PHYSICAL REVIEW VOLUME 65, NUMBERS 7 AND 8 APRIL 1 AND 15, 1944

A Generalized Electrodynamics

Part II—Quantum

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(Received January 20, 1944)

When the Lagrangian from which the field equations are derived contains second and higher derivatives of the generalized field coordinates, the method of quantizing the field equations developed by Heisenberg and Pauli cannot be immediately applied. By generalizing a method due to Ostrogradsky for converting Lagrange's equations of motion of a particle, when higher derivatives are present, into canonical Hamiltonian form, it becomes possible to perform a similar transformation of the field equations. Applying this method to Podolsky's generalized electrodynamics, we obtain the Hamiltonian of the field and double the usual number of generalized coordinates and momenta. The quantization of the field follows without any special assumptions. The last two sections are devoted to the discussion of the auxiliary conditions and some of their consequences.

1. PRELIMINARIES. EXTENSION OF HEISENBERG AND PAULI'S[1] METHOD

THE basis of generalizing the Heisenberg and Pauli method of quantizing fields to the case when the Lagrangian of the field contains second derivatives of the potentials is contained in a suggestion due to Ostrogradsky.[2] He showed how the Lagrange equations of motion of a particle, when higher derivatives are present, can be transformed into the Hamiltonian form.

Suppose the Lagrangian L is a function of the potentials $\varphi_\alpha=(\mathbf{A}, i\varphi)$ as well as their first and second derivatives:[3]

$$L=L(\varphi_\alpha,\ \varphi_{\alpha,\beta},\ \varphi_{\alpha,\beta\gamma}),$$

where φ_α are functions of the space-time coordinates $x_\alpha=(x_1,\ x_2,\ x_3,\ x_4=ix_0=ict)$, and

$$\varphi_{\alpha,\beta}=\partial\varphi_\alpha/\partial x_\beta,\quad \varphi_{\alpha,\beta\gamma}=\partial^2\varphi_\alpha/\partial x_\beta\partial x_\gamma.$$

The variational equation

$$\delta W=\delta\int\int L dVdt=0,\quad dV=dx_1dx_2dx_3,$$

or

$$ic\delta W=\delta\int L d\Omega=0,\quad d\Omega=dVdx_4, \tag{1.1}$$

leads to the field equation

$$\frac{\partial L}{\partial\varphi_\alpha}-\frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial\varphi_{\alpha,\mu}}+\frac{\partial^2}{\partial x_\mu\partial x_\nu}\frac{\partial L}{\partial\varphi_{\alpha,\mu\nu}}=0, \tag{1.2}$$

provided φ_α and $\varphi_{\alpha,\mu}$ are specified and are unvaried over the boundaries of the four-dimensional manifold Ω over which the integration is performed.

We introduce as the new generalized coordinates

$$q_\alpha=\varphi_\alpha,\quad\text{and}\quad Q_\alpha=\partial\varphi_\alpha/\partial t\equiv\dot{q}_\alpha, \tag{1.3}$$

and define the momenta conjugate to q_α and Q_α by

$$p_\alpha=(\partial L/\partial\dot{q}_\alpha)-\partial/\partial t(\partial L/\partial\ddot{q}_\alpha)-\partial/\partial x_i(\partial L/\partial\dot{q}_{\alpha,i}) \tag{1.4}$$

and

$$P_\alpha=\partial L/\partial\ddot{q}_\alpha, \tag{1.5}$$

respectively. The Hamiltonian is then conveniently defined by

$$H=-L+p_\alpha\dot{q}_\alpha+P_\alpha\dot{Q}_\alpha. \tag{1.6}$$

The time derivatives of the coordinates, $\dot{q}_\alpha$ and $\dot{Q}_\alpha$, can in general be eliminated from the Hamiltonian by using Eqs. (1.3) and (1.5). The result is

$$H=H(q_\alpha,\ p_\alpha,\ q_{\alpha,i},\ q_{\alpha,ij},\ Q_\alpha,\ P_\alpha,\ Q_{\alpha,i}). \tag{1.7}$$

Taking the differentials of Eqs. (1.6) and (1.7)

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[1] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929).
[2] See E. T. Whittaker, *Analytical Dynamics* (1927), Chapter X.
[3] Greek indices will range from 1 to 4, while Latin subscripts from 1 to 3. Repeated indices are summed.

and equating coefficients we obtain:

$$\begin{aligned}\partial H/\partial Q_\alpha &= -\partial L/\partial \dot{q}_\alpha + p_\alpha,\\ \partial H/\partial q_{\alpha,i} &= -\partial L/\partial q_{\alpha,i},\\ \partial H/\partial q_{\alpha,ij} &= -\partial L/\partial q_{\alpha,ij},\\ \partial H/\partial Q_{\alpha,i} &= -\partial L/\partial \dot{q}_{\alpha,i},\end{aligned} \tag{1.8}$$

and

$$\partial H/\partial p_\alpha = Q_\alpha = \dot{q}_\alpha, \quad \partial H/\partial P_\alpha = \dot{Q}_\alpha = \ddot{q}_\alpha. \tag{1.9}$$

