tentials. It follows from (34) and (19) (lower sign) that solutions exist only for

$$\alpha_1 \geq -u_0. \tag{36}$$

Both branches of the curves $u_0=$const. begin with the value $\alpha_1=-u_0$ at the value $\xi_0=\delta$ and with vertical tangent. They form together a loop which contracts around the point G on Fig. 1 ($\alpha_1=0$, $\xi_0=4/3$) as u_0 decreases from $u_0=1$ to $u_0=0$, and for $u_0=0$ there remains only the branch I. In Fig. 1 the curves for $u_0=\frac{1}{2}$ and $u_0=0$ have been added.

It is now easy to discuss the behavior of the solution. Three intervals have to be distinguished:

(a) $0\leq\xi_0\leq 4/3$. The solution starts, with $V_0=0$, on branch II until, with decreasing potentials, the value $\xi_0=\delta$ is reached. Then the solution changes to branch I (no minimum) and ends on the curve $u_0=0$. The current has the saturation value throughout.

(b) $4/3\leq\xi_0\leq 1.886$. The upper limit of this interval is set by the maximum of δ which occurs for $\alpha_1=-0.5$. The solution begins, with $V_0=0$, on branch II, changes to branch I when the δ curve is intersected for the first time, changes back to branch II when the δ curve is met for the second time, and finally becomes space-charge limited.

(c) $1.886\leq\xi_0\leq 3.771$. In this interval the solution is on branch II throughout. The current is saturated at first and becomes space-charge limited as soon as the critical value (26) is reached. All space-charge limited solutions [in cases (b) and (c)] end at the point G of Fig. 1, i.e., at $\xi_0=4/3$ or $j=E_0^{\frac{3}{2}}$.

In Fig. 5 we have represented the potential distribution in the case of a negative potential V_0. We have taken the value $u_0=\frac{1}{2}$ corresponding to $V_0=-(3/4)E_0$, and have chosen ξ_0 on the δ curve; i.e., $\xi_0=(4/3)2^{\frac{1}{2}}=1.886$. Whenever, in the intervals (a) and (b), the limiting case $\xi_0=\delta$ is realized, the solution of branch I coincides with the (stable) solution of branch II. This case, therefore, corresponds to the case $(d\eta/d\xi)_0=0$ for positive potentials, but the horizontal tangent is now at the receiver plate.

The two solutions on branch II correspond to the values $\alpha_1=-\frac{1}{2}$ and $\alpha_1=-0.025$, respectively. Both potential distributions are represented in Fig. 5, the unstable one by a broken line.

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The Electrical Oscillations of a Prolate Spheroid. Paper II

Prolate Spheroidal Wave Functions

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The vector wave equation in prolate spheroidal coordinates ξ, η, ϕ is set up, the variables are separated, and the characteristic values (eigenvalues) and characteristic functions (eigenfunctions) of the resulting ordinary differential equations are obtained in series which converge rapidly in the neighborhood of resonance for spheroids of eccentricity near to unity. The coefficients of the two independent primitives in the linear combinations representing diverging and converging waves at infinity are calculated. The zeros of certain of the characteristic values are investigated.

IN a previous paper[1] both the free oscillations of a perfectly conducting prolate spheroid and the oscillations forced by a plane wave with the electric field parallel to the long axis of the spheroid were discussed. The forced oscillations were treated by an approximate method valid only for very eccentric spheroids. For the exact investigation of these oscillations the complete solutions of the vector wave equation in prolate spheroidal coordinates are needed. The object of the present paper is

[1] L. Page and N. I. Adams, Jr., Phys. Rev. 53, 819 (1938). This paper will be referred to as I and the present paper as II.

to obtain these solutions in the form of series which converge rapidly in the neighborhood of resonance. In a following paper these solutions will be applied to the problem of the antenna.

I. FIELD EQUATIONS

If the time factor is taken as $e^{-i\omega\tau}$, where τ is the time, the field equations for simple harmonic waves of angular frequency ω in a medium of permittivity κ and permeability μ are

$$\nabla\times\mathbf{E}=i\frac{\omega\mu}{c}\mathbf{H},\quad \nabla\times\mathbf{H}=-i\frac{\omega\kappa}{c}\mathbf{E}, \tag{II-1}$$

in Heaviside-Lorentz symmetrical units, leading to the wave equations

$$\nabla\times\nabla\times\mathbf{E}=\frac{\omega^2\kappa\mu}{c^2}\mathbf{E},\quad \nabla\times\nabla\times\mathbf{H}=\frac{\omega^2\kappa\mu}{c^2}\mathbf{H}. \tag{II-2}$$

In the right-handed orthogonal prolate spheroidal coordinates ξ, η, ϕ defined in I, either of these vector wave equations yields the three scalar equations

$$\frac{\partial}{\partial\eta}\left[\frac{1-\xi^2}{\eta^2-\xi^2}\frac{\partial F_\eta}{\partial\xi}\right]-\frac{\partial}{\partial\eta}\left[\frac{\eta^2-1}{\eta^2-\xi^2}\frac{\partial F_\xi}{\partial\eta}\right]-\frac{1}{(1-\xi^2)(\eta^2-1)}\frac{\partial^2 F_\xi}{\partial\phi^2}+\frac{1}{\eta^2-1}\frac{\partial^2 F_\phi}{\partial\xi\partial\phi}=\epsilon^2 F_\xi, \tag{II-3}$$

$$\frac{1}{1-\xi^2}\frac{\partial^2 F_\phi}{\partial\eta\partial\phi}-\frac{1}{(1-\xi^2)(\eta^2-1)}\frac{\partial^2 F_\eta}{\partial\phi^2}-\frac{\partial}{\partial\xi}\left[\frac{1-\xi^2}{\eta^2-\xi^2}\frac{\partial F_\eta}{\partial\xi}\right]+\frac{\partial}{\partial\xi}\left[\frac{\eta^2-1}{\eta^2-\xi^2}\frac{\partial F_\xi}{\partial\eta}\right]=\epsilon^2 F_\eta, \tag{II-4}$$

$$\frac{\partial}{\partial\xi}\left[\frac{1}{(1-\xi^2)(\eta^2-1)}\frac{\partial F_\xi}{\partial\phi}\right]-\frac{1}{\eta^2-1}\frac{\partial^2 F_\phi}{\partial\xi^2}-\frac{1}{1-\xi^2}\frac{\partial^2 F_\phi}{\partial\eta^2}+\frac{\partial}{\partial\eta}\left[\frac{1}{(1-\xi^2)(\eta^2-1)}\frac{\partial F_\eta}{\partial\phi}\right]=\epsilon^2\frac{\eta^2-\xi^2}{(1-\xi^2)(\eta^2-1)}F_\phi, \tag{II-5}$$

where

$$F_\xi\equiv\{(\eta^2-\xi^2)(1-\xi^2)\}^{\frac{1}{2}}\begin{Bmatrix}E_\xi\\H_\xi\end{Bmatrix},$$

$$F_\eta\equiv\{(\eta^2-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}\begin{Bmatrix}E_\eta\\H_\eta\end{Bmatrix},$$

$$F_\phi\equiv\{(1-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}\begin{Bmatrix}E_\phi\\H_\phi\end{Bmatrix},$$

and

$$\epsilon\equiv\frac{\omega f}{c}(\kappa\mu)^{\frac{1}{2}}=\frac{2\pi f}{\lambda},$$

f being the semi-interfocal distance, and λ the wave-length.

