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Theory of Counter Experiments on Collision Electrons Ejected in Air by Mesotrons

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It was found in the preceding paper that mesotrons are associated with particles present at several meters distance from the mesotron trajectories. At least part of these particles are collision electrons ejected by these mesotrons along their path in the air. The theoretical results obtained in the present paper show that: (a) Only mesotrons of momenta $p > p_0 \sim 10^8$ ev/c can eject in air collision electrons sufficiently energetic to reach and to traverse a counter tube (C) placed at a distance D (a few meters) from the mesotron trajectory. (b) The number ν of collision electrons arising from a single mesotron and which are able to discharge (C) is practically independent of the mesotron momentum p if the influence of the spin of the mesotron is neglected and is given, in first approximation, by $\nu_0 = (BS/\pi D) \times \theta_{max}(D, E_1)$. In this formula, if the effective area of (C) is expressed in m² and D in meters, then $B = 10^{-2}$ for standard air. θ_{max} is a function of D and of the minimum energy E_1 necessary for an electron to penetrate the wall of the counter. For $E_1 = 2$ Mev, for example, θ_{max} decreases from 0.50 for $D = 2$ m, to 0.30 for $D = 10$ m. (c) When the mesotron spin (0 or $\frac{1}{2}$) is taken into account, ν is about equal to $\frac{1}{2}\nu_0$ for the lowest mesotron momenta, but approaches asymptotically ν_0 as p increases. (d) A fraction of these collision electrons can give, by their subsequent cascade multiplication in air, a certain contribution to the counting rate in (C), in excess of ν. However, when averaged adequately over the mesotron spectrum, this shower effect appears as a small correction only.

THE preceding paper,[1] here referred to as II, describes experiments (III) which show that mesotrons are associated with particles present at several meters from the mesotron trajectories. At least a part of these particles are collision electrons ejected by these mesotrons along their path in the air.

The method used in these experiments consists of measuring the frequency of discharges occurring simultaneously (a) in a group of counter tubes, responding in principle only to mesotrons, and (b) in a counter tube located at a variable distance from that group. This group of counters, which will be called here mesotron selector or, in short, selector, is surrounded by a lead shield 10 cm thick; it consists of two counters in coincidence placed in a vertical plane, and of two side counters, connected in anticoincidence to the circuit of the two first counters. The principle of this selector is based on the assumption that any event in which the two counters in coincidence are discharged, but not any of the side counters, should be attributed to a penetrating particle, such as a mesotron, since a high energy soft particle would almost always emerge below the lead with a shower discharging at least one of the side counters.

The purpose of the present paper is to calculate the mean frequency of discharges in the distant counter, which arise from collision electrons ejected in air by those mesotrons which traverse simultaneously the selector.

A. NOTATIONS AND FORMULAS

1. The basic formulas for collision processes between mesotrons and electrons will be applied here in the form given by B. Rossi and K. Greisen.[2] The notations used below are similar to those of the authors quoted.

The velocity of light will be put $c = 1$; the velocity of a particle: $v = \beta c = \beta$; 1 Mev will be taken as a unit of energy and 1 meter as a unit of length. In that way, the rest mass of an electron μ_e or of a mesotron μ will be expressed by the same symbol as their respective rest energies. These energies are assumed to have the following values: $\mu_e = 0.5$ and $\mu = 100$. Instead of the mesotron momentum p, the ratio $r = p/\mu = \beta/(1-\beta^2)^{\frac{1}{2}}$ will mostly be used. This ratio r will be called the equivalent momentum of the mesotron.

2. As a result of a collision between a mesotron of momentum p and an electron, supposed to be free and initially at rest, the latter is projected at an angle θ with respect to the initial trajectory of the mesotron and acquires an energy E' given by

$$E' = 2\mu_e \frac{p^2 \cos^2\theta}{[\mu_e{}^2 + (p^2+\mu^2)^{\frac{1}{2}}] - p^2\cos^2\theta}. \quad (1)$$

[1] Anatole Rogozinski, Phys. Rev. **65**, 291 (1944).