From Eqs. (1.4), (1.5), (1.8), (1.9), and (1.2) it now follows that

$$\partial H/\partial Q_\alpha = -\partial P_\alpha/\partial t - \partial/\partial x_i(\partial L/\partial \dot{q}_{\alpha,i}), \tag{1.10}$$

$$\partial p_\alpha/\partial t = -\partial H/\partial q_\alpha + \partial/\partial x_i(\partial H/\partial q_{\alpha,i}) - \partial^2/\partial x_i\partial x_j(\partial H/\partial q_{\alpha,ij}), \tag{1.11}$$

$$\partial P_\alpha/\partial t = -\partial H/\partial Q_\alpha + \partial/\partial x_i(\partial H/\partial Q_{\alpha,i}). \tag{1.12}$$

But, if we put

$$\delta\int H dV \equiv \delta\bar{H} = \int\{(\delta\bar{H}/\delta q_\alpha)\delta q_\alpha + (\delta\bar{H}/\delta Q_\alpha)\delta Q_\alpha + (\delta\bar{H}/\delta p_\alpha)\delta p_\alpha + (\delta\bar{H}/\delta P_\alpha)\delta P_\alpha\}dV, \tag{1.13}$$

the quantities

$$\delta\bar{H}/\delta q_\alpha = \partial H/\partial q_\alpha - \partial/\partial x_i(\partial H/\partial q_{\alpha,i}) + \partial^2/\partial x_i\partial x_j(\partial H/\partial q_{\alpha,ij}), \tag{1.14}$$

$$\delta\bar{H}/\delta Q_\alpha = \partial H/\partial Q_\alpha - \partial/\partial x_i(\partial H/\partial Q_{\alpha,i}), \tag{1.15}$$

$$\delta\bar{H}/\delta p_\alpha = \partial H/\partial p_\alpha, \text{ and } \delta\bar{H}/\delta P_\alpha = \partial H/\partial P_\alpha \tag{1.16}$$

are the functional derivatives of $\bar{H}$. Hence from Eqs. (1.9), (1.11), (1.12), (1.14), and (1.15), the canonical equations

$$\dot{p}_\alpha = -\delta\bar{H}/\delta q_\alpha, \quad \dot{P}_\alpha = -\delta\bar{H}/\delta Q_\alpha, \tag{1.17}$$

$$\dot{q}_\alpha = \delta\bar{H}/\delta p_\alpha, \quad \dot{Q}_\alpha = \delta\bar{H}/\delta P_\alpha, \tag{1.18}$$

follow. We note that

$$d\bar{H}/dt = \int\{(\delta\bar{H}/\delta q_\alpha)\dot{q}_\alpha + (\delta\bar{H}/\delta p_\alpha)\dot{p}_\alpha + (\delta\bar{H}/\delta Q_\alpha)\dot{Q}_\alpha + (\delta\bar{H}/\delta P_\alpha)\dot{P}_\alpha\}dV = 0. \tag{1.19}$$

The energy-momentum tensor can be obtained from the variational equation, where now not only the potentials φ_α but also the boundaries are varied. Let the four-dimensional manifold Ω be bounded by two open 3 spaces, having coordinates ξ_α and x_α, respectively. The boundary ξ_α remains fixed; the other x_α is to be displaced by the amount δx_α, while (a) the potentials themselves and their derivatives $\varphi_{\alpha,\mu}$ remain unchanged over the boundary, and (b) Eqs. (1.2) are satisfied for every interior point. Then

$$ic\delta W = \int_{\xi_\beta}^{x_\beta+\delta x_\beta}(L+\delta L)d\Omega - \int_{\xi_\beta}^{x_\beta} Ld\Omega, \tag{1.20}$$

where ξ_β and x_β symbolize the original boundaries and ξ_β and $x_\beta+\delta x_\beta$ the new boundaries. Therefore,

$$ic\delta W = \int_{\xi_\beta}^{x_\beta}\delta L d\Omega + \int_{\xi_\beta} L dS_\mu \delta x_\mu, \tag{1.21}$$

where the last integral is a surface integral over the old x_β boundary. From condition (a),

$$\delta\varphi_\alpha = -(\partial\varphi_\alpha/\partial x_\mu)\delta x_\mu$$

and

$$\partial/\partial x_\mu(\delta\varphi_\alpha) = -(\partial^2\varphi_\alpha/\partial x_\lambda\partial x_\mu)\delta x_\lambda. \tag{1.22}$$

By use of these results and

$$\int_{\xi_\beta}^{x_\beta}\delta L d\Omega = \int_{\xi_\beta}^{x_\beta}\{(\partial L/\partial\varphi_\alpha)\delta\varphi_\alpha + (\partial L/\partial\varphi_{\alpha,\mu})\delta\varphi_{\alpha,\mu} + (\partial L/\partial\varphi_{\alpha,\mu\nu})\delta\varphi_{\alpha,\mu\nu}\}d\Omega \tag{1.23}$$