Evidently the solutions are of the form

$$F_\xi=G_\xi\begin{Bmatrix}\sin m\phi\\\cos m\phi\end{Bmatrix},\quad F_\eta=G_\eta\begin{Bmatrix}\sin m\phi\\\cos m\phi\end{Bmatrix},\quad F_\phi=G_\phi\begin{Bmatrix}\cos m\phi\\-\sin m\phi\end{Bmatrix},$$

where m is a positive integer, and G_ξ, G_η, G_ϕ are functions of ξ and η only, satisfying the equations

$$\frac{\partial}{\partial\eta}\left[\frac{1-\xi^2}{\eta^2-\xi^2}\frac{\partial G_\eta}{\partial\xi}\right]-\frac{\partial}{\partial\eta}\left[\frac{\eta^2-1}{\eta^2-\xi^2}\frac{\partial G_\xi}{\partial\eta}\right]+\frac{m^2}{(1-\xi^2)(\eta^2-1)}G_\xi-\frac{m}{\eta^2-1}\frac{\partial G_\phi}{\partial\xi}=\epsilon^2 G_\xi, \tag{II-6}$$

$$-\frac{m}{1-\xi^2}\frac{\partial G_\phi}{\partial\eta}+\frac{m^2}{(1-\xi^2)(\eta^2-1)}G_\eta-\frac{\partial}{\partial\xi}\left[\frac{1-\xi^2}{\eta^2-\xi^2}\frac{\partial G_\eta}{\partial\xi}\right]+\frac{\partial}{\partial\xi}\left[\frac{\eta^2-1}{\eta^2-\xi^2}\frac{\partial G_\xi}{\partial\eta}\right]=\epsilon^2 G_\eta, \tag{II-7}$$

$$\frac{\partial}{\partial\xi}\left[\frac{mG_\xi}{(1-\xi^2)(\eta^2-1)}\right]-\frac{1}{\eta^2-1}\frac{\partial^2 G_\phi}{\partial\xi^2}-\frac{1}{1-\xi^2}\frac{\partial^2 G_\phi}{\partial\eta^2}+\frac{\partial}{\partial\eta}\left[\frac{mG_\eta}{(1-\xi^2)(\eta^2-1)}\right]=\epsilon^2\frac{\eta^2-\xi^2}{(1-\xi^2)(\eta^2-1)}G_\phi. \tag{II-8}$$

If we differentiate (II-6) with respect to ξ, (II-7) with respect to η, and combine the two equations so obtained with (II-8), we get

$$\frac{\partial G_\xi}{\partial \xi}+\frac{\partial G_\eta}{\partial \eta}=\left[\frac{m}{1-\xi^2}+\frac{m}{\eta^2-1}\right]G_\phi, \tag{II-9}$$

which is just the solenoidal condition.

Eliminating G_ϕ from (II-6) and (II-7), we find

$$\left[\frac{1}{1-\xi^2}+\frac{1}{\eta^2-1}\right][SG_\xi+NG_\xi]-\frac{2\xi}{1-\xi^2}\left[\frac{\partial G_\xi}{\partial \xi}+\frac{\partial G_\eta}{\partial \eta}\right]+\frac{2\eta}{\eta^2-1}\left[\frac{\partial G_\xi}{\partial \eta}+\frac{\partial G_\eta}{\partial \xi}\right]=0, \tag{II-10}$$

$$\left[\frac{1}{1-\xi^2}+\frac{1}{\eta^2-1}\right][SG_\eta+NG_\eta]-\frac{2\xi}{1-\xi^2}\left[\frac{\partial G_\eta}{\partial \xi}+\frac{\partial G_\xi}{\partial \eta}\right]+\frac{2\eta}{\eta^2-1}\left[\frac{\partial G_\eta}{\partial \eta}+\frac{\partial G_\xi}{\partial \xi}\right]=0, \tag{II-11}$$

where S and N are the operators

$$S\equiv(1-\xi^2)\frac{\partial^2}{\partial \xi^2}-\frac{m^2}{1-\xi^2}+\epsilon^2(1-\xi^2),$$

$$N\equiv(\eta^2-1)\frac{\partial^2}{\partial \eta^2}-\frac{m^2}{\eta^2-1}+\epsilon^2(\eta^2-1).$$

Equations (II-10) and (II-11) admit two pairs of solutions in which the variables are separable. The first pair is obtained by putting $G_\xi=\xi G$, $G_\eta=-\eta G$ and the second pair by putting $G_\xi=\eta G$, $G_\eta=-\xi G$. In both cases G satisfies the equation

$$\frac{\partial}{\partial \xi}\left[(1-\xi^2)\frac{\partial G}{\partial \xi}\right]-\frac{m^2}{1-\xi^2}G-\epsilon^2\xi^2G+\frac{\partial}{\partial \eta}\left[(\eta^2-1)\frac{\partial G}{\partial \eta}\right]-\frac{m^2}{\eta^2-1}G+\epsilon^2\eta^2G=0. \tag{II-12}$$

So, if we put $G=u(\xi)v(\eta)$, we are left with the two ordinary differential equations

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{du}{d\xi}\right]-\frac{m^2}{1-\xi^2}u+\alpha u+\epsilon^2(1-\xi^2)u=0, \quad -1\leq\xi\leq 1, \tag{II-13}$$

$$\frac{d}{d\eta}\left[(\eta^2-1)\frac{dv}{d\eta}\right]-\frac{m^2}{\eta^2-1}v-\alpha v+\epsilon^2(\eta^2-1)v=0, \quad 1<\eta<\infty, \tag{II-14}$$

which differ only in the range of the independent variable.

Finally, after G_ξ and G_η have been obtained, G_ϕ may be determined from (II-9).

Evidently the method fails when $m=0$. In this case, however, F_ξ, F_η, F_ϕ are not functions of ϕ, and (II-5) reduces to an equation in F_ϕ alone in which the variables are separable. If we put $F_\phi\equiv\{(1-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}u(\xi)v(\eta)$, the differential equations satisfied by u and v are found to be just (II-13) and (II-14) with $m=1$. The functions F_ξ and F_η corresponding to this solution for F_ϕ are both zero since the field component proportional to F_ϕ is itself solenoidal. A second solution for the case where $m=0$ is obtained by taking the curl of the field intensity corresponding to the first solution.

For a given m, solutions of (II-13) and (II-14) exist only for a discrete set of values of the constant of separation α, corresponding to the various harmonics possible. In fact, we conclude from the similarity of these differential equations to that for the associated Legendrian functions that α is a function of two positive integral indices l and m such that $m\leqslant l$. The solutions of (II-13) and (II-14) for a given α_{lm} we shall designate by $u_{lm}(\xi)$ and $v_{lm}(\eta)$. Evidently each group of functions corresponding to a specified m constitutes an orthogonal set. If, then, we put

$$\begin{Bmatrix}E_\xi\\H_\xi\end{Bmatrix}\equiv\Xi(\xi,\eta)\begin{Bmatrix}\sin m\phi\\\cos m\phi\end{Bmatrix}, \quad \begin{Bmatrix}E_\eta\\H_\eta\end{Bmatrix}\equiv\mathrm{H}(\xi,\eta)\begin{Bmatrix}\sin m\phi\\\cos m\phi\end{Bmatrix}, \quad \begin{Bmatrix}E_\phi\\H_\phi\end{Bmatrix}\equiv\Phi(\xi,\eta)\begin{Bmatrix}\cos m\phi\\-\sin m\phi\end{Bmatrix},$$

with omission of the time factor, we have the following pairs of solutions, distinguished by single and double accents: For $m=0$,

$$\left.\begin{aligned}
\Xi'_{l0}&=\frac{1}{(\eta^2-\xi^2)^{\frac{1}{2}}}u_{l1}(\xi)\frac{d}{d\eta}[(\eta^2-1)^{\frac{1}{2}}v_{l1}(\eta)],\\
\mathrm{H}'_{l0}&=-\frac{1}{(\eta^2-\xi^2)^{\frac{1}{2}}}\frac{d}{d\xi}[(1-\xi^2)^{\frac{1}{2}}u_{l1}(\xi)]v_{l1}(\eta),\\
\Phi'_{l0}&=0;
\end{aligned}\right\}\tag{II-15}$$