[2] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

It will be seen later that the values of p to be taken into account for the present case will always be appreciably larger than μ. Therefore, neglecting μ_e^2 as well as μ^2 in comparison with p^2, and remembering that $2\mu_e=1$, (1) will be expressed by

$$E'=\frac{\cos^2\theta}{\sin^2\theta+1/r^2+1/p}. \quad (2)$$

3. It can be seen from (2) that, whatever the energy of the primary mesotron, one has always

$$E'<\cot^2\theta.$$

Thus, if, for example, $\theta=\pi/4$, the maximum energy which can be transferred to the collision electron by a mesotron of any energy is always less than 1 (Mev).

When $1/r^2\gg 1/p$, or $p\ll\mu^2=10^4$, that is, for momenta $p\ll 10^{10}$ ev/c, (2) can be written in the simpler form

$$E'=\frac{\cos^2\theta}{\sin^2\theta+1/r^2}. \quad (2')$$

On the other hand, for $p\gg 10^{10}$ ev/c, (2) becomes

$$E'=\frac{\cos^2\theta}{\sin^2\theta+1/p}. \quad (2'')$$

For head-on collisions ($\theta=0$), E' reaches its maximum value E_m', given by

$$E_m'=1/(1/r^2+1/p)=\frac{p^2}{\mu^2+p}, \quad (3)$$

$$E_m'=r^2, \quad \text{for} \quad p\ll\mu^2, \quad (3')$$

$$E_m'=p, \quad \text{for} \quad p\gg\mu^2. \quad (3'')$$

We shall use the expression (2′) since it can be applied to the majority of the mesotrons present in the cosmic radiation. It should be understood, however, that for $p\gg\mu^2$, r^2 should eventually be replaced by p wherever it occurs in a formula which is derived originally from (2′).

B. ASSUMPTIONS

The following assumptions are made: (a) The energy loss of the mesotron is neglected over a distance H_p which will be specified in the calculations; (b) the mesotron trajectory is a straight line, regardless of any collision processes which occur over the same distance H_p; (c) the scattering effects for both the mesotron and the collision electron are neglected; (d) the collision electrons lose energy only by ionization; the average ionization loss k per meter of standard air is taken as $k=0.25$ (Mev); the minimum energy E_1 necessary for the collision electron to penetrate the wall of a counter tube is of the order of a few Mev; (e) the atmosphere is homogeneous over the distance H_p considered in (a).

C. GEOMETRY; ENERGY FACTORS

Let us consider now a mesotron of momentum p, directed downwards along the axis Oz. At A, in a plane perpendicular to Oz at O, a counter tube (C) is placed whose effective dimensions are assumed to be small in comparison with $OA=D$ (see Fig. 1A).

This counter can be discharged by a collision electron of energy E' ejected at a point T, located at a distance $OT=H$ from the origin.

The total energy loss ($kD/\sin\theta+E_1$) suffered by the collision electron along the path $TA=D/\sin\theta$ in air, and by traversing the wall of the counter tube, should be such that

$$E'-(kD/\sin\theta+E_1)\geqslant 0, \quad (4)$$

or, replacing E' by the expression (2′) and putting $\sin\theta=x$ one has

$$f(x)=x^3+ax^2+bx+c\leqslant 0, \quad (5)$$

with

$$a=kD/(E_1+1), \quad b=-(r^2-E_1)/r^2(E_1+1)$$

and

$$c=kD/r^2(E_1+1).$$

Collision electrons which satisfy (4) or (5) will be called "efficient" collision electrons.

It can be proved that the equation $f(x)=0$ has in general 3 roots: one negative, x_1, and two positive, x_2 and x_3, and that x should lie in between the positive roots in order to satisfy the physical conditions of the problem.

The general solution of (5) involves functions of the three parameters D, r, and E_1 requiring rather laborious calculations. It is more practical to proceed in the following way:

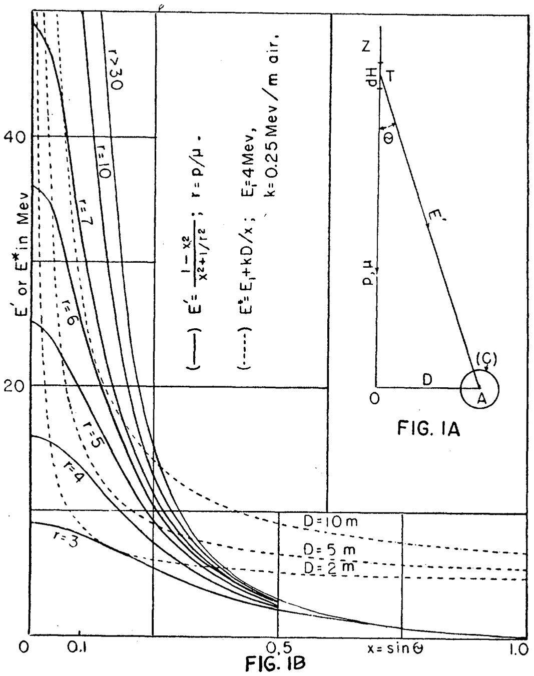

FIG. 1A. A mesotron of momentum p traveling along Oz ejects in T a collision electron of energy E' which discharges the counter tube (C) placed at a distance $OA=D$ from the mesotron trajectory.

FIG. 1B. Families of curves $E'=\cos^2\theta/(\sin^2\theta+1/r^2)$ and $E^*=E_1+kD/\sin\theta$ as a function of $x=\sin\theta$ for different values of the parameters $r=p/\mu$ and D [Eqs. (6) and (7)]. The minimum energy E_1 necessary for an electron to penetrate the wall of the counter (C) is taken $E_1=4$ Mev. In order to be able to discharge (C), a collision electron must be ejected at an angle θ such that $\sin\theta$ lies between the respective abscissae of the two points at which E' and E^* intersect.

Consider the two families of curves:

$$E'=(1-x^2)/(x^2+1/r^2) \tag{6}$$

and

$$E^*=E_1+kD/x \tag{7}$$

for different values of the parameters r, D, and E_1 as shown in Fig. 1B. The curves show that, in general, E' and E^* intersect at two points, the abscissae of which are identical with the two positive roots, x_2 and x_3, of the equation $f(x)=0$. One can also see that for any values of the parameters D and E_1 there exists a critical value $r_c(D, E_1)$ for which the corresponding curves E' and E^* are tangent to one another. The abscissa for the point of tangency gives directly the double root $x_d=x_2=x_3$, which $f(x)=0$ admits in this case.

For $r<r_c$ the roots are imaginary: This means, from the physical point of view, that mesotrons of momenta smaller than a certain critical momentum $p_c=\mu r_c$ will never project a collision electron which would be able to discharge the counter (C). On the other hand, it is easy to see that, for a given E_1, and for mesotrons of, say, $r=r_0$, collision electrons can discharge the counter only at distances D smaller than a certain critical distance $D_c(E_1, r_0)$. Beyond D_c, no collision electrons arising from mesotrons of $r<r_0$ are able to discharge the counter (C). The critical distance D_c can be determined best from the two families of curves E' and E^*, plotted as a function of $y=1/\sin\theta=1/x$,

$$E'=(y^2-1)/(y^2/r^2+1) \tag{6'}$$

and

$$E^*=E_1+kDy \tag{7'}$$

since then the family E^* is composed of straight lines (Fig. 2).

Table I gives approximate values of $r_c(D, E_1)$ for different distances D and for $E_1=4$ (Mev).

According to this table, if, for example, $D=5$ m, only mesotrons of momenta $p>4\times10^8$ ev/c will have to be considered at all.

It can be seen also from Figs. 1B and 2 that for $r>r_c$ the corresponding values of x_2 become small rapidly in comparison with x_3 and that x_3 can be considered as *independent of* r for values of r only

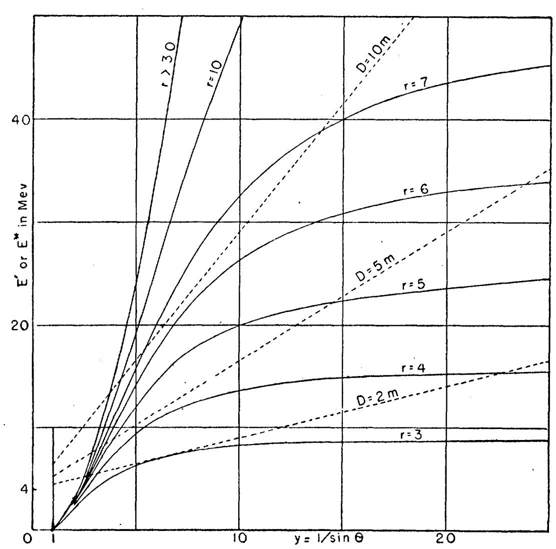

FIG. 2. Same families of curves E' and E^* as in Fig. 1B, but plotted as a function of $y=1/\sin\theta$ instead of $x=\sin\theta$ [Eqs. (6′) and (7′)].

