and σ is smaller than a few percent. The factor σ increases with increasing p. For $p \gg 1/\theta_s^2$, the integral

$$\int_{E_{\psi_0}'}^{E_m'} \theta dE' \rightarrow \frac{\pi}{2} p^{\frac{1}{2}},$$

hence,

$$\sigma < \frac{1}{\nu} \frac{BS}{2\pi D} \frac{\pi}{12E_{cr}} p^{\frac{1}{2}}. \quad (28)$$

Since the mesotron momenta to be considered in this case are very high, ν can be replaced by ν_0. Now, $BS/\nu_0 2\pi D$ is approximately equal to 1. Therefore

$$\sigma < (\pi/12E_{cr}) p^{\frac{1}{2}}. \quad (29)$$

It can be seen from (29) that σ will still remain smaller than 1 even for $p = (12E_{cr}/\pi)^2$, that is, for mesotron momenta as high as 1.5×10^{11} ev/c.

It will be sufficient to remember that the frequency of the mesotron momentum in the differential mesotron spectrum decreases as $p^{-\gamma} (\gamma > 2)$ to make clear that the shower effect of the collision electrons, when averaged adequately over the mesotron spectrum, adds a very small contribution only to the frequency of discharges in the counter tube (C).

G. ALTITUDE EFFECT

Concerning ν, the only terms which in (20) or (26) depend upon the altitude, that is, upon the density of air, are B, θ_{max}, and θ_{min}. The variation of θ_{min}, which is anyway very small, can be neglected. The coefficient B is directly proportional to the density of air, and θ_{max} depends upon this density only through the coefficient k. But, as can be seen from Figs. 1B or 2, θ_{max} depends, at least for small distances D, very little upon k. For example, for $E_1 = 4$ Mev and $D = 2$ m, θ_{max} would increase from ~ 0.40 at sea level ($k = 0.25$), to ~ 0.45 only at the top of the atmosphere ($k = 0$). Therefore, in first approximation, one can admit that ν is proportional to the density of air or to the atmospheric pressure. On the other hand, the effect of the altitude on the frequency n of discharges [see (21)] in the counter tube (C) follows, in the general case, the combined effects of: (a) the altitude increase of the intensity of the mesotron radiation, (b) the altitude dependence of the mesotron spectrum and, (c) the decrease of ν which, as has been seen above, is roughly proportional to the atmospheric pressure.

PHYSICAL REVIEW VOLUME 65, NUMBERS 11 AND 12 JUNE 1 AND 15, 1944

On the Perturbation of Boundary Conditions

HERMAN FESHBACH
George Eastman Research Laboratories, Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received February 25, 1944)

The solution of both the scalar and vector wave equations in regions which are bounded by irregular surfaces which have non-uniform physical properties has been reduced to the solution of a secular equation. The secular determinant is Hermitian. The solution to the secular equation has been expressed in a form suitable for obtaining its value to any approximation. Similar results are given for the corresponding eigenfunctions. Extension of these results to the problem of scattering and to the situation where the bounding surfaces move is indicated. The description of a source located in such a region is also discussed.

I. INTRODUCTION

THE solution to many physical problems can be reduced to the solution of a system of partial differential equations with prescribed boundary conditions. In the past, exact solutions were limited to those cases in which the boundary conditions were "simple" and were satisfied on "simple" surfaces. "Simple" boundary conditions correspond to uniform physical properties of the surface involved. "Simple" surfaces are coordinate surfaces of coordinate systems in which the partial differential equations will separate. As a result, a great many physical problems of interest have not been treated theoretically.

This paper will discuss solutions of the scalar and vector wave equations satisfying non-simple boundary conditions on non-simple surfaces. The scalar wave equation has been investigated previously by means of a perturbation technique in which results were sought for conditions in which the variation from simple boundaries and boundary conditions were small. First-order corrections to the unperturbed eigenvalue have been found by Froelich,[1] Brillouin,[2] and Cabrera.[3] First-order corrections to the wave functions, and second-order corrections to the eigenvalue in some cases, have been given in I.[4] Results have been given in the latter both for the case of discrete and continuous eigenvalues. Applications of I have been made to the acoustics of irregularly-shaped rooms yielding good agreement with experiment.[5]

The methods used here are not a continuation or extension of those used in I. Rather, a new method based on the use of Green's functions is employed. Formulae are given permitting the calculation not only of improved first-order results, but also of all the higher orders. The problem of discrete eigenvalues is reduced to the solution of a secular determinant.

The scalar and vector wave equations can be used to describe phenomena occurring in the fields of acoustics, elasticity, and electromagnetic theory. For example, our results can be used to treat the acoustics of irregularly-shaped rooms, scattering from irregularly-shaped objects, propagation down irregularly-shaped pipes. Moreover, simple boundary conditions need not be obeyed. For example, the walls of an irregularly-shaped room might be absorbing.

The same technique can be applied to the determination of source functions satisfying certain prescribed boundary conditions. These functions can be used in the solution of inhomogeneous wave equations. Application can be made, for example, to the calculation of the sound field due to a source placed in an irregularly-shaped room.

II. STATEMENT OF PROBLEM

The equations to be discussed can be grouped according to type of boundary condition. The methods used in their solution vary in the same way although the general basic idea remains the same. In all cases, the equation is to be satisfied in a region R bounded by a surface S. The boundary conditions are to be satisfied on S.

The scalar wave equation to be considered is:

$$\nabla^2\varphi+k^2\varphi=0. \tag{2.1}$$

The boundary conditions satisfied by φ can be stated generally as $\partial\varphi/\partial n=F\varphi$ where F is a function which may vary on surface S. In acoustics, F is related to the acoustic impedance. The outwardly drawn normal to the surface is designated by n. We distinguish two cases:

$$\begin{aligned} &\text{B.C.I} \quad \partial\varphi/\partial n=F\varphi \qquad F \text{ small,}\\ &\text{B.C.II} \quad \varphi=(1/F)(\partial\varphi/\partial n) \qquad F \text{ large.}\end{aligned} \tag{2.2}$$

Simple boundary conditions are limiting cases of (2.2). They are:

$$\begin{aligned} &\text{B.C.IA} \quad \partial\varphi/\partial n=0,\\ &\text{B.C.IIA} \quad \varphi=0.\end{aligned} \tag{2.3}$$