Eq. (1.21) can be transformed to

$$ic\delta W = \int\{L\delta_{\mu\nu} - \varphi_{\alpha,\mu}p_{\alpha\nu} - \varphi_{\alpha,\mu\lambda}P_{\alpha\lambda\nu}\}dS_\nu\delta x_\mu, \tag{1.24}$$

where

$$p_{\alpha\beta} = \partial L/\partial\varphi_{\alpha,\beta} - \partial/\partial x_\mu(\partial L/\partial\varphi_{\alpha,\beta\mu}), \tag{1.25}$$

$$P_{\alpha\beta\gamma} = \partial L/\partial\varphi_{\alpha,\beta\gamma}. \tag{1.26}$$

From the definition of the total momentum $\mathbf{P}_\mu$

$$\delta W = \mathbf{P}_\mu\delta x_\mu.$$

We obtain upon comparison with Eq. (1.24)

$$\mathbf{P}_\mu = \int t_{\mu\nu}dS_\nu, \tag{1.27}$$

where the energy-momentum tensor $t_{\mu\nu}$ is given by

$$ict_{\mu\nu} = L\delta_{\mu\nu} - \varphi_{\alpha,\mu}p_{\alpha\nu} - \varphi_{\alpha,\mu\lambda}P_{\alpha\lambda\nu}. \tag{1.28}$$

It can then be shown that this tensor satisfies the equation of conservation of energy and momentum, namely:

$$t_{\mu\nu,\nu}=0. \tag{1.29}$$

2. PRELIMINARIES. SPECIAL LAGRANGIANS

As an application of the foregoing discussion, consider the Lagrangian proposed by Podolsky.[4]

$$L=-\tfrac{1}{2}\{\tfrac{1}{2}F_{\alpha\beta}F_{\alpha\beta}+a^2F_{\alpha\beta,\beta}F_{\alpha\gamma,\gamma}\}, \tag{2.1}$$

where the field quantities

$$F_{\alpha\beta}=\varphi_{\beta,\alpha}-\varphi_{\alpha,\beta}$$

are now in Heaviside-Lorentz units. The field equations

$$(1-a^2\Box)F_{\alpha\mu,\mu}=0 \tag{2.2}$$

now follow from Eq. (1.2). To compute the energy-momentum tensor we need the quantities

$$\partial L/\partial\varphi_{\alpha,\nu}=F_{\alpha\nu} \tag{2.3}$$

and

$$\partial L/\partial\varphi_{\alpha,\nu\lambda}=-a^2(F_{\nu\beta,\beta}\delta_{\alpha\lambda}-F_{\alpha\beta,\beta}\delta_{\nu\lambda}). \tag{2.4}$$

Hence from Eqs. (1.25) and (1.26) we obtain

$$p_{\alpha\nu}=(1-a^2\Box)F_{\alpha\nu}, \tag{2.5}$$

$$P_{\alpha\lambda\nu}=-a^2(F_{\lambda\beta,\beta}\delta_{\alpha\nu}-F_{\alpha\beta,\beta}\delta_{\lambda\nu}). \tag{2.6}$$

Equation (1.28) thus becomes

$$ict_{\mu\nu}=L\delta_{\mu\nu}-\varphi_{\alpha,\mu}(1-a^2\Box)F_{\alpha\nu}+a^2F_{\alpha\nu,\mu}F_{\alpha\beta,\beta}, \tag{2.7}$$

from which the Hamiltonian is

$$H=-ict_{44}=L-\varphi_{\alpha,4}(1-a^2\Box)F_{\alpha4}+a^2F_{\alpha4,4}F_{\alpha\beta,\beta}. \tag{2.8}$$

Care must be exercised in computing the momenta canonically conjugate to φ_α and $\dot{\varphi}_\alpha$ because of the ambiguity in the partial differentiation with respect to $\dot{\varphi}_{\alpha,i}$. We have from Eq. (1.4)

$$p_\alpha=\frac{1}{ic}\left[\frac{\partial L}{\partial\varphi_{\alpha,4}}-\frac{\partial}{\partial x_4}\frac{\partial L}{\partial\varphi_{\alpha,44}}-\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial\varphi_{\alpha,4j}}+\frac{\partial L}{\partial\varphi_{\alpha,j4}}\right)\right]=\frac{1}{ic}[F_{\alpha4}+a^2(F_{4\beta,\beta j}\delta_{\alpha j}-F_{\alpha\beta,\beta4})]. \tag{2.9}$$

[4] B. Podolsky, Phys. Rev. **62**, 68 (1942), Eq. (3.8) Note the typographical error. This equation should read

$$L_f=-\frac{1}{8\pi}\{\tfrac{1}{2}F_{\alpha\beta}{}^2+a^2(\partial F_{\alpha\beta}/\partial x_\beta)^2\}.$$

Henceforth this paper will be designated by GE.