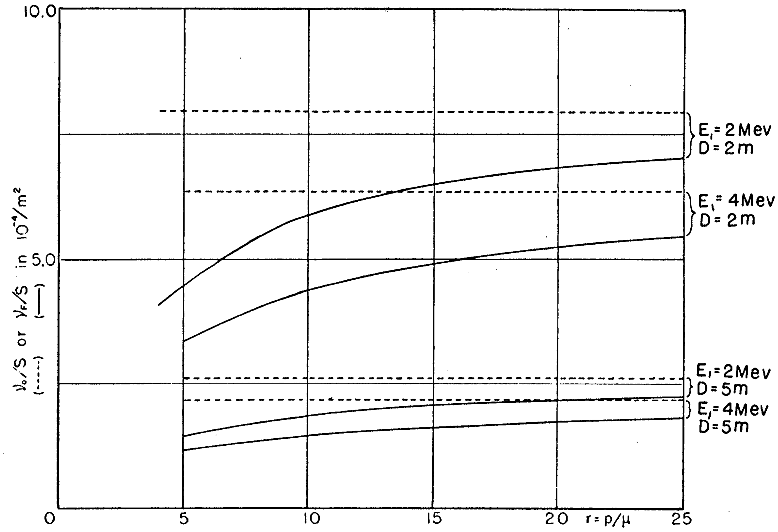

FIG. 3. Curves of ν/S expressed in m^{-2}, as a function of $r=p/\mu$ for different values of the parameters D and E_1. S is the effective area, in m^2, of the counter tube (C). In case of ν_F the mesotron spin (0 or $\frac{1}{2}$) is taken into account. When the spin term is neglected, and other less important approximations are made, ν reduces to ν_0, which is practically independent of the mesotron momentum.

slightly larger than r_c. In first approximation, this value of x_3 can be taken as equal to its asymptotic value $(x_3)_{\lim}$ for $r\rightarrow\infty$. $(x_3)_{\lim}$ can be calculated easily in this case from (5), which becomes then:

$$f(x)=x^2+\frac{kD}{E_1+1}x+\frac{1}{E_1+1}=0. \qquad (8)$$

For $D<\sim 20$ meters, one obtains

$$(x_3)_{\lim}\simeq\frac{1}{(E_1+1)^{\frac{1}{2}}}\left[1+\frac{k^2D^2}{8(E_1+1)}\right]-\frac{kD}{2(E_1+1)}. \qquad (9)$$

If D is small, $(x_3)_{\lim}\rightarrow(E_1+1)^{-\frac{1}{2}}$.

Table II gives certain values of $(x_3)_{\lim}$, for different values of E_1 and D.

From the fact that x_2 becomes small rapidly in comparison with x_3, for $r>r_c$, one can show that, with a good approximation,

$$x_2=\frac{kD}{r^2-E_1}. \qquad (10)$$

The values of x_2 and x_3 determine the "useful" portion $H_p=H_2-H_3$ of the mesotron trajectory which, through the collision electrons ejected along H_p, contributes to the discharges of the counter (C). The collision electrons ejected below $H_3=D\cot\theta_3$ and above $H_2=D\cot\theta_2=D/x_2-(1/k)(r^2-E_1)$, have insufficient energies to reach the counter and to penetrate its wall. Since $\cot\theta_3$ has a value comparable to unity and is practically independent of the mesotron momentum for $r>r_c$, H_3 is practically of the same order of magnitude as D, for all momenta of the mesotron considered. On the other hand, H_2 is a strongly increasing function of r, but it is practically independent of D. For example, for $r=10$, that is, for $p=10^9$ ev/c, $H_2=400$ meters. Hence, the useful portion H_p of the mesotron trajectory is approximately given by H_2.

TABLE I. Values of $r_c(D, E_1)$ for $E_1=4$ (Mev).

D meters	2	5	10
r_c	3	4	6

TABLE II. Values of $(x_3)_{\lim}$.