A general vector wave equation occurs in the theory of elasticity and may be written in the following form:

$$\alpha \text{ grad div } \mathbf{s}+\beta\nabla^2\mathbf{s}+\kappa^2\mathbf{s}=0, \tag{2.4}$$

where α and β are constants of the material. The vector $\mathbf{s}$ can always be split into two vectors $\mathbf{A}$ and $\mathbf{B}$ such that div $\mathbf{A}=0$, curl $\mathbf{B}=0$. In the latter case, a scalar potential can be defined and the vector equation reduced to a scalar one. In the first case, the vector equation can be written

$$\text{curl curl } \mathbf{A}-k^2\mathbf{A}=0 \tag{2.5}$$

where $k^2=\kappa^2/\beta$. This equation is also satisfied by the vector potential of the electromagnetic field. The boundary conditions satisfied by $\mathbf{A}$ can be stated generally as

$$(\mathbf{n}\times\mathbf{A})=(\mathbf{n}\times\text{curl } \mathbf{A})\cdot\mathfrak{Z} \tag{2.6}$$

where $\mathfrak{Z}$ is a dyadic. Two cases can again be distinguished

$$\begin{aligned} &\text{B.C.III} \quad (\mathbf{n}\times\mathbf{A})\\ &\qquad =(\mathbf{n}\times\text{curl } \mathbf{A})\cdot\mathfrak{Z} \qquad \mathfrak{Z} \text{ small,}\\ &\text{B.C.IV} \quad (\mathbf{n}\times\text{curl } \mathbf{A})\\ &\qquad =(\mathbf{n}\times\mathbf{A})\cdot\boldsymbol{\zeta} \qquad \boldsymbol{\zeta} \text{ small,}\end{aligned} \tag{2.7}$$

[1] H. Froelich, Phys. Rev. **54**, 945 (1938).
[2] L. Brillouin, Comptes rendus **204**, 1863 (1937).
[3] N. Cabrera, Comptes rendus **207**, 1175 (1938).
[4] H. Feshbach and A. M. Clogston, Phys. Rev. **59**, 189 (1941). This paper will be referred to as I.
[5] Bolt, Feshbach, and Clogston, J. Acous. Soc. Am. **14**, 65 (1942).

where $\zeta = (\mathfrak{Z})^{-1}$ and we have assumed that the determinant of the dyadic $\mathfrak{Z}$ does not vanish. In electromagnetic theory $\mathfrak{Z}$ is related to the dyadic impedance. The simple boundary conditions are

$$\begin{aligned} &\text{B.C.IIIA} \qquad (\mathbf{n}\times\mathbf{A})=0, \\ &\text{B.C.IVA} \qquad (\mathbf{n}\times\text{curl }\mathbf{A})=0. \end{aligned} \tag{2.8}$$

Case III can be reduced to case IV and vice versa. Since div $\mathbf{A}=0$, a vector $\mathbf{D}$ can be defined satisfying the relation $\mathbf{A}=\text{curl }\mathbf{D}$. The wave equation obeyed by $\mathbf{D}$ is the same as that for $\mathbf{A}$. Boundary conditions III become $(\mathbf{n}\times\text{curl }\mathbf{D}) = k^2(\mathbf{n}\times\mathbf{D})\cdot\mathfrak{Z}$. This is in the form given by boundary condition IV.

All these cases will be considered separately for both the discrete and continuous eigenvalue problems. The general technique to be used consists of reducing equations (2.1) and (2.5) to their corresponding integral equations by means of a Green's function. This has the advantage of introducing the boundary condition directly into the equation to be solved rather than its usual use as an auxiliary condition imposed on the solution. From these integral equations approximate formulae can be immediately derived. Solutions of the equations will then be derived by the method of successive approximations, and by expansion of the solution in eigenfunctions. The latter method leads to a secular equation.

Boundary condition I will be discussed in detail. The procedure to be used in the other cases will be outlined. Only specific differences from the treatment for boundary condition I will be noted. General remarks, unless otherwise noted, will be valid throughout.

III. BOUNDARY CONDITION I $\partial\phi/\partial n = F\phi$

The reduction of Eq. (2.1) to its equivalent integral equation is made possible by the use of a Green's function satisfying the inhomogeneous equation

$$\nabla_\xi^2 G_k(x,\xi) + k^2 G_k(x,\xi) = \delta(x-\xi), \tag{3.1}$$

where x represents the observation point, ξ the source point. $\delta(x-\xi)$ is the Dirac δ function. The subscript k is used to show that G_k is the Green's function corresponding to the wave number k.

G_k must satisfy Eq. (3.1) throughout a region R_0 which includes the region R. Moreover, it can satisfy any convenient boundary condition on the surface S_0 of R_0. Usually, S_0 is a simple surface and the boundary conditions simple ones.

Combining (2.1) and (3.1) we obtain:

$$\varphi(x) = \int\left[\varphi(\xi)\frac{\partial G_k(x,\xi)}{\partial n_\xi} - G_k(x,\xi)\frac{\partial\varphi(\xi)}{\partial n_\xi}\right]dS_\xi \tag{3.2}$$

where the integration is performed over surface S.

Introducing boundary condition I we have

$$\varphi(x) = \int\varphi(\xi)\left[\frac{\partial G_k(x,\xi)}{\partial n_\xi} - F(\xi)G_k(x,\xi)\right]dS_\xi. \tag{3.3}$$

The result for the simple boundary condition IA follows immediately.

An important special case occurs when S is a simple surface. Choosing a G_k satisfying boundary condition IA on S we obtain:

$$\varphi(x) = -\int\varphi(\xi)F(\xi)G_k(x,\xi)dS_\xi. \tag{3.4}$$

Equation (3.3) satisfies Eq. (3.1) as can be seen immediately by substitution of (3.3) in (2.1). Thus we can conclude that this integral equation is equivalent to the original differential equation and automatically includes the boundary condition.

Equation (3.3) states that it is possible to predict the required wave function in the interior of R by means of a proper distribution of sources on the surface S. This is essentially a statement of Huygens' principle differing from the usual formulation in the use of a general Green's function and the direct introduction of the boundary conditions. Huygens' principle has been previously used in discussing scattering and diffraction problems.[6]

Equations (3.3) and (3.4) can be reduced to a "true" integral equation involving only values of the unknown function on the surface S by noting that they can be used to predict the value

[6] See, for example: H. Lamb, *Hydrodynamics* (Cambridge University Press, 1932), p. 517; A. Sommerfeld, *Die Differential und Integralgluchungen der Mechanik und Physik* (Friedr. Vieweg. und Sohn, Braunschweig, 1935), Vol. 2, p. 853.

of φ on the surface. The resulting equation has a symmetric kernel and will involve one dimension less than the original integral equation. This procedure is used in discussions of potential theory.[7] It is particularly advantageous in those cases for which the boundary perturbation has a simple geometry, for then the equation may often be solved in closed form. It is also useful in two-dimensional problems since the resulting integral equation will be in one dimension and therefore susceptible to numerical techniques. Finally, if the perturbation is small compared to the wavelength, the kernel can be replaced by the simpler kernel of potential theory and the problem reduced to a potential problem. Once the value of φ on S is known, the value of φ in the interior can be immediately calculated from (3.3).