Thus we obtain

$$p_4=\frac{a^2}{ic}F_{j4,j4} \tag{2.10}$$

and

$$p_j=\frac{1}{ic}(1-a^2\Box)F_{j4}; \tag{2.11}$$

having made use of the symmetry properties of $F_{\alpha\beta}$, namely:

$$F_{\alpha\beta}=-F_{\beta\alpha}\quad\text{and}\quad F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}=0.$$

Similarly

$$P_\alpha=(a^2/c^2)(F_{4\beta,\beta}\delta_{\alpha4}-F_{\alpha\beta,\beta}) \tag{2.12}$$

or

$$P_4=0,\quad P_j=-(a^2/c^2)F_{j\beta,\beta}. \tag{2.13}$$

In vector notation these quantities become:

$$\begin{aligned}&p_4=i(a^2/c^2)\,\mathrm{div}\,\mathbf{E},\\&\mathbf{p}=-(1/c)(1-a^2\Box)\mathbf{E},\\&P_4=0,\quad \mathbf{P}=-(a^2/c^2)(\mathrm{curl}\,\mathbf{H}-(1/c)\dot{\mathbf{E}}).\end{aligned} \tag{2.14}$$

We note that p_4 no longer vanishes, as it does in the usual electrodynamics, but because of the vanishing P_4, we shall encounter the usual difficulty in quantization.

This difficulty can be avoided by using an artifice similar to that used by Fermi[5] and Rosenfeld.[6] Using a modified Lagrangian,

$$L=-\tfrac{1}{2}(\varphi_{\alpha,\beta}\varphi_{\alpha,\beta}+a^2\varphi_{\alpha,\beta\beta}\varphi_{\alpha,\gamma\gamma}), \tag{2.15}$$

we find for the field equations

$$(1-a^2\Box)\Box\varphi_\alpha=0. \tag{2.16}$$

If we further require that these potentials φ_α give the same field equations, Eq. (2.2), we must have

$$(1-a^2\Box)F_{\alpha\mu,\mu}=(1-a^2\Box)\varphi_{\mu,\alpha\mu}-(1-a^2\Box)\varphi_{\alpha,\mu\mu}=0 \tag{2.17}$$

or, using Eq. (2.16), we have

$$(1-a^2\Box)\varphi_{\mu,\alpha\mu}=0 \tag{2.18}$$

for the additional conditions to be imposed on the potentials. Since

$$\begin{aligned}\partial L/\partial\varphi_{\alpha,\nu}&=-\varphi_{\alpha,\nu},\\ \partial L/\partial\varphi_{\alpha,\lambda\nu}&=-a^2\varphi_{\alpha,\beta\beta}\delta_{\lambda\nu},\end{aligned} \tag{2.19}$$

[5] E. Fermi, Rev. Mod. Phys. **4**, 131 (1932).
[6] L. Rosenfeld, Zeits. f. Physik **76**, 729 (1932).

we have, by Eqs. (1.25) and (1.26),

$$p_{\alpha\nu}=-(1-a^2\Box)\varphi_{\alpha,\nu},\quad P_{\alpha\lambda\nu}=-a^2\delta_{\lambda\nu}\Box\varphi_\alpha. \quad (2.20)$$

Therefore, by Eq. (1.28),

$$ict_{\mu\nu}=L\delta_{\mu\nu}+\varphi_{\alpha,\mu}(1-a^2\Box)\varphi_{\alpha,\nu}+a^2\varphi_{\alpha,\mu\nu}\Box\varphi_\alpha, \quad (2.21)$$

from which follows the Hamiltonian

$$H=-L-\varphi_{\alpha,4}(1-a^2\Box)\varphi_{\alpha,4}-a^2\varphi_{\alpha,44}\Box\varphi_\alpha. \quad (2.22)$$

The momenta canonically conjugate to φ_α and $\dot{\varphi}_\alpha$ are

$$p_\alpha=-(1-a^2\Box)\dot{\varphi}_\alpha \text{ and } P_\alpha=(a^2/c^2)\Box\varphi_\alpha, \quad (2.23)$$

and satisfy the equations

$$\Box p_\alpha=0 \quad\text{and}\quad (1-a^2\Box)P_\alpha=0. \quad (2.24)$$

3. REDUCTION OF FIELD QUANTITIES AND THE HAMILTONIAN TO FOURIER AMPLITUDES

Following Fock and Podolsky,[7] we shall find it convenient to express the field quantities and the Hamiltonian in terms of the Fourier amplitudes of the potentials. Since,[8] in order to satisfy Eq. (2.16),

$$\varphi_\alpha(\mathbf{r},x_0)=\int(M_\alpha+\tilde{M}_\alpha)dk, \quad (3.1)$$

where

$$\begin{aligned} M_\alpha&=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\{\varphi_\alpha(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-kx_0)]\\ &\quad+\varphi_\alpha^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}-kx_0)]\},\\ \tilde{M}_\alpha&=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\{\tilde{\varphi}_\alpha(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)]\\ &\quad+\tilde{\varphi}_\alpha^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)]\}, \end{aligned} \quad (3.2)$$

the notation being the same as in GE, Eq. (6.2). Further, for each field variable F we introduce the Fourier amplitudes defined by the equation

$$\begin{aligned} F=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int&\{F(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-kx_0)]\\ &+F^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}-kx_0)]\\ &+\tilde{F}(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)]\\ &+\tilde{F}^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)]\}dk. \end{aligned} \quad (3.3)$$

[7] V. Fock and B. Podolsky, Physik. Zeits. Sowjetunion 1, 801 (1932). This paper will be designated as FP.