$$\left.\begin{aligned}
\Xi''_{l0}&=0,\\
\mathrm{H}''_{l0}&=0,\\
\Phi''_{l0}&=u_{l1}(\xi)v_{l1}(\eta);
\end{aligned}\right\}\tag{II-16}$$

for $m>0$

$$\left.\begin{aligned}
\Xi'_{lm}&=\frac{\xi}{\{(1-\xi^2)(\eta^2-\xi^2)\}^{\frac{1}{2}}}u_{lm}(\xi)v_{lm}(\eta),\\
\mathrm{H}'_{lm}&=-\frac{\eta}{\{(\eta^2-1)(\eta^2-\xi^2)\}^{\frac{1}{2}}}u_{lm}(\xi)v_{lm}(\eta),\\
\Phi'_{lm}&=\frac{\{(1-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}}{m(\eta^2-\xi^2)}\left[\xi\frac{du_{lm}}{d\xi}v_{lm}(\eta)-\eta u_{lm}(\xi)\frac{dv_{lm}}{d\eta}\right];
\end{aligned}\right\}\tag{II-17}$$

$$\left.\begin{aligned}
\Xi''_{lm}&=\frac{\eta}{\{(1-\xi^2)(\eta^2-\xi^2)\}^{\frac{1}{2}}}u_{lm}(\xi)v_{lm}(\eta),\\
\mathrm{H}''_{lm}&=-\frac{\xi}{\{(\eta^2-1)(\eta^2-\xi^2)\}^{\frac{1}{2}}}u_{lm}(\xi)v_{lm}(\eta),\\
\Phi''_{lm}&=\frac{\{(1-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}}{m(\eta^2-\xi^2)}\left[\eta\frac{du_{lm}}{d\xi}v_{lm}(\eta)-\xi u_{lm}(\xi)\frac{dv_{lm}}{d\eta}\right].
\end{aligned}\right\}\tag{II-18}$$

2. SOLUTION OF THE EQUATION FOR $u_{lm}(\xi)$

The form of Eq. (II-13) for $u_{lm}(\xi)$ suggests a solution in the associated Legendrian functions $P_{lm}(\xi)$. Hence we put

$$\alpha_{lm}=l(l+1)+\epsilon^2\alpha'_{lm}+\epsilon^4\alpha''_{lm}+\epsilon^6\alpha'''_{lm}+\cdots,$$
$$u_{lm}=P_{lm}(\xi)+\epsilon^2\sum_k a_k'P_{km}(\xi)+\epsilon^4\sum_k a_k''P_{km}(\xi)+\epsilon^6\sum_k a_k'''P_{km}(\xi)+\cdots,$$

substitute in the differential equation,[2] and equate to zero the coefficient of each separate power of ϵ^2 after getting rid of the factor $1-\xi^2$ in the last term by means of the recurrence formula

$$(1-\xi^2)P_{lm}=\Gamma(-l-2)\Gamma(-l-1)[P_{lm}-P_{l+2,m}]+\Gamma(l-1)\Gamma(l)[P_{lm}-P_{l-2,m}],$$

where $\Gamma(x)\equiv(x+m)/(2x+1)$.

In this way we find the following formulas for the first four terms following $l(l+1)$ in the series for the characteristic value (eigenvalue) α_{lm}:

$$\alpha'_{lm}=-2\frac{l(l+1)+m^2-1}{(2l-1)(2l+3)},\tag{II-19}$$

$$\alpha''_{lm}=\frac{\mathrm{H}(l-1)\mathrm{H}(l)}{2(2l-1)}-\frac{\mathrm{H}(l+1)\mathrm{H}(l+2)}{2(2l+3)},\tag{II-20}$$

[2] This is a much simpler and easier method of solution than that given in I.

TABLE I. Characteristic values (eigenvalues).

$$\alpha_{11}=2-\frac{4}{5}\epsilon^2-\frac{4}{5^3\cdot 7}\epsilon^4+\frac{8}{3\cdot 5^5\cdot 7}\epsilon^6-\frac{124}{5^6\cdot 7^3\cdot 11}\epsilon^8+\cdots=2-0.800{,}000\epsilon^2-0.004{,}571\epsilon^4+0.000{,}122\epsilon^6-0.000{,}002\epsilon^8+\cdots,$$

$$\alpha_{21}=6-\frac{4}{7}\epsilon^2-\frac{4}{3\cdot 7^3}\epsilon^4+\frac{8}{3\cdot 7^5\cdot 11}\epsilon^6+\frac{5420}{3^4\cdot 7^7\cdot 11\cdot 13}\epsilon^8+\cdots=6-0.571{,}429\epsilon^2-0.003{,}887\epsilon^4+0.000{,}014\epsilon^6+0.000{,}001\epsilon^8+\cdots,$$

$$\alpha_{31}=12-\frac{8}{3\cdot 5}\epsilon^2+\frac{152}{3^4\cdot 5^3\cdot 11}\epsilon^4-\frac{115{,}568}{3^7\cdot 5^5\cdot 11\cdot 13}\epsilon^6+\frac{34{,}094{,}936}{3^{10}\cdot 5^6\cdot 11^3\cdot 13}\epsilon^8+\cdots=12-0.533{,}333\epsilon^2+0.001{,}365\epsilon^4-0.000{,}118\epsilon^6+0.000{,}002\epsilon^8+\cdots,$$

$$\alpha_{41}=20-\frac{40}{7\cdot 11}\epsilon^2+\frac{7064}{7^3\cdot 11^3\cdot 13}\epsilon^4-\frac{462{,}736}{7^5\cdot 11^5\cdot 13}\epsilon^6+\cdots=20-0.519{,}481\epsilon^2+0.001{,}190\epsilon^4-0.000{,}013\epsilon^6+\cdots,$$

$$\alpha_{51}=30-\frac{20}{3\cdot 13}\epsilon^2+\frac{1108}{3^4\cdot 7\cdot 13^3}\epsilon^4-\frac{301{,}160}{3^7\cdot 7\cdot 13^5\cdot 17}\epsilon^6+\cdots=30-0.512{,}821\epsilon^2+0.000{,}889\epsilon^4-0.000{,}003\epsilon^6+\cdots,$$

$$\alpha_{22}=6-\frac{6}{7}\epsilon^2-\frac{2}{3\cdot 7^3}\epsilon^4+\frac{20}{3\cdot 7^5\cdot 11}\epsilon^6-\frac{5554}{3^4\cdot 7^7\cdot 11\cdot 13}\epsilon^8+\cdots=6-0.857{,}143\epsilon^2-0.001{,}944\epsilon^4+0.000{,}036\epsilon^6-0.000{,}001\epsilon^8+\cdots,$$

$$\alpha_{32}=12-\frac{2}{3}\epsilon^2-\frac{2}{3^4\cdot 11}\epsilon^4+\frac{4}{3^7\cdot 11\cdot 13}\epsilon^6+\frac{62}{3^{10}\cdot 11^3\cdot 13}\epsilon^8+\cdots=12-0.666{,}667\epsilon^2-0.002{,}245\epsilon^4+0.000{,}013\epsilon^6+0.000{,}000\epsilon^8+\cdots$$

$$\alpha'''_{lm}=-\frac{4m^2-1}{(2l-1)(2l+3)}\left[\frac{\mathrm{H}(l-1)\mathrm{H}(l)}{(2l-5)(2l-1)^2}-\frac{\mathrm{H}(l+1)\mathrm{H}(l+2)}{(2l+3)^2(2l+7)}\right], \tag{II-21}$$

$$\begin{aligned}\alpha^{IV}_{lm}=&\frac{\mathrm{H}(l-1)\mathrm{H}(l)}{2(2l-1)}\left[\frac{4(4m^2-1)^2}{(2l-5)^2(2l-1)^4(2l+3)^2}-\frac{\alpha''_{lm}}{2(2l-1)}+\frac{\mathrm{H}(l-3)\mathrm{H}(l-2)}{2\cdot 4(2l-1)(2l-3)}\right]\\&-\frac{\mathrm{H}(l+1)\mathrm{H}(l+2)}{2(2l+3)}\left[\frac{4(4m^2-1)^2}{(2l-1)^2(2l+3)^4(2l+7)^2}+\frac{\alpha''_{lm}}{2(2l+3)}+\frac{\mathrm{H}(l+3)\mathrm{H}(l+4)}{2\cdot 4(2l+3)(2l+5)}\right],\end{aligned} \tag{II-22}$$

where $\mathrm{H}(x)\equiv\Gamma(x)\Gamma(-x)=(x^2-m^2)/(4x^2-1)$.