This method is complicated in the case of (3.3) by the fact that φ, being zero outside of R is discontinuous at the surface S. (Cf. Appendix I.) We shall, therefore, specify the value of φ at a point on the surface by approaching the point from the interior of R. In the case of (3.4), the solution φ has a discontinuity in slope at S. These discontinuities limit the rapidity of convergence of an expansion of φ in orthonormal functions.

It is very useful to separate the integral in Eq. (3.3) into two parts, one of which is known. This is particularly true if the known part is a good approximation to the unknown φ.

Suppose the known part ψ_n is a member of an orthonormal set of functions 0 satisfying (2.1) in a region R_0 including region R and satisfying convenient boundary conditions on S_0. G_k can then be expressed by means of an expansion in set 0. The term containing ψ_n can then be separated from the remaining terms of the expansion and its coefficient adjusted to one by means of a proper choice of normalization condition. Equation (3.3) becomes

$$\varphi(x)=\psi_n(x)+\int'\varphi(\xi)\times\left[\frac{\partial G_k(x,\xi)}{\partial n_\xi}-F(\xi)G_k(x,\xi)\right]dS_\xi, \quad (3.5)$$

where the prime on the integral sign indicates that the term involving ψ_n in the surface integral is to be omitted. The normalization condition is

$$\int_{R_0}\varphi\psi_n dV=1. \quad (3.6)$$

This is stated correctly only for the case of discrete eigenvalues. It can be extended to continuous eigenvalues by integrating the integral in (3.6) over a small volume element in wave number space corresponding to the wave numbers associated with ψ_n.

A second method employs the fact that (3.2) is satisfied by any solution of the wave equation and therefore by ψ_n. We must note, however, that (3.2) yields ψ_n only inside R and yields zero outside R. Let us call this discontinuous function Ψ_n. We now find that

$$\varphi(x)=\Psi_n(x)+\int\left\{[\varphi(\xi)-\psi_n(\xi)]\frac{\partial G_k(x,\xi)}{\partial n_\xi}-G_k(x,\xi)\left[F(\xi)\varphi(\xi)-\frac{\partial\psi_n}{\partial n_\xi}\right]\right\}dS_\xi. \quad (3.7)$$

Forms (3.5) and (3.7) yield exactly the same values for φ within region R. However, they behave differently outside of R leading to a different analytical behavior at S. Expansions of φ in orthonormal functions will have different rates of convergence depending on whether (3.5) or (3.7) is used.

Our attention, so far, has been focused on the wave function φ. However, the eigenvalue k^2 is of equal importance since it is needed for the specification of G_k. Suppose the eigenvalue associated with ψ_n is K_n^2. Using Green's theorem, it can be shown that

$$k^2=K_n^2+\int\left[\varphi\frac{\partial\psi_n}{\partial n}-\psi_n\frac{\partial\varphi}{\partial n}\right]dS\Big/\int_R\psi_n\varphi dV. \quad (3.8)$$

Introducing boundary condition I we find:

$$k^2=K_n^2+\int\varphi\left(\frac{\partial\psi_n}{\partial n}-F\psi_n\right)dS\Big/\int_R\psi_n\varphi dV. \quad (3.9)$$

A similar formula can be derived for the case of the continuous spectrum by performing a second integration in wave number space as described for formula (3.6).

[7] O. D. Kellogg, *Foundations of Potential Theory* (Frederick Unger Publishing Company, New York, 1929), p. 286.

Our problem has now been reduced to the solution of integral equation (3.3) which implicitly contains the boundary condition to be satisfied. Approximate formulae can be derived at once. If $\varphi \simeq \psi_n$, then from (3.5)

$$\varphi(x) \simeq \psi_n(x) + \int' \psi_n(\xi) \times\left[\frac{\partial G_k(x,\xi)}{\partial n_\xi} - F(\xi)G_k(x,\xi)\right] dS_\xi. \quad (3.10a)$$

From (3.7), one finds

$$\varphi(x) \simeq \Psi_n(x) - \int G_k(x,\xi) \times\left[F(\xi)\psi_n(\xi) - \frac{\partial \psi_n(\xi)}{\partial n_\xi}\right] dS_\xi. \quad (3.10b)$$

Formulae (3.10a) and (3.10b) are equivalent as has been pointed out above. The eigenvalue becomes:

$$k^2 \simeq K_n^2 + \int \psi_n\left(\frac{\partial \psi_n}{\partial n} - F\psi_n\right) dS. \quad (3.11)$$

This formula has been given previously. The eigenvalue can be found to a second approximation by using the first approximation results for the wave function.

The use of these results depends upon our knowledge of G_k. The latter is sometimes known in closed form, but then, the integrations are often unmanageable. In such case, it is useful to expand G_k in a series in set 0 as follows:

$$G_k(x,\xi) = \sum_t \frac{\psi_t(x)\psi_t(\xi)}{k^2 - K_t^2}. \quad (3.12)$$

This expansion will now be introduced into (3.10b) for this yields the more rapidly converging result.

$$\varphi(x) \simeq \Psi_n(x) + \sum_t{}' \frac{A_{tn}\psi_t(x)}{K_n^2 - K_t^2} \quad (3.13)$$

where the prime indicates omission of the $n=t$ term in the summation and

$$A_{tn} = \int \psi_t\left(\frac{\partial \psi_n}{\partial n} - F\psi_n\right) dS. \quad (3.14)$$

Expansion (3.13) can be introduced into (3.9) to yield a second approximation for the eigenvalue:

$$k^2 = K_n^2 + A_{nn}/N_{nn} + \sum_t{}' \frac{A_{tn}^2}{K_n^2 - K_t^2} \quad (3.15)$$

where

$$N_{tn} = \int_R \psi_t \psi_n dV. \quad (3.16)$$

Formulae equivalent to (3.13) and (3.15) have been given previously in I. They have been applied to the acoustics of trapezoidally shaped rooms and agree with experiment within the range of their validity.

The convergence of the series for φ given by (3.13) is fairly good, the tth term behaving asymptotically as $1/K_t^2$. This is sufficient to permit its introduction into (3.5) or (3.7) to yield a next approximation for the wave function. The resulting series will behave asymptotically as $1/K_t$ and could not be used to find either the next approximation to the eigenfunction φ or eigenvalue k^2.

One method of improving convergence would be to find a function which converges asymptotically as $1/K_t^2$ and subtract it from (3.13). The function χ satisfying

$$\chi(x) = -\int G_0(x,\xi) \times\left[F(\xi)\psi_n(\xi) - \frac{\partial \psi_n(\xi)}{\partial n_\xi}\right] dS_\xi \quad (3.17)$$

is of this type. On substitution of (3.12) into (3.17) one finds an expansion of the (3.13) type with k replaced by 0. χ is the solution of Laplace's equation for region R, with a source distribution $F\psi_n - (\partial\psi_n/\partial n)$. χ can often be calculated in closed form especially for two-dimensional problems. If this is so, the next approximation for φ can be found by introducing the amended (3.13) for Ψ_n in (3.7). Successive approximations could be found by continuing this procedure.