E_1 (Mev)	D meters 2	5	10
2	0.50	0.41	0.30
4	0.40	0.34	0.27

D. COLLISION PROBABILITIES

The probability for a mesotron of momentum p and of velocity β to project, over a distance dH of its trajectory, collision electrons of energy between E' and $E'+dE'$ is given by

$$PdE'dH=\frac{B}{\beta^2}\frac{1}{E'^2}(1-F)dE'dH. \qquad (11)$$

For standard air, if E' is expressed in Mev and H in meters, one obtains $B=0.98\times 10^{-2}\simeq 10^{-2}$.

The term F depends upon the mesotron spin which is not yet known. However, the results of experiments[3] on bursts occurring in an ionization chamber shielded by absorbers of different atomic number show a considerable disagreement with the hypothesis of mesotrons of spin 1; the agreement is much better for spin 0 or spin $\frac{1}{2}$. For these reasons, only the latter values of the mesotron spin will be considered here.

One has

$$F=\beta^2E'/E_m' \quad \text{for spin } 0 \qquad (12)$$

and

$$F=\beta^2E'/E_m'-E'^2/2(p^2+\mu^2) \quad \text{for spin } \tfrac{1}{2}.$$

In case of spin $\frac{1}{2}$, the maximum value of the second term of F is equal to $p^4/2(p+\mu^2)^2(p^2+\mu^2)$ [see (3)]; for $p\ll\mu^2$, this term is always negligible; it is equal to $\frac{1}{8}$, for $p=\mu^2$, and approaches $\frac{1}{2}$, for $p\gg\mu^2$. When averaged over the mesotron spectrum, this term will constitute only a very small correction, and therefore, no distinction will be made in the present calculations between spin 0 and spin $\frac{1}{2}$.

For small values of E', one obtains in all cases practically $F=0$, and P then reduces to the Rutherford formula

$$P=B/\beta^2E'^2. \qquad (13)$$

E. FREQUENCY OF THE EFFICIENT COLLISION ELECTRONS

(a) Approximate Formulas

1. Let N be the mean number of mesotrons per unit time following the axis Oz and $Ng(r)dr$ their fraction with equivalent momenta between r and $r+dr$. The function $g(r)$ represents the differential mesotron spectrum. The integral mesotron spectrum will be represented by the symbol $G(r)=\int_0^r g(r)dr$.

The differential spectrum $g(r)$ is normalized by

$$\int_0^\infty g(r)dr=1. \qquad (14)$$

2. It will be assumed first $F=0$.

The mean number ν of collision electrons arising from a mesotron and able to discharge the counter tube (C) is given by

$$\nu=\int_{H_3}^{H_2} dH\int_{(\Delta E')} PdE'. \qquad (15)$$

In (15), H_2 and H_3 determine the useful portion $H_p=H_2-H_3$ of the mesotron trajectory and are given, in first approximation, by (9) and (10), respectively. The energy difference $(\Delta E')$ is defined in the following way:

Let us consider first two cones forming angles θ and $\theta+d\theta$ with respect to their common axis Oz, and having T as their common vertex (Fig. 1A). The energy interval of the collision electrons ejected along dH into the solid angle $d\omega=2\pi\sin\theta d\theta$ which is comprised between the two cones is

$$dE'=\frac{1}{2\pi\sin\theta}\frac{dE'}{d\theta}d\omega. \qquad (16)$$

Let d be the effective diameter of the counter tube (C), l, its effective length, and $S=d\times l$, its effective area. Since we have assumed that the two dimensions of the counter are small in comparison with D, the values of E', P, and $dE'/d\theta$ can be considered as constant all over the area S which defines a solid angle $\Delta\omega=S/(AT)^2=S\sin^2\theta/D^2$; thus,

$$\int_{(\Delta E')} PdE'=\int_{(\Delta\omega)}\frac{P}{2\pi\sin\theta}\frac{dE'}{d\theta}d\omega =\frac{S\sin\theta}{2\pi D^2}P\frac{dE'}{d\theta}. \qquad (17)$$

On the other hand $dH=-(D/\sin^2\theta)d\theta$; hence,

$$\nu_{F=0}=\int_{H_3}^{H_2} dH\int_{(\Delta E')} PdE' =\frac{BS}{\pi D}\frac{1+r^2}{r^2}\int_{\theta_{\min}}^{\theta_{\max}}\frac{d\theta}{\cos^3\theta}, \qquad (18)$$