[8] GE, Eq. (6.2).

In terms of these quantities then the relations between the field strengths and the potentials become:

$$\begin{aligned} \mathbf{H}(\mathbf{k})&=i\mathbf{k}\times\mathbf{A}(\mathbf{k}),\\ \tilde{\mathbf{H}}(\mathbf{k})&=i\mathbf{k}\times\tilde{\mathbf{A}}(\mathbf{k}),\\ \mathbf{H}^*(\mathbf{k})&=-i\mathbf{k}\times\mathbf{A}^*(\mathbf{k}),\\ \tilde{\mathbf{H}}^*(\mathbf{k})&=-i\mathbf{k}\times\tilde{\mathbf{A}}^*(\mathbf{k}),\\ \mathbf{E}(\mathbf{k})&=i(\mathbf{k}\varphi(\mathbf{k})-k\mathbf{A}(\mathbf{k})),\\ \tilde{\mathbf{E}}(\mathbf{k})&=i(\mathbf{k}\tilde{\varphi}(\mathbf{k})-\tilde{k}\tilde{\mathbf{A}}(\mathbf{k})),\\ \mathbf{E}^*(\mathbf{k})&=-i(\mathbf{k}\varphi^*(\mathbf{k})-k\mathbf{A}^*(\mathbf{k})),\\ \tilde{\mathbf{E}}^*(\mathbf{k})&=-i(\mathbf{k}\tilde{\varphi}^*(\mathbf{k})-\tilde{k}\tilde{\mathbf{A}}^*(\mathbf{k})). \end{aligned} \quad (3.4)$$

The reduction of the Hamiltonian to the Fourier amplitudes is a little more difficult. Putting

$$\begin{aligned} N_\alpha&=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\{\varphi_\alpha(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-kx_0)]\\ &\quad-\varphi_\alpha^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}-kx_0)]\} \end{aligned}$$

and

$$\begin{aligned} \tilde{N}_\alpha&=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\{\tilde{\varphi}_\alpha(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)]\\ &\quad-\tilde{\varphi}_\alpha^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)]\}, \end{aligned} \quad (3.5)$$

we obtain for the derivatives of the potentials

$$\begin{aligned} \varphi_{\alpha,4}&=-\int\{kN_\alpha+\tilde{k}\tilde{N}_\alpha\}dk,\\ \varphi_{\alpha,j}&=i\int\{k_jN_\alpha+k_j\tilde{N}_\alpha\}dk,\\ \Box\varphi_\alpha&=(1/a^2)\int\tilde{M}_\alpha dk,\\ (1-a^2\Box)\varphi_\alpha&=\int M_\alpha dk. \end{aligned} \quad (3.6)$$

Adapting FP, Eq. (15) to our need, and after some manipulations, we find

$$\begin{aligned} \bar{H}=&\int\{\varphi_\alpha^*(\mathbf{k})\varphi_\alpha(\mathbf{k})+\varphi_\alpha(\mathbf{k})\varphi_\alpha^*(\mathbf{k})\}k^2dk\\ &-\int\{\tilde{\varphi}_\alpha^*(\mathbf{k})\tilde{\varphi}_\alpha(\mathbf{k})+\tilde{\varphi}_\alpha(\mathbf{k})\tilde{\varphi}_\alpha^*(\mathbf{k})\}\tilde{k}^2dk. \end{aligned} \quad (3.7)$$

4. QUANTIZATION

In order to pass from the classical to the quantum equations, without having to make some new postulate for the commutation rules of the potentials, two methods are available. In one, due to Heisenberg and Pauli, the canonically conjugate coordinates and momenta of the field satisfy commutation rules similar to those of the quantum mechanics of a particle, i.e.,

$$\begin{aligned}[][p_\alpha(\mathbf{r}, x_0), q_\beta(\mathbf{r}', x_0)] &= -\hbar i\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'),\\ [P_\alpha(\mathbf{r}, x_0), Q_\beta(\mathbf{r}', x_0)] &= -\hbar i\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}');\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}[][q_\alpha(\mathbf{r}, x_0), q_\beta(\mathbf{r}', x_0)] &= [p_\alpha(\mathbf{r}, x_0), p_\beta(\mathbf{r}', x_0)]\\ &= [Q_\alpha(\mathbf{r}, x_0), Q_\beta(\mathbf{r}', x_0)]\\ &= [P_\alpha(\mathbf{r}, x_0), P_\beta(\mathbf{r}', x_0)]\\ &= [p_\alpha(\mathbf{r}, x_0), Q_\beta(\mathbf{r}', x_0)]\\ &= [P_\alpha(\mathbf{r}, x_0), q_\beta(\mathbf{r}', x_0)]\\ &= [q_\alpha(\mathbf{r}, x_0), Q_\beta(\mathbf{r}', x_0)]\\ &= [p_\alpha(\mathbf{r}, x_0), P_\beta(\mathbf{r}', x_0)] = 0.\end{aligned} \tag{4.2}$$