These difficulties do not appear when S is a simple surface. It is possible to find the solution of the subsequent integral equation by means of the method of successive substitutions. This yields:

$$\varphi=\psi_n+\sum_t{}'\frac{F_{tn}}{k^2-K_t^2}\psi_t+\sum_{t,s}{}''\frac{F_{ts}F_{sn}}{(k^2-K_t^2)(k^2-K_s^2)}\psi_t+\sum_{s,t,n}{}'''\frac{F_{ts}F_{sr}F_{rn}}{(k^2-K_t^2)(k^2-K_s^2)(k^2-K_r^2)}\psi_t+\cdots,\quad(3.18)$$

$$k^2=K_n^2+F_{nn}+\sum{}'\frac{(F_{tn})^2}{k^2-K_t^2}+\sum{}''\frac{F_{ts}F_{sn}F_{tn}}{(k^2-K_s^2)(k^2-K_t^2)}+\sum{}'''\frac{F_{ts}F_{sr}F_{rn}F_{tn}}{(k^2-K_s^2)(k^2-K_r^2)(k^2-K_t^2)}+\cdots\quad(3.19)$$

where

$$F_{tn}=-\int\psi_tF\psi_ndS.\quad(3.20)$$

The primes indicate omission of the nth term in the summation over each index. It will be noted that the unknown k appears in these results so that (3.18) is really an equation which must be solved for k. It is, however, in a form which permits the evaluation of k^2 by means of successive approximations leading to a continued fraction solution for k^2 as compared to the usual power series. Its chief advantage over the latter lies in a considerably simplified formula.[8] Moreover, it is interesting to note that in acoustics F is a function of k, and so it would serve no useful purpose to eliminate the explicit appearance of k on the right-hand side of (3.18).

We shall now attempt a solution of the more general case of boundary condition I. It is possible to write down general solutions of (3.5) or (3.7) in terms of iterated kernels by means of the method of successive approximations. As a rule, the evaluation of the multiple integrals involved in closed form, i.e., not by expansion techniques, is possible only by numerical methods.

Another method which suggests itself would involve the substitution of a general expansion of φ in orthonormal functions ψ_n of set 0 on both sides of (3.3). The resulting linear equations for the unknown coefficients would yield a secular determinant. However, in this case this procedure is not valid since the $\varphi(x)$ occurring on the left-hand side of (3.3) is discontinuous at S. An expansion of φ in terms of set 0 would therefore have poor convergence. It could not be substituted in the right side of (3.3) since the expansion would not converge to the correct value at the surface S. This invalidates the above procedure for the general case.

However, this difficulty does not occur when S is a simple surface. This yields the secular equation:

$$|F_{tn}-\delta_{tn}(k^2-K_n^2)|=0.\quad(3.21)$$

A solution of this equation is given by formula (3.19) with associated wave function (3.18).

It is also possible to derive a secular equation for the more general case boundary condition I. Two general requirements should be noted. First, the secular determinant should be Hermitian. Secondly, its convergence should be sufficient to yield a well-converging expansion for φ, the tth term decreasing at least as strongly as $1/K_t^2$.

We shall expand φ in terms of the non-orthogonal set $\{\Psi_n\}$. Each member of this member of this set is discontinuous so that it is no longer necessary for the expansions to represent discontinuous functions. Note that this set is complete in region R.

Let $\varphi=\sum_t a_t\Psi_t$. Equation (3.3) yields

$$\sum_t a_t[A_{tn}-N_{tn}(k^2-K_n^2)]=0.\quad(3.22)$$

The resulting secular equation is:

$$|A_{tn}-N_{tn}(k^2-K_n^2)|=0.\quad(3.23)$$

From the equation:

$$N_{tn}=\frac{A_{nt}-A_{tn}}{K_n^2-K_t^2}\quad(3.24)$$

it follows that the secular determinant is Hermitian. Secondly, good convergence can be expected since the values of the non-diagonal elements decrease as their distance from a diagonal element increases.

[8] A short note giving similar results for secular perturbation theory is in preparation.

Equation (3.22) can be solved by the method of successive approximations for the coefficients a_t yielding:

$$\varphi=\Psi_n+\sum_{t\neq n} W_{nt}\Psi_t+\sum_{s\neq t\neq n} W_{ns}W_{st}\Psi_t + \sum_{r\neq s\neq t\neq n} W_{nr}W_{rs}W_{st}\Psi_t+\cdots \tag{3.25}$$

where

$$W_{nt}=-\frac{A_{nt}-N_{nt}(k^2-K_t^2)}{A_{tt}-N_{tt}(k^2-K_t^2)}. \tag{3.26}$$

The corresponding eigenvalue can be obtained from (3.9).

$$k^2=K_n^2 + \frac{A_{nn}+\sum_{t\neq n} W_{nt}A_{tn}+\sum_{s\neq t\neq n} W_{ns}W_{st}A_{tn}+\cdots}{N_{nn}+\sum_{t\neq n} W_{nt}N_{tn}+\sum_{s\neq t\neq n} W_{ns}W_{st}N_{tn}+\cdots}. \tag{3.27}$$

The rapid convergence previously demanded is exhibited by series (3.25). All the terms involved in these formulae involve only surface integrals except the normalization integrals N_{nn}. These formulae reduce in first approximation to those given by (3.13) and (3.15).

The results given above apply to those situations for which the eigenvalues are discrete. A great many problems, e.g., scattering, require the construction of a theory on which the eigenvalues have a continuous spectrum. The above theory can, with suitable modification, be applied to this case. Integral equation (3.3) applies to the continuous spectrum case as well as to the discrete spectrum. However, in view of the usual boundary condition of incident plus outgoing scattered waves, forms (3.5) and (3.7) are more convenient.

The first approximation (3.10) can be used directly. The usefulness of this approximation depends upon the proper choice of the Green's function and the closeness of ψ_n to the solution φ. No general rules can be made as to the latter since it depends mostly on physical intuition. However, as to the former, it can be said that G_k should be as close as possible to the Green's function for region R. This applies particularly over the region where the perturbation is most important. If the G_k chosen were actually the Green's function for region R, the first approximation would yield exact results.

In many cases, a second or higher approximation will be needed. The formulae developed above can be used if all the summations are replaced by integrations over the continuous spectrum. The function ψ_n will satisfy the normalization condition characteristic of continuous spectrum theory:

$$\int \psi_n\psi_{n'}dV=\delta(n-n').$$

The integrations over the continuous spectrum must be carried out in such a manner as to yield a solution satisfying the boundary conditions mentioned above. This usually requires the use of contour integrals in the complex plane for one of the wave numbers.