In fact, we can obtain by this method an equation for the entire characteristic value α_{lm}. Putting

$$k_s\equiv(l+s)(l+s+1)-\epsilon^2[\Gamma(-l-s-2)\Gamma(-l-s-1)+\Gamma(l+s-1)\Gamma(l+s)]-\alpha_{lm},$$

and

$$x_s\equiv\frac{\mathrm{H}(l+s)\mathrm{H}(l+s+1)}{k_{s-1}k_{s+1}},$$

this equation, involving two continued fractions, is

$$\frac{\epsilon^2x_{-1}}{1-\dfrac{\epsilon^2x_{-3}}{1-\epsilon^2x_{-5}/1-}}+\frac{\epsilon^2x_1}{1-\dfrac{\epsilon^2x_3}{1-\epsilon^2x_5/1-}}=1. \tag{II-23}$$

However, the first five terms of α_{lm} are sufficient for most purposes, and they are more easily calculated from (II-19) to (II-22) than from (II-23), the apparently simple form of which is deceptive.

We list in Table I characteristic values for $m=1$ and $l=1, 2, 3, 4, 5$, and for $m=2$ and $l=2, 3$.

When the characteristic functions in $u_{lm}(\xi)$ are expanded as a series in the associated Legendrian functions $P_{lm}(\xi)$, the coefficients are found to be much more complicated than when they are expanded in a simple power series in $s\equiv(1-\xi^2)^{\frac{1}{2}}$. So the latter expansion has been adopted. We have to consider separately the cases $l-m$ even and $l-m$ odd. In both cases it is found convenient to put

$$\beta_{lm}\equiv\frac{\alpha_{lm}-l(l+1)}{\epsilon^2}.$$

TABLE II. Characteristic functions (eigenfunctions).

$$u_{11}(\xi)=3!\sum_{p=1}^{\infty}\frac{p}{(2p+1)!}\epsilon^{2(p-1)}A_p s^{2p-1},$$

$$p_{11}(\eta)=3!\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p}{(2p+1)!}\epsilon^{2(p-1)}A_p t^{2p-1};$$

$$f_{11}(\eta)=-\frac{12}{\alpha_{11}}(1+t^2)^{\frac{1}{2}}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p}{(2p+1)!}\epsilon^{2(p-1)}B_p t^{2p-3};$$

where

$$A_1\equiv 1,\quad A_2\equiv -\frac{5}{4}\beta_{11},\quad A_3\equiv 1+\frac{2}{3^2\cdot 5^2}\epsilon^2-\frac{677}{3^2\cdot 5^4\cdot 7^2\cdot 11}\epsilon^4+\cdots,\quad A_4\equiv 1+\frac{3}{5^2\cdot 11}\epsilon^2+\cdots,\quad A_5\equiv 1+\cdots;$$

$$B_1\equiv 1,\quad B_2\equiv 1+\frac{81}{5^2\cdot 7}\epsilon^2-\frac{7474}{3^4\cdot 5^4\cdot 7}\epsilon^4+\cdots,\quad B_3\equiv 1+\frac{1858}{3^4\cdot 5^2}\epsilon^2+\cdots,\quad B_4\equiv 1+\cdots$$

$$u_{21}(\xi)=\frac{5!}{2!}(1-s^2)^{\frac{1}{2}}\sum_{p=1}^{\infty}\frac{p(p+1)}{(2p+3)!}\epsilon^{2(p-1)}A_p s^{2p-1},$$

$$p_{21}(\eta)=\frac{5!}{2!}(1+t^2)^{\frac{1}{2}}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)}{(2p+3)!}\epsilon^{2(p-1)}A_p t^{2p-1},$$

$$f_{21}(\eta)=-\frac{12}{\alpha_{21}}\left[\frac{1}{3!}\frac{1}{t}+\frac{5!}{2\cdot 2!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)}{(2p+3)!}\epsilon^{2(p-1)}B_p t^{2p-1}\right];$$

where

$$A_1\equiv 1,\quad A_2\equiv -\frac{7}{4}\beta_{21},\quad A_3\equiv 1+\frac{6}{7^2\cdot 11}\epsilon^2+\cdots,\quad A_4\equiv 1+\cdots;$$

$$B_1\equiv 1-\frac{5}{2\cdot 3\cdot 7}\epsilon^2+\frac{125}{2\cdot 3^3\cdot 7^3}\epsilon^4+\cdots,\quad B_2\equiv 1-\frac{517}{2^2\cdot 3^3\cdot 7^2}\epsilon^2+\cdots,\quad B_3\equiv 1+\cdots.$$

$$u_{31}(\xi)=s-\frac{5\cdot 7!}{2^2\cdot 3!}\sum_{p=1}^{\infty}\frac{p(p+1)(p+2)}{(2p+5)!}\epsilon^{2(p-1)}A_p s^{2p+1},$$

$$p_{31}(\eta)=t+\frac{5\cdot 7!}{2^2\cdot 3!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)(p+2)}{(2p+5)!}\epsilon^{2(p-1)}A_p t^{2p+1},$$

$$f_{31}(\eta)=-\frac{12}{\alpha_{31}}(1+t^2)^{\frac{1}{2}}\left[\frac{1}{3!}\frac{1}{t}+\frac{5\cdot 7!}{2^2\cdot 3!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)(p+2)}{(2p+5)!}\epsilon^{2(p-1)}B_p t^{2p-1}\right];$$

where

$$A_1\equiv 1+\frac{1}{2\cdot 5}\epsilon^2\beta_{31},\quad A_2\equiv 1-\frac{166}{3^3\cdot 5^2\cdot 11}\epsilon^2+\frac{1172}{3^6\cdot 5^4\cdot 13}\epsilon^4+\cdots,\quad A_3\equiv 1-\frac{98}{3^3\cdot 5^2\cdot 13}\epsilon^2+\cdots,\quad A_4\equiv 1+\cdots;$$

$$B_1\equiv 1-\frac{41}{3^3\cdot 5^2}\epsilon^2+\frac{4109}{3^6\cdot 5^4\cdot 11}\epsilon^4+\cdots,\quad B_2\equiv 1-\frac{767}{2\cdot 3^3\cdot 5^2\cdot 11}\epsilon^2+\cdots,\quad B_3\equiv 1+\cdots.$$

For $l-m$ even we put $u_{lm}(\xi)=s^m z$, getting

$$(1-s^2)\frac{d^2z}{ds^2}+\frac{2m+1}{s}\frac{dz}{ds}-2(m+1)s\frac{dz}{ds}+(l-m)(l+m+1)z+\epsilon^2(\beta_{lm}+s^2)z=0. \qquad \text{(II-24)}$$

Assuming a solution $z=\sum_p c_p s^p$, the recurrence formula for the coefficients is

$$p(p+2m)c_p-\{(p-l+m-2)(p+l+m-1)-\epsilon^2\beta_{lm}\}c_{p-2}+\epsilon^2 c_{p-4}=0,$$

giving a series that starts with c_0.

