourth and fifth columns of Table V.

It is seen that the present results are in better agreement with the experimental results than those given by Bethe and Bacher. If we take $V=-(B/r)e^{K/r}$, the results do not differ appreciably from that given above.

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Momentum and Energy of Photon and Electron in the Čerenkov Radiation

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The conservation of momentum and energy between an emitted photon and an electron in the Čerenkov radiation gives an equation for the direction of emission. This equation differs from the one which Frank and Tamm derived from electromagnetic theory only by a negligible term involving the ratio of the wave-length of the electron to that of the photon.

It has been observed by Čerenkov[1] that fast electrons traversing a transparent medium emit a radiation, continuous through the visible spectrum and beyond, which differs in respect to its polarization and angular distribution from radiation previously found with similar sources. Frank and Tamm[2] have given an electrodynamic theory of this radiation similar to the dynamic theory of the bow wave of a ship or the conical wave around the path of a bullet moving through the air at a speed greater than the speed of sound. According to this theory the emitted radiation has a component of every wave-length of which the speed in the medium is less than the speed of the electron. The energy per unit length of path radiated by the electron in the frequency range $d\nu$ is $(2\pi e/c)^2(1-1/\beta^2n^2)\nu d\nu$, where e is the electronic charge, c the speed of light in vacuum, β the ratio to this speed of the speed of the electron, and n the index of refraction of the medium for light of frequency ν. The emitted rays make an angle θ with the electron velocity given by $\cos\theta=1/n\beta$, and they are polarized with the electric vector in the plane of this angle. The theory accords with the observations of Čerenkov, and it has been confirmed by observations under more favorable experimental conditions made by Collins and Reiling[3] and, over a considerable range of electron speeds, by Wyckoff and Henderson.[4]

Although the phenomenon has thus an adequate classical explanation, it is interesting also to treat it by considering the conservation of momentum and energy between the electron and an emitted photon of the radiation. Let it be supposed that the electron, traversing the medium with a speed u, emits a photon of energy $h\nu$ in a direction making an angle θ with the initial velocity of the electron. After the emission, let the speed of the electron be v and let the angle between its final and initial velocities be ϕ. By the conservation of momentum, the angles θ and ϕ will be coplanar and on opposite sides of the initial direction of the velocity of the electron.

Also, by the conservation of momentum,

$$mv(1-v^2/c^2)^{-\frac{1}{2}}\cos\phi+(h/\lambda)\cos\theta=mu(1-u^2/c^2)^{-\frac{1}{2}},$$

$$mv(1-v^2/c^2)^{-\frac{1}{2}}\sin\phi-(h/\lambda)\sin\theta=0.$$

[1] P. A. Čerenkov, Comptes rendus Acad. Sci. U.S.S.R. **8**, 451 (1934); **12**, 413 (1936); **14**, 102 (1937); **14**, 105 (1937); Phys. Rev. **52**, 378 (1937).

[2] I. Frank and Ig. Tamm, Comptes rendus Acad. Sci. U.S.S.R. **14**, 109 (1937). The subject has also been treated by Fermi, Phys. Rev. **57**, 485 (1940), in the development of a classical theory of the retardation of charged particles in gaseous and condensed media.

[3] G. B. Collins and V. G. Reiling, Phys. Rev. **54**, 499 (1938).

[4] H. O. Wyckoff and J. E. Henderson, Phys. Rev. **64**, 1 (1943).