IV. BOUNDARY CONDITION II $\phi=1/F(\partial\phi/\partial n)$

As in the case of boundary condition I, this problem can be reduced to the integral equation:

$$\varphi(x)=\int \frac{\partial\varphi(\xi)}{\partial n_\xi} \times\left[\frac{1}{F(\xi)}\frac{\partial G_k(x,\,\xi)}{\partial n_\xi}-G_k(x,\,\xi)\right]dS_\xi. \tag{4.1}$$

The solution of this equation will be a function which has a discontinuous value at surface S. In the special case boundary condition IIA, the solution is discontinuous in slope at S.

Formulae corresponding to (3.5), (3.7), (3.9) can be derived for this case. They are

$$\varphi(x)=\psi_n(x)+\int' \frac{\partial\varphi(\xi)}{\partial n_\xi} \times\left[\frac{1}{F(\xi)}\frac{\partial G_k(x,\,\xi)}{\partial n_\xi}-G_k(x,\,\xi)\right]dS_\xi, \tag{4.2}$$

$$\varphi(x)=\Psi_n(x)+\int\left\{\frac{\partial G_k(x,\,\xi)}{\partial n_\xi} \times\left[\frac{1}{F(\xi)}\frac{\partial\varphi(\xi)}{\partial n_\xi}-\psi_n(\xi)\right] -G_k(x,\,\xi)\left[\frac{\partial\varphi(\xi)}{\partial n_\xi}-\frac{\partial\psi_n(\xi)}{\partial n_\xi}\right]\right\}dS_\xi, \tag{4.3}$$

$$k^2=K_n^2+\int\frac{\partial\psi}{\partial n}\times\left(\frac{1}{F}\frac{\partial\psi_n}{\partial n}-\psi_n\right)dS\bigg/\int_R\psi_n\varphi dV. \quad (4.4)$$

If one assumes that $\varphi\simeq\psi_n$ within the surface integrals of (4.2), (4.3), (4.4) one can find approximate formulae analogous to (3.10) and (3.11).

Using expansion (3.12) we obtain the expression:

$$\varphi(x)\simeq\psi_n(x)+\sum_t{}'\frac{B_{nt}}{K_n^2-K_t^2}\psi_t \quad (4.5)$$

where

$$B_{nt}=\int\frac{\partial\psi_n}{\partial n}\left(\frac{1}{F}\frac{\partial\psi_t}{\partial n}-\psi_t\right)dS. \quad (4.6)$$

Convergence of this series is poor. Equation (4.5) can no longer be substituted in (4.4) to obtain the next approximation for the eigenvalue, nor can it be introduced with either (4.2) or (4.3) to obtain higher approximations for the eigenfunctions. Moreover, it is not possible to improve the convergence by means of an expansion in terms of the discontinuous set $\{\Psi_n\}$. For example, in the special case B.C.IIA, one needs an expansion in terms of functions which have discontinuous slope rather than value. Instead of considering such a set directly, it is more convenient to consider the vector $\mathbf{v}=-\nabla\varphi/k$. It satisfies the vector wave equation:

$$\nabla\nabla\cdot\mathbf{v}+k^2\mathbf{v}=0. \quad (4.7)$$

Its boundary conditions are:

$$\nabla\cdot\mathbf{v}=-k^2/F(\mathbf{n}\cdot\mathbf{v}). \quad (4.8)$$

In order to convert (4.7) into an integral equation, it is necessary to develop a vector Green's theorem and define the proper Green's function. The former is:

$$\int[\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{v})-\mathbf{v}\cdot\nabla(\nabla\cdot\mathbf{u})]dV=\int[(\mathbf{n}\cdot\mathbf{u})(\nabla\cdot\mathbf{v})-(\mathbf{n}\cdot\mathbf{v})(\nabla\cdot\mathbf{u})]dS. \quad (4.9)$$

The Green's function to be used with a vector wave equation is a dyadic satisfying

$$\nabla_\xi(\nabla_\xi\cdot\mathfrak{G}_k(x,\xi))+k^2G_k(x,\xi)=\delta(x-\xi)\mathfrak{J} \quad (4.10)$$

where $\mathfrak{J}$ is the idemfactor. In acoustics, $\mathbf{a}\cdot\mathfrak{G}$ gives the velocity due to an elementary source located at ξ, having the direction and magnitude of $\mathbf{a}$. $\mathfrak{G}_k(x,\xi)$ can be expanded in terms of the complete orthonormal set found by the vectors:

$$\mathbf{u}_t=-\nabla\psi_t/K_t \quad (4.11)$$

satisfying the orthogonality relation $\int_{R_0}\mathbf{u}_n\cdot\mathbf{u}_m dV=\delta_{nm}$. Then:

$$\mathfrak{G}_k(x,\xi)=\sum_t\frac{\mathbf{u}_t(x)\mathbf{u}_t(\xi)}{k^2-K_t^2}. \quad (4.12)$$

Combining (4.7), (4.9), and (4.10) we obtain

$$\mathbf{v}(x)=\int[\mathbf{n}_\xi\cdot\mathbf{v}(\xi)\nabla_\xi\cdot\mathfrak{G}_k(x,\xi)-\nabla_\xi\cdot\mathbf{v}(\xi)\mathbf{n}_\xi\cdot\mathfrak{G}_k(x,\xi)]dS_\xi. \quad (4.13)$$

Introducing boundary condition (4.8) we obtain the integral equation:

$$\mathbf{v}(x)=\int\mathbf{n}_\xi\cdot\mathbf{v}(\xi)\left[\frac{k^2}{F(\xi)}\mathbf{n}_\xi\cdot\mathfrak{G}_k(x,\xi)+\nabla_\xi\cdot\mathfrak{G}_k(x,\xi)\right]dS_\xi. \quad (4.14)$$

Comparison of (4.14) and (3.3) reveals their strong similarity enabling us to use procedures similar to those of Sec. III. The first approximation to $\mathbf{v}$ is

$$\mathbf{v}\simeq\mathbf{u}_n+\sum_t{}'\frac{C_{nt}}{K_n^2-K_t^2}\mathbf{u}_t \quad (4.15)$$

where

$$C_{nt}=\frac{K_t}{K_n}\int\frac{\partial\psi_n}{\partial n}\left(\frac{k^2}{K_t^2F}\frac{\partial\psi_t}{\partial n}-\psi_t\right)dS. \quad (4.16)$$

Convergence of (4.15) is sufficient to permit its introduction into (4.2) and (4.4) to yield higher approximation for both eigenvalue and eigenfunction. The eigenvalue result is of value:

$$k^2\simeq K_n^2+B_{nn}/N_{nn}+\sum{}'\frac{K_n}{K_t}\frac{C_{nt}B_{tn}}{K_n^2-K_t^2}. \quad (4.17)$$