[3] M. Schein and P. S. Gill, Rev. Mod. Phys. **11**, 267 (1939). R. F. Christy and S. Kusaka, Phys. Rev. **59**, 405, 414 (1941). S. Kusaka, Phys. Rev. **64**, 256 (1943). R. E. Lapp, Phys. Rev. **64**, 255 (1943).

where $\theta_{\min}=\arcsin x_2 \simeq kD/(r^2-E_1)$ and $\theta_{\max}=\arcsin x_3$. The angle $\theta_{\max}$, as well as x_3, can be considered as independent of $r>r_c$.

Developing in (18) $\cos^3\theta$ in a series, one obtains

$$\nu_{F=0}=\frac{BS}{\pi D}\frac{1+r^2}{r^2}(\theta_{\max}+\tfrac{1}{2}\theta_{\max}{}^3-\theta_{\min}+\cdots), \quad (19)$$

where the terms neglected in the parenthesis are of power $\geqslant 5$ for $\theta_{\max}$, and of power $\geqslant 3$ for $\theta_{\min}$.

It has been seen that except for values of r very close to r_c, the value of x_2 or $\theta_{\min}$ is small in comparison with $\theta_{\max}$. Therefore, in first approximation, $\nu_{F=0}$ is equal to

$$\nu_0=\frac{BS}{\pi D}\theta_{\max}. \quad (20)$$

It should be emphasized that within the limits of the approximations assumed, the value of ν_0 which represents the contribution of a single mesotron to the number of discharges in the counter (C) is practically independent of the momentum of that mesotron; this is true in spite of the fact that the useful portion of the mesotron trajectory increases approximately proportional to r^2.

3. The mean frequency n of discharges in the counter (C) is given thus by:

$$n=\int_{r_c}^{\infty}Ng(r)dr\int_{H_3}^{H_2}dH\int_{(\Delta E')}PdE'$$
$$=N\int_{r_c}^{\infty}g(r)\nu_{F=0}dr. \quad (21)$$

In order to correlate n with those mesotrons which effectively discharge the selector, one should remark here that N is the frequency of the mesotrons which would be recorded by a hypothetical *unshielded* mesotron selector. In traversing the 10-cm thickness of the lead shield which, in fact, covers the selector (see II), all mesotrons of a range not larger than 10 cm of Pb are stopped. Let r_1 be the maximum equivalent momentum of these stopped mesotrons. The frequency of discharges in the mesotron selector N_{sel} will then be equal to

$$N_{\text{sel}}=N\int_{r_1}^{\infty}g(r)dr=N[1-G(r_1)]. \quad (22)$$

The exact shape of the differential mesotron spectrum, at least on the ground, is not yet known at its very low end. There are, however, experimental reasons to believe that the fraction of mesotrons with energies smaller than a few times 10^8 ev is rather small. Since for 10 cm of Pb one has $r_1\simeq 2.5$, corresponding to a mesotron energy of 1.7×10^8 ev, $G(r_1)$ can be considered as small and probably not larger than a few percent.

Taking into account (22), and using for ν its value from (20), one finds that n is given, in first approximation, by

$$n_0=N_{\text{sel}}\frac{1-G(r_c)}{1-G(r_1)}\frac{BS}{\pi D}\theta_{\max}. \quad (23)$$

According to Table I, the critical value r_c represents mesotron momenta of the order of a few times 10^8 ev/c. Therefore, $G(r_c)$ is, like $G(r_1)$, small and the ratio $[1-G(r_c)]/[1-G(r_1)]\simeq 1-G(r_c)+G(r_1)$ is approximately equal to 1. Hence,

$$n_0=N_{\text{sel}}\frac{BS}{\pi D}\theta_{\max}. \quad (24)$$

This (approximate) formula, which gives the frequency of discharges in a counter of effective area S placed at a distance D from the mesotron selector whose proper frequency of discharges is equal to N_{sel}, can be compared directly with the experimental data.

(b) Correction for the Mesotron Spin

In obtaining the above result the spin dependent factor F was neglected. It will now be taken into account (see Section D) in the case of spin 0, for which $F=\beta^2E'/E_m'$, and it will be assumed that $\beta=1$.