It is interesting to note that now the potentials q_α and their time derivatives Q_β commute, a result which is quite different from that of usual electrodynamics.

Alternatively, we may require that

$$\dot{F} = (i/\hbar)[\bar{H}, F] \tag{4.3}$$

is satisfied for every operator F in the Heisenberg representation. Either method will lead to the same results.

From Eq. (4.3) we have, as special cases,

$$\dot{p}_\alpha = (i/\hbar)[\bar{H}, p_\alpha] \quad\text{and}\quad \dot{P}_\alpha = (i/\hbar)[\bar{H}, P_\alpha]. \tag{4.4}$$

Using Eqs. (2.23) and (3.6) we obtain

$$\begin{aligned} p_\alpha &= (-i/c)\int kN_\alpha dk,\\ P_\alpha &= (1/c^2)\int \tilde{k}\tilde{N}_\alpha dk.\end{aligned} \tag{4.5}$$

The commutation rules for the Fourier amplitudes are now found by expressing both sides of Eq. (4.4) in terms of their Fourier amplitudes and equating coefficients of corresponding exponentials. Thus:

$$\begin{aligned}[][\varphi_\alpha^*(\mathbf{k}), \varphi_\beta(\mathbf{k}')] &= -\frac{c\hbar}{2k}\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}'),\\ [\tilde{\varphi}_\alpha^*(\mathbf{k}), \tilde{\varphi}_\beta(\mathbf{k}')] &= \frac{c\hbar}{2\tilde{k}}\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}').\end{aligned} \tag{4.6}$$

All other pairs of the Fourier component of the potentials commute.

The commutation rules for the potentials at different space-time points can be obtained by transforming back to coordinate space. The result is

$$\begin{aligned}[][\varphi_\alpha(\mathbf{r}, x_0), \varphi_\beta(\mathbf{r}', x_0')] &= -\frac{ic\hbar\delta_{\alpha\beta}}{(2\pi)^3}\int \exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]\\ &\quad\times\left[\frac{\sin k(x_0-x_0')}{k} - \frac{\sin \tilde{k}(x_0-x_0')}{\tilde{k}}\right]dk.\end{aligned} \tag{4.7}$$

The first term gives the well-known Jordan-Pauli invariant delta function:

$$\begin{aligned}\left(\frac{1}{2\pi}\right)^3\int e^{i\mathbf{k}\cdot\mathbf{R}}\frac{\sin kX_0}{k}dk &= \Delta(X)\\ &= \frac{1}{4\pi|\mathbf{R}|}[\delta(|\mathbf{R}|-X_0) - \delta(|\mathbf{R}|+X_0)].\end{aligned} \tag{4.8}$$

The second term is proportional to[9]

$$\begin{aligned}\left(\frac{1}{2\pi}\right)^3\int e^{i\mathbf{k}\cdot\mathbf{R}}\frac{\sin \tilde{k}X_0}{\tilde{k}}dk &= -\frac{1}{4\pi|\mathbf{R}|}\frac{\partial}{\partial|\mathbf{R}|}F(|\mathbf{R}|, X_0),\end{aligned} \tag{4.9}$$

where

$$F(|\mathbf{R}|, X_0) = \begin{cases} J_0\left\{\frac{1}{a}(X_0^2 - |\mathbf{R}|^2)^{\frac{1}{2}}\right\} & \text{for } X_0 > |\mathbf{R}|,\\ 0 & \text{for } |\mathbf{R}| > X_0 > -|\mathbf{R}|,\\ -J_0\left\{\frac{1}{a}(X_0^2 - |\mathbf{R}|^2)^{\frac{1}{2}}\right\} & \text{for } X_0 < -|\mathbf{R}|.\end{cases} \tag{4.10}$$

Equations (4.1) and (4.2) can be obtained immediately from Eq. (4.7). For example, partial differentiation with respect to x_0, and subsequent

[9] P. A. M. Dirac, Proc. Camb. Phil. Soc. **30**, 100 (1934)

substitution of $x_0'=x_0$, gives Eq. (4.2). Later we shall need the relation

$$[\varphi_\alpha(\mathbf{r}, x_0), \varphi_{\alpha,i}(\mathbf{r}', x_0)]=0, \tag{4.11}$$

which is easily obtained from Eqs. (4.1).