The particular case of S a simple surface permits the choice of $\mathfrak{G}_k$ so that $\nabla\cdot\mathfrak{G}_k=0$ (i.e.,

$\psi_n=0$ on S). The resulting integral equation can be solved exactly.

$$\mathbf{v}=\mathbf{u}_n+\sum_t{}' \frac{G_{nt}}{k^2-K_t^2}\mathbf{u}_t+\sum_{r,t}{}'' \frac{G_{nr}G_{rt}}{(k^2-K_r^2)(k^2-K_t^2)}\mathbf{u}_t+\cdots \quad (4.18)$$

where

$$G_{nt}=\frac{k^2}{K_nK_t}\int \frac{\partial\psi_n}{\partial n}\frac{1}{F}\frac{\partial\psi_t}{\partial n}dS. \quad (4.19)$$

Introducing (4.18) into (4.2) one finds

$$\varphi=\psi_n+\sum_t{}' \frac{K_nK_t}{k^2}\frac{G_{nt}}{(k^2-K_t^2)}\psi_t+\sum_{r,t}{}'' \frac{K_tK_nG_{nr}G_{rt}}{k^2(k^2-K_r^2)(k^2-K_t^2)}\psi_t+\cdots. \quad (4.20)$$

The value of k^2 is:

$$k^2=K_n^2+G_{nn}+\sum_t{}' \frac{G_{nt}G_{tn}}{k^2-K_t^2}+\sum_{r,t}{}'' \frac{G_{nr}G_{rt}G_{tn}}{(k^2-K_t^2)(k^2-K_r^2)}+\cdots. \quad (4.21)$$

Result (4.21) can be derived from the secular equation satisfied by k^2:

$$|G_{tn}-(k^2-K_n^2)\delta_{tn}|=0. \quad (4.22)$$

A secular determinant can be developed for the more general case. $\mathbf{v}$ is expanded in terms of a complete but non-orthogonal set $\{\mathbf{U}_n\}$. $\mathbf{U}_n$ is a vector function equal to $\mathbf{u}_n$ inside R but zero outside. The resulting secular determinant is:

$$|C_{tn}-(k^2-K_n^2)M_{tn}|=0 \quad (4.23)$$

where

$$M_{tn}=\int_R \mathbf{u}_n\cdot\mathbf{u}_t dV=\frac{C_{nt}-C_{tn}}{K_n^2-K_t^2}. \quad (4.24)$$

This determinant is Hermitian. Solution of (4.23) can be given as before [cf. (3.25)],

$$\mathbf{v}=\mathbf{U}_n+\sum_{t\neq n} V_{nt}\mathbf{U}_t+\sum_{t\neq s\neq n} V_{ns}V_{st}\mathbf{U}_t+\cdots \quad (4.25)$$

where

$$V_{nt}=-\frac{C_{nt}-M_{nt}(k^2-K_t^2)}{C_{tt}-M_{tt}(k^2-K_t^2)}. \quad (4.26)$$

This permits the computation of φ from (4.2):

$$\varphi=\psi_n+\sum_{t\neq n}\frac{B_{nt}}{k^2-K_t^2}\psi_t+\sum_{s\neq t\neq n}\frac{K_n}{K_t}\frac{V_{nr}B_{rt}}{(k^2-K_t^2)}\psi_t+\cdots. \quad (4.27)$$

Finally, the result for the eigenvalue k^2 has the same form as (3.27), with A_{tn} replaced by C_{tn}, W_{nt} by V_{nt}, and N_{tn} by M_{tn}. Convergence of (4.25) and (4.26) is good.

Remarks given in the preceding section on extension of these results to the problem of scattering hold here.

V. BOUNDARY CONDITIONS III AND IV: $(\mathbf{n}\times\text{curl }\mathbf{A})=(\mathbf{n}\times\mathbf{A})\cdot\boldsymbol{\zeta}$.

The reduction of wave equation (2.5) to an integral equation can be accomplished through the use of a dyadic Green's function satisfying the equation

$$\text{curl}_\xi\,\text{curl}_\xi\,\mathfrak{G}_k(x,\xi)-k^2\mathfrak{G}_k(x,\xi)=\delta(x-\xi)\mathfrak{J}. \quad (5.1)$$

We also need a vector Green's theorem:[9]

$$\int[\mathbf{A}\cdot\text{curl curl }\mathbf{B}-\mathbf{B}\cdot\text{curl curl }\mathbf{A}]dV=\int[\mathbf{A}\cdot(\mathbf{n}\times\text{curl }\mathbf{B})-\mathbf{B}\cdot(\mathbf{n}\times\text{curl }\mathbf{A})]dS. \quad (5.2)$$

It then follows that:

$$\mathbf{A}(x)=\int[\mathbf{A}(\xi)\cdot(\mathbf{n}_\xi\times\text{curl}_\xi\,\mathfrak{G}_k(x,\xi))-(\mathbf{n}_\xi\times\text{curl}_\xi\,\mathbf{A}(\xi))\cdot\mathfrak{G}_k(x,\xi)]dS_\xi. \quad (5.3)$$

Introducing B.C.IV we find the integral equation satisfied by $\mathbf{A}(x)$:

$$\mathbf{A}(x)=-\int(\mathbf{n}_\xi\times\mathbf{A}(\xi))\cdot[\text{curl}_\xi\,\mathfrak{G}_k(x,\xi)+\boldsymbol{\zeta}(\xi)\cdot\mathfrak{G}_k(x,\xi)]dS_\xi. \quad (5.4)$$

This equation is very similar to (3.3) and permits the use of the techniques developed in Sec. III.

[9] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 250.

Formulae corresponding to (3.5), (3.7), and (3.9) can be derived. They are:

$$\mathbf{A}(x)=\mathbf{D}_n(x)-\int'(\mathbf{n}_\xi\times\mathbf{A}(\xi))\cdot[\text{curl}_\xi\,\mathfrak{G}_k(x,\xi)+\boldsymbol{\zeta}(\xi)\cdot\mathfrak{G}_k(x,\xi)]dS_\xi, \quad (5.5)$$

$$\mathbf{A}(x)=\boldsymbol{\Delta}_n(x)-\int\{\text{curl}_\xi\,\mathfrak{G}_k(x,\xi)\cdot[\mathbf{n}_\xi\times\mathbf{A}(\xi)-\mathbf{n}_\xi\times\mathbf{D}_n(\xi)]+[\mathbf{n}_\xi\times\mathbf{A}(\xi)\cdot\boldsymbol{\zeta}(\xi)-(\mathbf{n}_\xi\times\text{curl}_\xi\,\mathbf{D}_n)]\cdot\mathfrak{G}_k(x,\xi)\}dS_\xi, \quad (5.6)$$

where $\mathbf{D}_n$ is any solution of Eq. (2.5). $\boldsymbol{\Delta}_n$ is a vector equal to $\mathbf{D}_n$ within R and equal to 0 outside of R. If K_n^2 is the eigenvalue corresponding to $\mathbf{D}_n$, then,

$$k^2=K_n^2+\int(\mathbf{n}\times\mathbf{A})\cdot[\text{curl}\,\mathbf{D}_n+\boldsymbol{\zeta}\cdot\mathbf{D}_n]dS\bigg/\int_R\mathbf{A}\cdot\mathbf{D}_n dV. \quad (5.7)$$

First-approximation results can be immediately obtained from (5.5), (5.6), (5.7) if $\mathbf{A}(x)\simeq\mathbf{D}_n(x)$ by replacing $\mathbf{A}(x)$ by $\mathbf{D}_n$ within the surface integrals.