TABLE II.—*Continued.*

$$u_{41}(\xi)=(1-s^2)^{\frac{1}{2}}\left[s-\frac{7\cdot 9!}{2^2\cdot 4!}\sum_{p=1}^{\infty}\frac{p(p+1)(p+2)(p+3)}{(2p+7)!}\epsilon^{2(p-1)}A_p s^{2p+1}\right],$$

$$p_{41}(\eta)=(1+t^2)^{\frac{1}{2}}\left[t+\frac{7\cdot 9!}{2^2\cdot 4!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)(p+2)(p+3)}{(2p+7)!}\epsilon^{2(p-1)}A_p t^{2p+1}\right],$$

$$f_{41}(\eta)=-\frac{12}{\alpha_{41}}\left[\frac{1}{3!}\left\{\frac{1}{t}+\frac{95}{2\cdot 3}B't\right\}+\frac{35\cdot 9!}{2^2\cdot 3\cdot 4!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)(p+2)(p+3)}{(2p+7)!}\epsilon^{2(p-1)}B_p t^{2p+1}\right];$$

where

$$A_1\equiv 1+\frac{1}{2\cdot 7}\epsilon^2\beta_{41},\quad A_2\equiv 1-\frac{1446}{7^2\cdot 11^2\cdot 13}\epsilon^2+\cdots,\quad A_3\equiv 1+\cdots;$$

$$B'\equiv 1-\frac{241}{5\cdot 7\cdot 11\cdot 19}\epsilon^2+\cdots,\quad B_1\equiv 1-\frac{253}{2\cdot 3\cdot 7^2\cdot 11}\epsilon^2+\cdots,\quad B_2\equiv 1+\cdots.$$

$$u_{51}(\xi)=s-\frac{7}{2}A's^3+\frac{21\cdot 11!}{2^3\cdot 5!}\sum_{p=1}^{\infty}\frac{p(p+1)\cdots(p+4)}{(2p+9)!}\epsilon^{2(p-1)}A_p s^{2p+3},$$

$$p_{51}(\eta)=t+\frac{7}{2}A't^3+\frac{21\cdot 11!}{2^3\cdot 5!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)\cdots(p+4)}{(2p+9)!}\epsilon^{2(p-1)}A_p t^{2p+3},$$

$$f_{51}(\eta)=-\frac{12}{\alpha_{51}}(1+t^2)^{\frac{1}{2}}\left[\frac{1}{3!}\left\{\frac{1}{t}+\frac{105}{2^2}B't\right\}+\frac{105\cdot 11!}{2^4\cdot 5!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)\cdots(p+4)}{(2p+9)!}\epsilon^{2(p-1)}B_p t^{2p+1}\right];$$

where

$$A'\equiv 1+\frac{1}{2^2\cdot 7}\epsilon^2\beta_{51},\quad A_1\equiv 1-\frac{22}{3^3\cdot 13}\epsilon^2+\frac{47{,}321}{3^6\cdot 7^2\cdot 13^3}\epsilon^4+\cdots,\quad A_2\equiv 1-\frac{142}{3^3\cdot 13^2}\epsilon^2+\cdots,\quad A_3\equiv 1+\cdots;$$

$$B'\equiv 1-\frac{268}{3^3\cdot 5\cdot 7\cdot 13}\epsilon^2+\cdots,\quad B_1\equiv 1-\frac{19}{3^3\cdot 13}\epsilon^2+\cdots,\quad B_2\equiv 1+\cdots.$$

$$u_{22}(\xi)=\frac{5!}{2!}\sum_{p=1}^{\infty}\frac{p(p+1)}{(2p+3)!}\epsilon^{2(p-1)}A_p s^{2p},$$

$$p_{22}(\eta)=\frac{5!}{2!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)}{(2p+3)!}\epsilon^{2(p-1)}A_p t^{2p},$$

$$f_{22}(\eta)=4(1+t^2)^{\frac{1}{2}}\left[\frac{1}{3!}B'-\frac{1}{t^2}-\frac{5!}{2^2\cdot 2!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)}{(2p+3)!}\epsilon^{2(p-1)}B_p t^{2p-2}\right.$$

where

$$A_1\equiv 1,\quad A_2\equiv -\frac{7}{6}\beta_{22},\quad A_3\equiv 1+\cdots;$$

$$B'\equiv 1+\frac{4}{3\cdot 7}\epsilon^2+\frac{20}{3^2\cdot 7^3}\epsilon^4+\cdots,\quad B_1\equiv 1+\frac{1}{3\cdot 7}\epsilon^2-\frac{65}{3^2\cdot 7^3}\epsilon^4+\cdots,\quad B_2\equiv 1+\frac{55}{2\cdot 3^3\cdot 7^2}\epsilon^2+\cdots,\quad B_3\equiv 1+\cdots.$$

For $l-m$ odd we put $u_{lm}(\xi)=\xi s^m Z$, getting

$$(1-s^2)\frac{d^2Z}{ds^2}+\frac{2m+1}{s}\frac{dZ}{ds}-2(m+2)s\frac{dZ}{ds}+(l-m-1)(l+m+2)Z+\epsilon^2(\beta_{lm}+s^2)Z=0, \qquad \text{(II-25)}$$

which gives, when we put $Z=\sum_p c_p s^p$, the recurrence formula

$$p(p+2m)c_p-\{(p-l+m-1)(p+l+m)-\epsilon^2\beta_{lm}\}c_{p-2}+\epsilon^2 c_{p-4}=0,$$

representing a series that starts with c_0.

TABLE II.—*Concluded.*

$$u_{32}(\xi)=\frac{7!}{3!}(1-s^2)^{\frac{1}{2}}\sum_{p=1}^{\infty}\frac{p(p+1)(p+2)}{(2p+5)!}\epsilon^{2(p-1)}A_p s^{2p},$$

$$p_{32}(\eta)=\frac{7!}{3!}(1+t^2)^{\frac{1}{2}}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)(p+2)}{(2p+5)!}\epsilon^{2(p-1)}A_p t^{2p},$$

$$f_{32}(\eta)=\frac{4}{5}\left[\frac{1}{3!}\left\{B'\frac{1}{t^2}-\frac{5}{2}B''\right\}-\frac{5\cdot 7!}{2^2\cdot 3!}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{p(p+1)(p+2)}{(2p+5)!}\epsilon^{2(p-1)}B_p t^{2p}\right];$$

where

$$A_1\equiv 1,\quad A_2\equiv -\tfrac{3}{2}\beta_{32},\quad A_3\equiv 1+\cdots;$$

$$B'\equiv 1+\frac{4}{3^2\cdot 5}\epsilon^2+\frac{28}{3^5\cdot 5^2}\epsilon^4+\cdots,\qquad B''\equiv 1+\frac{1}{3^2\cdot 5}\epsilon^2-\frac{103}{3^5\cdot 5^2\cdot 11}\epsilon^4+\cdots,$$

$$B_1\equiv 1-\frac{1}{2\cdot 3^3}\epsilon^2-\frac{37}{2^2\cdot 3^6\cdot 5\cdot 11}\epsilon^4+\cdots,\quad B_2\equiv 1-\frac{5}{2^2\cdot 3^3\cdot 11}\epsilon^2+\cdots,\quad B_3\equiv 1+\cdots$$

The characteristic functions $u_{lm}(\xi)$ obtained from (II-24) and (II-25) are listed in Table II together with the functions $v_{lm}(\eta)$, the arbitrary coefficient c_0 in the tabulated functions $u_{lm}(\xi)$ being taken as unity.

3. SOLUTION OF THE EQUATION FOR $v_{lm}(\eta)$

Since we are interested primarily in conducting spheroids of eccentricity very close to unity (antenna), the objective in solving Eq. (II-14) for $v_{lm}(\eta)$ is to obtain series solutions representing diverging waves at $\eta=\infty$ which converge rapidly in the neighborhood of resonance for η nearly equal to unity.