It is interesting to note that in our formulation the four-divergence of the potentials at different space-time points does not commute. In fact, it can be shown that

$$[\varphi_{\alpha,\alpha}(\mathbf{r}, x_0), \varphi_{\beta,\beta}(\mathbf{r}', x_0')] = \frac{-ic\hbar}{a^2(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\frac{\sin \tilde{k}(x_0-x_0')}{\tilde{k}}dk. \tag{4.12}$$

The commutation rules for the Fourier components of the field strengths can be obtained from Eqs. (3.4) and (4.6). Thus, since

$$\begin{aligned}
[A_i^*(\mathbf{k}), A_j(\mathbf{k}')] &= -\frac{c\hbar}{2k}\delta_{ij}\delta(\mathbf{k}-\mathbf{k}'),\\
[\varphi^*(\mathbf{k}), \varphi(\mathbf{k}')] &= \frac{c\hbar}{2k}\delta(\mathbf{k}-\mathbf{k}'),\\
[\tilde{A}_i^*(\mathbf{k}), \tilde{A}_j(\mathbf{k}')] &= \frac{c\hbar}{2\tilde{k}}\delta_{ij}\delta(\mathbf{k}-\mathbf{k}'),\\
[\tilde{\varphi}^*(\mathbf{k}), \tilde{\varphi}(\mathbf{k}')] &= -\frac{c\hbar}{2\tilde{k}}\delta(\mathbf{k}-\mathbf{k}'),
\end{aligned} \tag{4.13}$$

it follows that

$$\begin{aligned}
[E_i(\mathbf{k}), E_j^*(\mathbf{k}')] &= \frac{c\hbar}{2k}(k^2\delta_{ij}-k_ik_j)\delta(\mathbf{k}-\mathbf{k}'),\\
[\tilde{E}_i(\mathbf{k}), \tilde{E}_j^*(k')] &= -\frac{c\hbar}{2\tilde{k}}\times(\tilde{k}^2\delta_{ij}-k_ik_j)\delta(\mathbf{k}-\mathbf{k}'),\\
[H_i(\mathbf{k}), H_j^*(\mathbf{k}')] &= \frac{c\hbar}{2k}(k^2\delta_{ij}-k_ik_j)\delta(\mathbf{k}-\mathbf{k}'),\\
[\tilde{H}_i(\mathbf{k}), \tilde{H}_j^*(\mathbf{k}')] &= -\frac{c\hbar}{2\tilde{k}}\times(k^2\delta_{ij}-k_ik_j)\delta(\mathbf{k}-\mathbf{k}'),\\
[E_i(\mathbf{k}), H_j^*(\mathbf{k}')] &= [E_i^*(\mathbf{k}), H_j(\mathbf{k}')] = \frac{c\hbar}{2}k_m\epsilon_{ijm}\delta(\mathbf{k}-\mathbf{k}'),\\
[\tilde{E}_i(\mathbf{k}), \tilde{H}_j^*(\mathbf{k}')] &= [\tilde{E}_i^*(\mathbf{k}), \tilde{H}_j(\mathbf{k}')] = -\frac{c\hbar}{2}k_m\epsilon_{ijm}\delta(\mathbf{k}-\mathbf{k}').
\end{aligned} \tag{4.14}$$

All other combinations commute. In the last expression, ϵ_{ijk} is $+1$ or -1, according to whether ijk is an even or odd permutation of 123, and zero otherwise.

The commutation rules of the field quantities can be obtained by the method used in deriving Eq. (4.7). Thus:

$$\begin{aligned}
[E_i(\mathbf{r}, x_0), E_j(\mathbf{r}', x_0')] &= \frac{ic\hbar}{(2\pi)^3}\int\Big\{(k^2\delta_{ij}-k_ik_j)\frac{\sin k(x_0-x_0')}{k}\\
&\quad\times\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]-(\tilde{k}^2\delta_{ij}-k_ik_j)\\
&\quad\times\frac{\sin \tilde{k}(x_0-x_0')}{\tilde{k}}\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]\Big\}dk\\
&= \frac{ic\hbar}{(2\pi)^3}\left(\delta_{ij}\frac{\partial^2}{\partial x_0\partial x_0'}-\frac{\partial^2}{\partial x_i\partial x_j'}\right)\\
&\quad\times\int\left\{\frac{\sin k(x_0-x_0')}{k}-\frac{\sin \tilde{k}(x_0-x_0')}{\tilde{k}}\right\}\\
&\quad\times\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]dk.
\end{aligned} \tag{4.15}$$