Suppose $\mathbf{D}_n$ is normalized so that it is a member of an orthonormal set. The dyadic Green's function can then be expanded in terms of the set $\{\mathbf{D}_n\}$:

$$\mathfrak{G}_k(x,\xi)=-\sum_t\frac{\mathbf{D}_t(x)\mathbf{D}_t(\xi)}{k^2-K_t^2}. \quad (5.8)$$

Introducing this expansion into (5.6) and replacing $\mathbf{A}$ by $\mathbf{D}_n$ we obtain

$$\mathbf{A}(x)\simeq\boldsymbol{\Delta}_n(x)+\sum_t{}'\frac{D_{tn}}{K_n^2-K_t^2}\mathbf{D}_t(x) \quad (5.9)$$

where

$$D_{tn}=\int(\mathbf{n}\times\mathbf{D}_t)\cdot[\text{curl}_\xi\,\mathbf{D}_n+\boldsymbol{\zeta}(\xi)\cdot\mathbf{D}_n]dS_\xi. \quad (5.10)$$

A second approximation to the eigenvalue follows:

$$k^2\simeq K_n^2+\frac{D_{nn}}{P_{nn}}+\sum_t{}'\frac{(D_{tn})^2}{K_n^2-K_t^2} \quad (5.11)$$

where

$$P_{nt}=\int_R\mathbf{D}_n\cdot\mathbf{D}_t dV=\frac{D_{nt}-D_{tn}}{K_n^2-K_t^2}. \quad (5.12)$$

Poor convergence prevents calculation of further approximations from (5.9). However, in the special case where S is a simple surface choose $(\mathbf{n}\times\text{curl}\,\mathfrak{G}_k)=0$ on S (i.e., $\mathbf{n}\times\text{curl}\,\mathbf{D}_t=0$ on S). The resulting formulae have exactly the same form as (3.18). F_{tn} should be replaced by

$$\zeta_{tn}=\int(\mathbf{n}\times\mathbf{D}_n)\cdot\boldsymbol{\zeta}\cdot\mathbf{D}_t dS. \quad (5.13)$$

The corresponding secular determinant is

$$|\zeta_{tn}-(k^2-K_n^2)\delta_{tn}|=0. \quad (5.14)$$

A secular determinant can be developed for the more general case. $\mathbf{A}$ is expanded in the complete set of discontinuous vectors $\{\boldsymbol{\Delta}_n\}$. The resulting secular determinant is

$$|D_{tn}-(k^2-K_n^2)P_{tn}|=0. \quad (5.15)$$

The determinant is Hermitian. Solution of (5.15) can be found as in Sec. III. Formulae (3.25) and (3.27) will apply to this case if Ψ_t is replaced by $\boldsymbol{\Delta}_t$, A_{nt} by D_{nt}, N_{tn} by P_{tn} throughout.

VI. GREEN'S FUNCTIONS

The formulae developed in the preceding sections apply only when there are no sources present in the region R being considered. This is usually not the case experimentally since sources must be present to excite the modes being investigated, or in the scattering situation the source is frequently not at infinity.

The description of the effect of a source is given by the Green's function which satisfies the boundary conditions imposed by the physical situation at the surface S of R. This Green's function can be constructed in those cases for which the boundary conditions are such that the solutions of the corresponding wave equation form an orthonormal set. The Green's function can then be expanded in terms of this orthonormal set according to Eqs. (3.11), (4.19), or (5.12). The various possible solutions of the wave equations yielding the desired set are given in the discussion above. It should be noted that these form an orthogonal set, but are not normalized as given.

However, for some boundary conditions the solutions are not orthogonal, or the expansion is unwieldy. It is then interesting to note that the Green's function also satisfies an integral equation from which approximate results for the new Green's function may be deduced. We shall give the details for the case of boundary condition I, giving only the results for the others.

Let the desired Green's function be $G_k(x, \xi)$. Let any other satisfying the wave equation for other boundary conditions be $L_k(x, \xi)$. Then

$$G_k(x, \xi) = L_k(x, \xi) + \int \left[G_k(x, \rho) \frac{\partial L_k(\rho, \xi)}{\partial n_\rho} - L_k(\rho, \xi) \frac{\partial G_k(x, \rho)}{\partial n_\rho} \right] dS_\rho. \quad (6.1)$$

The introduction of boundary conditions I on G_k yields

$$G_k(x, \xi) = L_k(x, \xi) + \int G_k(x, \rho) \times \left[\frac{\partial L_k(\rho, \xi)}{\partial n_\rho} - F(\rho) L_k(\rho, \xi) \right] dS_\rho. \quad (6.2)$$

For the other boundary conditions we get

B.C.II $$G_k(x, \xi) = L_k(x, \xi) + \int \frac{\partial G_k(x, \rho)}{\partial n_\rho} \times \left[\frac{1}{F(\rho)} \frac{\partial L_k(\rho, \xi)}{\partial n_\rho} - L_k(\rho, \xi) \right] dS_\rho, \quad (6.3)$$

B.C.IV $$\mathfrak{G}_k(x, \xi) = \mathfrak{L}_k(x, \xi) - \int \mathbf{n}_\rho \times \mathfrak{G}_k(x, \rho) \cdot [\operatorname{curl}_\rho \mathfrak{L}_k(\rho, \xi) + \zeta(\rho) \cdot \mathfrak{L}_k(\rho, \xi)] dS_\rho. \quad (6.4)$$

Any attempt to solve these integral equations by means of expansions leads again to the expansion in eigenfunctions mentioned above. It is possible, however, to find an approximate value of G_k if $G_k \simeq L_k$ by substituting the known L_k for the unknown G_k in the right side of Eqs. (6.2), (6.3), (6.4). This represents the chief use of these integral equations in practice since usually it is impossible to evaluate a second approximation by the reintroduction of the first approximation into these equations. A similar approximation used in a discussion of the Laplace equation has been given by Hadamard.[10]

VII. CONCLUSIONS

It has been found possible to reduce the scalar and vector wave equations to the solutions of a corresponding secular determinant by means of the following procedure.