By changing the independent variable from η to $p=\epsilon\eta$ the differential equation can be thrown into such a form that the zero-order terms are satisfied by the wave functions

$$S_l=(-1)^l p^l\left[\frac{1}{p}\frac{d}{dp}\right]^l e^{ip}.$$

Therefore, a solution can be constructed in the form of a series of these functions, which meets the first requirement stated above. Unfortunately, this series converges so slowly in the neighborhood of resonance for η nearly equal to unity as to be quite useless for the antenna problem.

Another possibility is to construct two independent solutions in series of associated Legendrian functions of the first and second kinds, respectively. The coefficients, however, turn out to be complicated, and difficult to calculate in the second case, and there remains the problem of finding what linear combination of the two independent primitives represents a diverging wave at infinity.

Finally, it was concluded that the most satisfactory method was that employed in I. First, we obtain two independent primitives in the independent variable[3] $t\equiv(\eta^2-1)^{\frac{1}{2}}$ which goes to zero as η goes to unity, and then determine the coefficients of the desired linear combination by comparison with the solution in $\rho=\epsilon t$ which represents a diverging wave at infinity. As in the case of the equation for $u_{lm}(\xi)$ we have to consider separately the cases $l-m$ even and $l-m$ odd.

For $l-m$ even we put $v_{lm}(\eta)=t^m y$, getting

$$(1+t^2)\frac{d^2y}{dt^2}+\frac{2m+1}{t}\frac{dy}{dt}+2(m+1)t\frac{dy}{dt}-(l-m)(l+m+1)y-\epsilon^2(\beta_{lm}-t^2)y=0. \qquad \text{(II-26)}$$

The first primitive for small t is the power series $y_1=\sum_p a_p t^p$ where

$$p(p+2m)a_p+\{(p-l+m-2)(p+l+m-1)-\epsilon^2\beta_{lm}\}a_{p-2}+\epsilon^2 a_{p-4}=0.$$

[3] Note that t used in this paper is the square root of the t in I.

A second primitive is obtained by putting

$$y_2=\tfrac{1}{2}y_1\log\frac{(1+t^2)^{\frac{1}{2}}+1}{(1+t^2)^{\frac{1}{2}}-1}+(1+t^2)^{\frac{1}{2}}x,$$

where x satisfies the equation

$$(1+t^2)\frac{d^2x}{dt^2}+\frac{2m+1}{t}\frac{dx}{dt}+2(m+2)t\frac{dx}{dt}-(l-m-1)(l+m+2)x-\epsilon^2(\beta_{lm}-t^2)x=2\left[\frac{1}{t}\frac{dy_1}{dt}+\frac{m}{t^2}y_1\right]. \quad \text{(II-27)}$$

For $l-m$ odd we put $v_{lm}(\eta)=\eta t^m Y$, getting

$$(1+t^2)\frac{d^2Y}{dt^2}+\frac{2m+1}{t}\frac{dY}{dt}+2(m+2)t\frac{dY}{dt}-(l-m-1)(l+m+2)Y-\epsilon^2(\beta_{lm}-t^2)Y=0. \quad \text{(II-28)}$$

The power series $Y_1=\sum_p a_pt^p$ supplies the first primitive for small t, where

$$p(p+2m)a_p+\{(p-l+m-1)(p+l+m)-\epsilon^2\beta_{lm}\}a_{p-2}+\epsilon^2a_{p-4}=0,$$

and the second primitive is

$$Y_2=\tfrac{1}{2}Y_1\log\frac{(1+t^2)^{\frac{1}{2}}+1}{(1+t^2)^{\frac{1}{2}}-1}+\frac{1}{(1+t^2)^{\frac{1}{2}}}X,$$

where X satisfies the equation

$$(1+t^2)\frac{d^2X}{dt^2}+\frac{2m+1}{t}\frac{dX}{dt}+2(m+1)t\frac{dX}{dt}-(l-m)(l+m+1)X-\epsilon^2(\beta_{lm}-t^2)X$$
$$=2\left[\left(t+\frac{1}{t}\right)\frac{dY_1}{dt}+\left(m+1+\frac{m}{t^2}\right)Y_1\right]. \quad \text{(II-29)}$$

To obtain the solution for large t for the case $l-m$ even we change the independent variable in (II-26) to $\rho\equiv\epsilon t$ and put

$$y=\rho^{-(m+1)}\Psi(\rho)e^{i\rho},$$

getting

$$z^2\frac{d^2\Psi}{dz^2}+2z^2\frac{d\Psi}{dz}-\alpha_{lm}\Psi=\epsilon^2\left[\frac{d^2\Psi}{dz^2}+\left(2-\frac{1}{z}\right)\frac{d\Psi}{dz}+\left(1-\frac{1}{z}-\frac{m^2-1}{z^2}\right)\Psi\right], \quad \text{(II-30)}$$

where $z\equiv i\rho$. The solution is of the form

$$\Psi=\sum_{p=0}^{\infty}c_pz^{-p}$$

where

$$2pc_p=\{(p+l)(p-l-1)-\epsilon^2(\beta_{lm}+1)\}c_{p-1}+\epsilon^2(2p-3)c_{p-2}-\epsilon^2\{(p-2)^2-m^2\}c_{p-3}.$$

After calculating the coefficients c_p we write the solution in the form

$$y=\frac{1}{\epsilon}\sum_{n}^{\text{even}}(-\epsilon^2)^{n/2}C_nt^{-m+n-1}+i\sum_{n}^{\text{odd}}(-\epsilon^2)^{(n-1)/2}C_nt^{-m+n-1}$$

for comparison with the primitives for t small, where

$$C_n=\frac{1}{n!}+\frac{c_1}{(n+1)!}+\frac{c_2}{(n+2)!}+\cdots,$$

$$C_{-n}=c_n+\frac{c_{n+1}}{1!}+\frac{c_{n+2}}{2!}+\cdots.$$

For $l-m$ odd we make the same change in independent variable in (II-28) and put

$$Y=\rho^{-(m+2)}\Psi(\rho)e^{i\rho},$$

getting

$$z^2\frac{d^2\Psi}{dz^2}+2z^2\frac{d\Psi}{dz}-\alpha_{lm}\Psi=\epsilon^2\left[\frac{d^2\Psi}{dz^2}+\left(2-\frac{3}{z}\right)\frac{d\Psi}{dz}+\left(1-\frac{3}{z}-\frac{m^2-4}{z^2}\right)\Psi\right]. \tag{II-31}$$

The recurrence formula for the coefficients in the solution

$$\Psi=\sum_{p=0}^{\infty}c_pz^{-p}$$

is

$$2pc_p=\{(p+l)(p-l-1)-\epsilon^2(\beta_{lm}+1)\}c_{p-1}+\epsilon^2(2p-1)c_{p-2}-\epsilon^2\{(p-1)^2-m^2\}c_{p-3},$$

and Y may be written in the form

$$Y=\frac{1}{\epsilon}\sum_n^{\text{even}}(-\epsilon^2)^{n/2}C_nt^{-m+n-2}+i\sum_n^{\text{odd}}(-\epsilon^2)^{(n-1)/2}C_nt^{-m+n-2},$$

where C_n has the same significance as before.

In calculating the coefficients a_p of y_1 or Y_1 the manner in which successive coefficients become of higher and higher order in ϵ^2 provides a perfect check on the arithmetic as well as on the previous calculation of α_{lm}. The same is true of the calculation of the coefficients c_p in the solution for large t, although here we have an additional check in the vanishing of those C_n's corresponding to negative powers of t in y_1. In every case at least one such vanishing coefficient was calculated as an arithmetical check. The arbitrary coefficient in the solution for large t was chosen in each instance so as to make $v_{lm}(\eta)$ equal to $(1/\rho)e^{i\rho}$ at infinity, whereas that in the solution for small t was determined by making a_0 equal to unity.