Only the final result of the evaluation of

$$\nu_F=\int_{H_3}^{H_2}dH\int_{(\Delta E')}P(1-F)dE' \quad (25)$$

will be given here. One finds:

$$\nu_F=\frac{BS}{\pi D}\left[\theta_{\max}+\frac{1+r^2}{2r^2}\theta_{\max}{}^3-\theta_{\min}-\frac{1}{r}h(r)\right],$$
$$h(r)=\arctan\,(r\sin\theta_{\max})-\arctan\,(r\sin\theta_{\min}).\quad(26)$$

For $r>30$, $h(r)$ can be considered as equal to its asymptotic value of $\pi/2$. On the other hand, the formula (26) shows immediately that ν_F approaches asymptotically ν_0 as r increases.

Table III gives $h(r)/r$ for different values of E_1 and D.

TABLE III. Values of $10^2\times h(r)/r$.

E_1 (Mev)	D meters	r = 4	5	6	7	8	9	10	20	30
2	2	24.5	22	19.7	17.6	15.9	14.4	13.3	7.2	4.9
2	5	—	17.4	16.6	15.4	14.2	13.2	12.2	7.1	4.8
4	2	21	19	18	16	15	14	13	7.1	4.8
4	5	—	15.4	14.8	14	13	12.4	11.6	7.0	4.7

In Fig. 3 the values of ν_0/S and ν_F/S are plotted as a function of r.

When F is taken into account, a graphical integration of $\int g(r)\nu_F dr$ will give the final result for n since $g(r)$ is an empirical function.

F. INFLUENCE OF THE SHOWER PRODUCTION BY THE EFFICIENT COLLISION ELECTRONS

So far, we have considered single collision electrons losing energy only by ionization and discharging directly the counter tube (C). We know, however, that when the energy of such an electron is larger than its critical energy E_{cr}, which is equal to $\sim 10^8$ ev for air, it can undergo a multiplication by successive radiation and pair production processes. As a result of this multiplication, the average range of the produced shower is smaller than would be the range of its parent collision electron if it would not have multiplied. Therefore, in order to produce any effect on the counting rate in (C), *in excess of* ν, the energy of the shower particles arising from an efficient collision electron must be high enough to reach and to traverse the wall of the counter, otherwise the above effect could become even negative. On the other hand, to obtain an appreciable effect, the number of these shower particles should be, in addition, not too low; therefore, the energy E' of the collision electron and the distance $R=D/\sin\theta$, which separates the point of its ejection on the mesotron trajectory from the counter tube, would have to be appreciably larger than $E_{cr}(\sim 10^8$ ev) and λ_0, respectively, $\lambda_0(\sim 300$ meters) representing the radiation unit of length in air.

Let ν_s be the number of discharges in (C), *in excess of* ν, arising indirectly, by the shower particles, from a collision electron ejected in air by a mesotron of momentum p. Let, furthermore, $\sigma=\nu_s/\nu$ be the relative shower effect. One has $\sigma\equiv 0$ as long as $E'\ll E_{cr}$. On the other hand, it can be understood easily that provided $R>\sim\lambda_0$, the factor σ passes a negative minimum (a) when $E'>E_{cr}$ remains close to E_{cr} and (b) for E' remaining in the neighborhood of the highest energy [see Figs. 1B, 2 and Eq. (10)] which an efficient collision electron can possess and which is given by $1/(\theta_{\min}{}^2+1/r^2)\simeq E_m'$. Indeed, in both cases, if a multiplication of the collision electron takes place, the energy of any of the produced shower particles will be too small to reach and to discharge the counter.

An upper limit of σ can be obtained as follows: According to the theory of cascade showers the average maximum number ψ of shower particles of energy larger than E_{cr}, produced by an electron of energy $E'\gg E_{cr}$, is given by $\psi=E'/6E_{cr}$. If one considers only values of E' for which $\psi\geqslant\psi_0$, one has evidently $E'\geqslant E_{\psi_0}'=6\psi_0 E_{cr}$. The above maximum occurs for a thickness $t\simeq\lambda_0\log\,(E'/E_{cr})$ of material in which the shower develops. In the present case only small values of θ will have to be considered. At any rate, $\theta<\theta_s=D/\lambda_0$. For $D<10$ m, $\theta_s\simeq 10^{-2}$.