1. Reduction to an integral equation through the use of a Green's function. This equation includes the boundary conditions.
2. For boundary conditions I and IV, expansion of the solution in terms of a discontinuous set of functions which vanish outside region R. This leads to a secular equation.
3. For boundary conditions II and III, the introduction of another auxiliary quantity whose boundary conditions and equations are similar to I and IV is necessary. The technique employed for these boundary conditions is then valid.

We note that the secular determinant is Hermitian. Its non-diagonal terms involve only surface integrals whereas the diagonal terms involve also volume integrals of the N_{nn} type.

Extension of these results to the scattering problem has been indicated at the end of Sec. III. Similar considerations as applied to source functions are given in Sec. VI. The solution of these equations in regions bounded by moving surfaces is found to the second approximation in Appendix II.

The procedure developed here appears to apply to any system of partial differential equations whose solutions form a complete orthonormal set or are orthogonal to another set. There is, however, a restriction for boundary conditions of type B.C.II. In this case, it is necessary for the solution of the auxiliary equation to satisfy some orthogonality relation. This condition has not been found in the case of the Schroedinger equation. However, for B.C.I, no such restriction applies. In this problem, the solutions found for the wave equation apply equally well to the Schroedinger equation. It is this boundary condition which occurs in the Wigner-Seitz theory of metals.

The author wishes to thank Professor P. M.

[10] P. Levy, *Leçons d'Analyse Fonctionelle* (Gauthier-Villars, Paris, 1922), p. 181.

Morse for his active interest and aid. He is also indebted to Professor J. C. Slater and Professor L. I. Schiff for some illuminating discussions and criticisms.

APPENDIX I

Here we shall show that the surface integral (3.2) represents any solution of the wave equation within R but is zero outside. Any function $f(x)$ which is zero outside R can be expanded in terms of the set 0 as follows:

$$f(x)=\sum_n\left(\int_R f\psi_n dV\right)\psi_n. \tag{I.1}$$

If, furthermore, f is a solution of the wave equation with eigenvalue k^2, it follows that

$$f(x)=\sum_n\frac{\int\left(f\frac{\partial\psi_n}{\partial n}-\psi_n\frac{\partial f}{\partial n}\right)dS}{k^2-K_n^2}\psi_n. \tag{I.2}$$

Using the expansion of the Green's function (3.12) we obtain Eq. (3.2).

APPENDIX II

In this appendix, we shall consider the problem of moving boundary surfaces. We shall restrict the discussion to the scalar equation and B.C.I since the extension to other cases involves the same calculations given in the main body of this paper. The function φ will now satisfy the time dependent equation

$$\nabla^2\varphi=(1/c^2)(\partial^2\varphi/\partial t^2). \tag{II.1}$$

The following discussion will be restricted to cases involving nearly simple harmonic time dependence. The extension to a more general type of time dependence can be easily carried out in terms of a Fourier integral superposing the various possible frequencies. We, therefore, consider the Green's function $G_k(x,\xi)e^{-i\omega t}=g_k$ which satisfies the equation

$$\nabla_\xi^2 g_k=1/c^2\frac{\partial^2 g_k}{\partial t^2}+\delta(x-\xi)e^{-i\omega t}. \tag{II.2}$$

Combining (II.1) and (II.2) we have

$$\varphi(x)e^{-i\omega t}=\int\left[\varphi(\xi)\frac{\partial g_k}{\partial n_\xi}-g_k\frac{\partial\varphi(\xi)}{\partial n_\xi}\right]dS_\xi +1/c^2\int\left[g_k\frac{\partial^2\varphi(\xi)}{\partial t^2}-\frac{\partial^2 g_k}{\partial t^2}\varphi(\xi)\right]dV_\xi. \tag{II.3}$$

Another result which can be derived in the same manner is

$$\frac{\partial^2\varphi(x)}{\partial t^2}e^{-i\omega t}=\int\left[\frac{\partial^2\varphi(\xi)}{\partial t^2}\frac{\partial g_k}{\partial n_\xi}-\frac{\partial^2 g_k}{\partial t^2}\frac{\partial\varphi(\xi)}{\partial n_\xi}\right]dS_\xi. \tag{II.4}$$

It will be noted that the time enters explicitly in these surface and volume integrals as a parameter. Equation (II.3) will be used in the present discussion since it yields a value of φ directly, whereas (II.4) would involve further integrations before value of φ is found. However, the latter equation has definite advantages because it involves only surface integration.

Introducing boundary conditions and time dependence of g_k we have

$$\varphi(x)=\int\left[\varphi(\xi)\frac{\partial G_k(x,\xi)}{\partial n_\xi}-F(\xi)G_k(x,\xi)\right]dS_\xi +1/c^2\int G_k(x,\xi)\left[\frac{\partial^2\varphi(\xi)}{\partial t^2}+\omega^2\varphi(\xi)\right]dV_\xi. \tag{II.5}$$

The surface integral term is identical to that which occurs in (3.3). As a first approximation let $\varphi\simeq\psi_n e^{-i\omega t}$. The volume integral drops out. The surface integral can be evaluated in two forms:

$$\varphi=e^{-i\omega t}\left[\psi_n+\sum_r{}'\frac{A_{nr}}{k^2-K_r^2}\psi_r\right] \tag{II.6}$$

or

$$\varphi=e^{-i\omega t}\left[\Psi_n-\sum{}'\frac{A_{nr}-N_{nr}(k^2-K_r^2)}{A_{rr}-N_{rr}(k^2-K_r^2)}\Psi_r\right]. \tag{II.7}$$

Formula (II.7) is taken from (3.25) and converges more rapidly than (II.6). Note that the matrix elements A_{nr}, N_{rr} are functions of time. It is most convenient to introduce (II.6) into the volume term since the ψ_r are independent of time whereas Ψ_r in (II.7) are not. The second approximation for the surface terms will just be the second term in (3.25). We finally find

$$\varphi=e^{-i\omega t}\left[\Psi_n+\sum_{r\neq n}W_{nr}\Psi_r+\sum_{r\neq s\neq n}W_{ns}W_{sr}\Psi_r +\frac{1}{c^2}\sum_{r\neq s\neq n}\frac{\ddot{A}_{ns}-2i\omega\dot{A}_{ns}}{(k^2-K_s^2)(k^2-K_n^2)}N_{sr}\psi_r\right]. \tag{II.8}$$

The dot notation has its usual significance of time derivative. These results are useful when the time dependence of φ is approximately simple harmonic. More general types of time dependence can be handled by means of a superposition of simple harmonic terms as suggested above. It would also be possible to make use of a more general formulation of Huygens' principle for the time dependent equation.