We shall designate by $p_{lm}(\eta)$ and $q_{lm}(\eta)$ the two independent solutions of the equation for $v_{lm}(\eta)$ for small t corresponding, in the case $l-m$ even, to y_1 and y_2, respectively, and, in the case $l-m$ odd, to Y_1 and Y_2, respectively. The function $p_{lm}(\eta)$ remains finite at $t=0$, while $q_{lm}(\eta)$ becomes infinite. For all values of the indices the latter has the form

$$q_{lm}(\eta)=\tfrac{1}{2}p_{lm}(\eta)\log\frac{(1+t^2)^{\frac{1}{2}}+1}{(1+t^2)^{\frac{1}{2}}-1}+f_{lm}(\eta).$$

Hence, in tabulating these functions it is sufficient to give $p_{lm}(\eta)$ and $f_{lm}(\eta)$. In Table II the three functions $u_{lm}(\xi)$, $p_{lm}(\eta)$, and $f_{lm}(\eta)$ are given for those values of the indices for which the characteristic values are listed in Table I. They are expressed in such a form that the first term in each of the series A_p or B_p which has been calculated is unity.

Finally we designate the function $v_{lm}(\eta)$ representing the diverging wave $(1/\rho)e^{i\rho}$ at infinity by $r_{lm}(\eta)$ and write

$$r_{lm}(\eta)=\frac{(i)^{l+1}}{2\cdot4\cdot6\cdots2m}\left[\frac{(l+m)!\epsilon^l}{(2l+1)\{1^2\cdot3^2\cdot5^2\cdots(2l-1)^2\}}a_{lm}p_{lm}(\eta)\right.$$
$$\left.+(-1)^{m+1}i\frac{(2l+1)\{1^2\cdot3^2\cdot5^2\cdots(2l-1)^2\}}{(l-m)!\epsilon^{l+1}}b_{lm}q_{lm}(\eta)\right]. \tag{II-32}$$

The coefficients a_{lm} and b_{lm} are listed in Table III. Evidently the function representing the converging wave $(1/\rho)e^{-i\rho}$ at infinity is just the complex conjugate of $r_{lm}(\eta)$.

It will be observed that, for $m=1$, the equation

$$a_{l1}b_{l1}=\alpha_{l1}/l(l+1)$$

holds for all five cases. This relation provides a valuable check on the arithmetical computations. Also, for $m=2$, we have a check in the relation

$$a_{l2}b_{l2}=1/B',$$

although here each coefficient was calculated by two independent methods to preclude the possibility of arithmetical error.

4. ZEROS OF CERTAIN CHARACTERISTIC VALUES (EIGENVALUES)

The zeros of the coefficients b_{l1} are very important in the applications of the theory to the antenna problem. The series for b_{11} and for the characteristic value α_{11} have been carried far enough to show that they have a common zero $\epsilon=\pi/2$ accurate to five significant figures. To a lesser degree of accuracy the given series indicates that b_{21} and α_{21} have a common zero $\epsilon=\pi$ and b_{31} and α_{31} a common zero $\epsilon=3\pi/2$. We shall now prove rigorously that b_{l1} and α_{l1} have a common zero and shall find its exact value.

If we make $m=1$ in Eq. (II-14) for v and put $v=z/(\eta^2-1)^{\frac{1}{2}}$, the resulting equation for z is

$$(\eta^2-1)\left[\frac{d^2z}{d\eta^2}+\epsilon^2 z\right]-\alpha z=0.$$

If ϵ_0 is a zero of α, then, the complete solution of (II-14) for this value of the parameter is

$$v=\frac{A}{(\eta^2-1)^{\frac{1}{2}}}\exp(i\epsilon_0\eta)+\frac{B}{(\eta^2-1)^{\frac{1}{2}}}\exp(-i\epsilon_0\eta), \tag{II-33}$$

and the solution representing a diverging wave of amplitude $1/\rho$ at infinity, where $\rho\equiv\epsilon(\eta^2-1)^{\frac{1}{2}}$, is

$$v_{l1}(\eta)=\frac{1}{\rho(\epsilon_0)}\exp(i\epsilon_0\eta), \tag{II-34}$$

exact for all values of η. This function, therefore, must be identical with the function $r_{l1}(\eta)$ given by (II-32) for $\epsilon=\epsilon_0$, for all values of the index l. Consequently the coefficient of the logarithmic term in $q_{l1}(\eta)$ must vanish for $\epsilon=\epsilon_0$, that is,

$$b_{l1}(\epsilon_0)=0,$$

proving that α_{l1} and b_{l1} have a common zero.

Furthermore, if we replace η by $(1+t^2)^{\frac{1}{2}}$ in (II-34) and use the power series expansion for the exponential factor, we find, on separating real and imaginary parts,

$$\begin{aligned} v_{l1}(\eta)=&\frac{1}{\epsilon_0 t}\cos\epsilon_0-\tfrac{1}{2}t\sin\epsilon_0+\tfrac{1}{8}t^3\{\sin\epsilon_0-\epsilon_0\cos\epsilon_0\}+\cdots \\ &+i(1+t^2)^{\frac{1}{2}}\left[\frac{1}{\epsilon_0 t}\sin\epsilon_0-\frac{1}{2}\frac{t}{\epsilon_0}\{\sin\epsilon_0-\epsilon_0\cos\epsilon_0\}+\frac{1}{8}\frac{t^3}{\epsilon_0}\{(3-\epsilon_0^2)\sin\epsilon_0-3\epsilon_0\cos\epsilon_0\}+\cdots\right]. \end{aligned} \tag{II-35}$$

Now the function $p_{l1}(\eta)$ contains no term in negative powers of t. Therefore, comparing with (II-32), we find that when l is odd, $\cos\epsilon_0=0$ and consequently ϵ_0 is equal to $(n+\frac{1}{2})\pi$, whereas, when l is even, $\sin\epsilon_0=0$ and consequently $\epsilon_0=n\pi$. The value of the integer n is easily ascertained from the series for the characteristic values given in Table I. We find, for l odd or even, that $\epsilon_0=l\pi/2$.

When l is odd the part of $r_{l1}(\eta)$ containing $p_{l1}(\eta)$ is real, and the first term in the power series for this function is t. Hence, comparing (II-32) with (II-35), we find that

$$a_{l1}(\epsilon_0)=\frac{(2l+1)\{1^2\cdot 3^2\cdot 5^2\cdots(2l-1)^2\}}{(l+1)!\epsilon_0^{\,l}}. \tag{II-36}$$

TABLE III. Coefficients of diverging wave functions.

$$a_{11}=1-\frac{1}{2\cdot 5^2}\epsilon^2+\frac{187}{2^3\cdot 5^4\cdot 7^2}\epsilon^4-\frac{26{,}021}{2^4\cdot 3^4\cdot 5^6\cdot 7^2}\epsilon^6+\cdots=1-0.020{,}000\epsilon^2+0.000{,}763\epsilon^4-0.000{,}026\epsilon^6+\cdots,$$

$$b_{11}=1-\frac{19}{2\cdot 5^2}\epsilon^2-\frac{2609}{2^3\cdot 5^4\cdot 7^2}\epsilon^4+\frac{32{,}593}{2^4\cdot 3^4\cdot 5^5\cdot 7^2}\epsilon^6+\cdots=1-0.380{,}000\epsilon^2-0.010{,}649\epsilon^4+0.000{,}164\epsilon^6+\cdots;$$

$$a_{21}=1-\frac{3}{2\cdot 7^2}\epsilon^2+\frac{389}{2^3\cdot 3^3\cdot 7^4}\epsilon^4+\cdots=1-0.030{,}612\epsilon^2+0.000{,}750\epsilon^4+\cdots,$$

$$b_{21}=1-\frac{19}{2\cdot 3\cdot 7^2}\epsilon^2-\frac{1751}{2^3\cdot 3^3\cdot 7^4}\epsilon^4+\cdots=1-0.064{,}626\epsilon^2-0.003{,}376\epsilon^4+\cdots;$$