Thus [compare (17)], neglecting θ^2, θ^3, $\cdots$ in comparison with θ,

$$\sigma<\frac{1}{\nu}\int_{D\cot\theta_s}^{D\cot\theta_{\min}}dH\int_{(\Delta E')}P(1-F)\psi dE'$$
$$=\frac{1}{\nu}\frac{BS}{2\pi D\times 6E_{cr}}\int_{E_{\psi_0}'}^{E_m'}\theta dE'.\quad(27)$$

It can be shown that for $p\ll 1/\theta_s{}^2$, that is, $p\ll\sim 10^{10}$ ev/c, and consequently also for $E_m'\ll 1/\theta_s{}^2$, the integral

$$\int_{E_{\psi_0}'}^{E_m'}\theta dE'<\theta_s(E_m'-E_{\psi_0}')$$

and σ is smaller than a few percent. The factor σ increases with increasing p. For $p \gg 1/\theta_s^2$, the integral

$$\int_{E_{\psi_0}'}^{E_m'} \theta dE' \rightarrow \frac{\pi}{2} p^{\frac{1}{2}},$$

hence,

$$\sigma < \frac{1}{\nu} \frac{BS}{2\pi D} \frac{\pi}{12E_{cr}} p^{\frac{1}{2}}. \tag{28}$$

Since the mesotron momenta to be considered in this case are very high, ν can be replaced by ν_0. Now, $BS/\nu_0 2\pi D$ is approximately equal to 1. Therefore

$$\sigma < (\pi/12E_{cr}) p^{\frac{1}{2}}. \tag{29}$$

It can be seen from (29) that σ will still remain smaller than 1 even for $p = (12E_{cr}/\pi)^2$, that is, for mesotron momenta as high as 1.5×10^{11} ev/c.

It will be sufficient to remember that the frequency of the mesotron momentum in the differential mesotron spectrum decreases as $p^{-\gamma} (\gamma > 2)$ to make clear that the shower effect of the collision electrons, when averaged adequately over the mesotron spectrum, adds a very small contribution only to the frequency of discharges in the counter tube (C).

G. ALTITUDE EFFECT

Concerning ν, the only terms which in (20) or (26) depend upon the altitude, that is, upon the density of air, are B, θ_{max}, and θ_{min}. The variation of θ_{min}, which is anyway very small, can be neglected. The coefficient B is directly proportional to the density of air, and θ_{max} depends upon this density only through the coefficient k. But, as can be seen from Figs. 1B or 2, θ_{max} depends, at least for small distances D, very little upon k. For example, for $E_1 = 4$ Mev and $D = 2$ m, θ_{max} would increase from ~ 0.40 at sea level ($k = 0.25$), to ~ 0.45 only at the top of the atmosphere ($k = 0$). Therefore, in first approximation, one can admit that ν is proportional to the density of air or to the atmospheric pressure. On the other hand, the effect of the altitude on the frequency n of discharges [see (21)] in the counter tube (C) follows, in the general case, the combined effects of: (a) the altitude increase of the intensity of the mesotron radiation, (b) the altitude dependence of the mesotron spectrum and, (c) the decrease of ν which, as has been seen above, is roughly proportional to the atmospheric pressure.

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On the Perturbation of Boundary Conditions

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The solution of both the scalar and vector wave equations in regions which are bounded by irregular surfaces which have non-uniform physical properties has been reduced to the solution of a secular equation. The secular determinant is Hermitian. The solution to the secular equation has been expressed in a form suitable for obtaining its value to any approximation. Similar results are given for the corresponding eigenfunctions. Extension of these results to the problem of scattering and to the situation where the bounding surfaces move is indicated. The description of a source located in such a region is also discussed.

I. INTRODUCTION

THE solution to many physical problems can be reduced to the solution of a system of partial differential equations with prescribed boundary conditions. In the past, exact solutions were limited to those cases in which the boundary conditions were "simple" and were satisfied on "simple" surfaces. "Simple" boundary conditions correspond to uniform physical properties of the surface involved. "Simple" surfaces are coordinate surfaces of coordinate systems in which the partial differential equations will separate. As a result, a great many physical problems of interest have not been treated theoretically.

This paper will discuss solutions of the scalar and vector wave equations satisfying non-simple