$$a_{31}=1-\frac{23}{2\cdot 3^3\cdot 5^2}\epsilon^2-\frac{113{,}549}{2^3\cdot 3^6\cdot 5^4\cdot 11^2}\epsilon^4+\cdots=1-0.017{,}037\epsilon^2-0.000{,}257\epsilon^4+\cdots,$$

$$b_{31}=1-\frac{37}{2\cdot 3^3\cdot 5^2}\epsilon^2-\frac{42{,}233}{2^3\cdot 3^6\cdot 5^4\cdot 11^2}\epsilon^4+\cdots=1-0.027{,}407\epsilon^2-0.000{,}096\epsilon^4+\cdots;$$

$$a_{41}=1-\frac{127}{2\cdot 7^2\cdot 11^2}\epsilon^2+\cdots=1-0.010{,}710\epsilon^2+\cdots,\quad b_{41}=1-\frac{181}{2\cdot 7^2\cdot 11^2}\epsilon^2+\cdots=1-0.015{,}264\epsilon^2+\cdots;$$

$$a_{51}=1-\frac{67}{2\cdot 3^3\cdot 13^2}\epsilon^2+\cdots=1-0.007{,}342\epsilon^2+\cdots,\quad b_{51}=1-\frac{89}{2\cdot 3^3\cdot 13^2}\epsilon^2+\cdots=1-0.009{,}752\epsilon^2+\cdots;$$

$$a_{22}=1-\frac{1}{2\cdot 7^2}\epsilon^2+\frac{121}{2^3\cdot 3^3\cdot 7^4}\epsilon^4+\cdots=1-0.010{,}204\epsilon^2+0.000{,}233\epsilon^4+\cdots,$$

$$b_{22}=1-\frac{53}{2\cdot 3\cdot 7^2}\epsilon^2+\frac{14{,}381}{2^3\cdot 3^3\cdot 7^4}\epsilon^4+\cdots=1-0.180{,}272\epsilon^2+0.027{,}730\epsilon^4+\cdots;$$

$$a_{32}=1-\frac{1}{2\cdot 3^3}\epsilon^2+\frac{223}{2^3\cdot 3^6\cdot 11^2}\epsilon^4+\cdots=1-0.018{,}519\epsilon^2+0.000{,}316\epsilon^4+\cdots,$$

$$b_{32}=1-\frac{19}{2\cdot 3^3\cdot 5}\epsilon^2+\frac{5903}{2^3\cdot 3^6\cdot 5\cdot 11^2}\epsilon^4+\cdots=1-0.070{,}370\epsilon^2+0.001{,}673\epsilon^4+\cdots$$

When l is even, the part of $r_{l1}(\eta)$ containing $p_{l1}(\eta)$ is imaginary, and the first term in the series for this function is $(1+t^2)^{\frac{1}{2}}t$. Therefore, comparing again with (II-35), we find that (II-36) holds for l even as well as l odd.

When l is odd, it is evident from Table II that the first term in $f_{l1}(\eta)$ is $-\{2(1+t^2)^{\frac{1}{2}}\}/\{\alpha_{l1}t\}$. Hence

$$\frac{b_{l1}(\epsilon_0)}{\alpha_{l1}(\epsilon_0)}=\frac{(l-1)!\epsilon_0{}^l}{(2l+1)\{1^2\cdot 3^2\cdot 5^2\cdots(2l-1)^2\}}. \tag{II-37}$$

On the other hand, when l is even, the first term in $f_{l1}(\eta)$ is $-2/\{\alpha_{l1}t\}$. Therefore, (II-37) holds for l even as well as for l odd.

Similarly we can find exact values of the other coefficients in $p_{l1}(\eta)$ and $f_{l1}(\eta)$ for $\epsilon=\epsilon_0$. Thus we have a criterion of the rapidity with which the series given in the tables converge in the neighborhood of ϵ_0. This is illustrated in Table IV, where the exact values and the series values of some of the coefficients are given for $\epsilon=\epsilon_0$. The figures in parentheses indicate the number of terms used in the series. Of course the agreement becomes worse the larger l, partly because ϵ_0 becomes larger and partly because fewer terms of the series have been calculated. In fact, the functions for $l>1$ were not computed

TABLE IV. Coefficients at zero of α_{l1}.

l	ϵ_0	Coefficient	Exact value	Series value	
1	$\pi/2$	a_{11}	$0.95493\cdots$	0.95491	(4)
		A_2	$1.013212\cdots$	1.013210	(4)
		b_{11}/α_{11}	$0.5236\cdots$	0.5241	(4)
		B_2	$2.026\cdots$	2.014	(3)
		B_3	$3.06\cdots$	3.26	(2)
2	π	a_{21}	$0.76\cdots$	0.77	(3)
		A_2	$1.0639\cdots$	1.0637	(4)
		b_{21}/α_{21}	$0.219\cdots$	0.220	(3)
		B_1	0	0.5	(3)
3	$3\pi/2$	a_{31}	$0.63\cdots$	0.71	(4)
		A_1	-0.200	-0.195	(4)
		b_{31}/α_{31}	$0.133\cdots$	0.137	(3)
		B_1	$-0.07\cdots$	0.05	(3)

primarily for evaluation at the zeros of their characteristic values, but rather as correction terms in the neighborhood of the important first resonance ($\epsilon=\pi/2$) for $t>0$. Here the convergence is entirely satisfactory. Therefore, the coefficients for which it is important that the series should give accurate values in the neighborhood of ϵ_0 are those for which $l=1$. In judging these, it must be remembered that the terms in $f_{11}(\eta)$ involving B_2 and B_3, as compared with that in B_1, have numerical coefficients 1/10 and 1/280, respectively. Hence, for small t, the accuracy is not less than that of the series for b_{11}/α_{11}, the first four terms of which show an error of only one part in one thousand.

Turning now to Eq. (II-13) for u for the case $m=1$, we note that, if we put $u=z/(1-\xi^2)^{\frac{1}{2}}$, we get for z the equation

$$(1-\xi^2)\left[\frac{d^2z}{d\xi^2}+\epsilon^2 z\right]+\alpha z=0,$$

which yields a solution, when $\alpha=0$, of the same form as that previously obtained in η. Hence we conclude, when l is odd, that

$$[u_{l1}(\xi)]_{\epsilon=\epsilon_0}=(-1)^{(l-1)/2}\frac{2(2l+1)\{1^2\cdot3^2\cdot5^2\cdots(2l-1)^2\}}{(l+1)!\epsilon_0^{l+1}a_{l1}(\epsilon_0)}\frac{\cos\epsilon_0\xi}{(1-\xi^2)^{\frac{1}{2}}},$$

and, when l is even, that

$$[u_{l1}(\xi)]_{\epsilon=\epsilon_0}=(-1)^{(l-2)/2}\frac{2(2l+1)\{1^2\cdot3^2\cdot5^2\cdots(2l-1)^2\}}{(l+1)!\epsilon_0^{l+1}a_{l1}(\epsilon_0)}\frac{\sin\epsilon_0\xi}{(1-\xi^2)^{\frac{1}{2}}}.$$

Using the value of $a_{l1}(\epsilon_0)$ specified by (II-36) these reduce to

$$[u_{l1}(\xi)]_{\epsilon=\epsilon_0}=(-1)^{(l-1)/2}\frac{2}{\epsilon_0}\frac{\cos\epsilon_0\xi}{(1-\xi^2)^{\frac{1}{2}}} \tag{II-38}$$

for l odd, and

$$[u_{l1}(\xi)]_{\epsilon=\epsilon_0}=(-1)^{(l-2)/2}\frac{2}{\epsilon_0}\frac{\sin\epsilon_0\xi}{(1-\xi^2)^{\frac{1}{2}}} \tag{II-39}$$

for l even.

The method followed in this section does not succeed when $m>1$.