Also

$$\begin{aligned}
[H_i(\mathbf{r}, x_0), H_j(\mathbf{r}', x_0')] &= \frac{ic\hbar}{(2\pi)^3}\left(\delta_{ij}\frac{\partial^2}{\partial x_0\partial x_0'}-\frac{\partial^2}{\partial x_i\partial x_j'}\right)\\
&\quad\times\int\left\{\frac{\sin k(x_0-x_0')}{k}-\frac{\sin \tilde{k}(x_0-x_0')}{\tilde{k}}\right\}\\
&\quad\times\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]dk
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
[E_i(\mathbf{r}, x_0), H_j(\mathbf{r}', x_0')] &= \frac{c\hbar\epsilon_{ijm}}{(2\pi)^3}\\
&\quad\times\frac{\partial}{\partial x_m}\int\left\{\frac{\sin k(x_0-x_0')}{k}-\frac{\sin \tilde{k}(x_0-x_0')}{\tilde{k}}\right\}\\
&\quad\times\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]dk.
\end{aligned} \tag{4.17}$$

5. AUXILIARY CONDITIONS

It is customary in the usual quantum electrodynamics to assume, in addition to the wave equation, that the Schrödinger functional Ψ satisfies the condition

$$(\partial\varphi_\alpha/\partial x_\alpha)\Psi=0.$$

However, it is required of an auxiliary condition of the form $A\Psi=0$ that $A(\mathbf{r},t)$ commute with $A(\mathbf{r}',t')$ taken at another space-time point. This condition, according to Eq. (4.12), is not satisfied by $(\partial\varphi_\alpha/\partial x_\alpha)$.

A suitable generalization of the above equation can be obtained as follows: Eq. (2.18) is satisfied if we put

$$a(\partial\varphi_\alpha/\partial x_\alpha)+B=0, \tag{5.1}$$

where B is any function satisfying the equation

$$(1-a^2\Box)B=0. \tag{5.2}$$

In terms of the Fourier amplitudes, Eq. (5.2) is equivalent to

$$B=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\{\tilde{B}(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)] +\tilde{B}^*(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r}-\tilde{k}x_0)]\}dk. \tag{5.3}$$

Following the suggestion due to Stückelberg,[10] we shall take as the auxiliary condition

$$[a(\partial\varphi_\alpha/\partial x_\alpha)+B]\Psi=0, \tag{5.4}$$

and assume that $\tilde{B}(\mathbf{k})$ and $\tilde{B}^*(\mathbf{k})$ satisfy the same commutation rules as the other tilde functions, i.e.,

$$[\tilde{B}(\mathbf{k}),\tilde{B}^*(\mathbf{k}')]=-(c\hbar/2\tilde{k})\delta(\mathbf{k}-\mathbf{k}'), \tag{5.5}$$

and commute with the amplitudes of the potentials. Hence from Eqs. (5.3) and (5.5)

$$[B(\mathbf{r},x_0),B(\mathbf{r}',x_0')]=\frac{-ic\hbar}{(2\pi)^3}\times\int\frac{\sin\tilde{k}(x_0-x_0')}{\tilde{k}}\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]dk. \tag{5.6}$$

Using this result and Eq. (4.12), we have

$$[a\varphi_{\alpha,\alpha}(\mathbf{r},x_0)+B(\mathbf{r},x_0), a\varphi_{\beta,\beta}(\mathbf{r}',x_0')+B(\mathbf{r}',x_0')]=0. \tag{5.7}$$

Equating coefficients of corresponding exponentials, the auxiliary condition can be written in terms of the Fourier amplitudes, i.e.,

$$\begin{aligned}[][\mathbf{k}\cdot\mathbf{A}(\mathbf{k})-k\varphi(\mathbf{k})]\Psi&=0,\\ [a(\mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k})-\tilde{k}\tilde{\varphi}(\mathbf{k}))-i\tilde{B}(\mathbf{k})]\Psi&=0,\\ [\mathbf{k}\cdot\mathbf{A}^*(\mathbf{k})-k\varphi^*(\mathbf{k})]\Psi&=0,\\ [a(\mathbf{k}\cdot\tilde{\mathbf{A}}^*(\mathbf{k})-\tilde{k}\tilde{\varphi}^*(\mathbf{k}))+i\tilde{B}^*(\mathbf{k})]\Psi&=0.\end{aligned} \tag{5.8}$$

The operators occurring in these equations commute among themselves and also with the Hamiltonian.

6. FIELD WITH CHARGED PARTICLES

The present formulation can be extended to fields containing charged particles exactly as was done by Dirac, Fock, and Podolsky.[11] As in DFP, we shall find again that the operators in Eqs. (5.8) do not commute with the operator[12]

$$R_s-i\hbar(\partial/\partial t_s). \tag{6.1}$$

This difficulty can be overcome by introducing operators

$$\begin{aligned}C(\mathbf{k})&=i[\mathbf{k}\cdot\mathbf{A}(\mathbf{k})-k\varphi(\mathbf{k})]+f(\mathbf{r}_s,t_s),\\ \tilde{C}(\mathbf{k})&=i[\mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k})-\tilde{k}\tilde{\varphi}(\mathbf{k})]+\frac{1}{a}\tilde{B}(\mathbf{k})+\tilde{f}(\mathbf{r}_s,t_s),\end{aligned} \tag{6.2}$$

and their complex conjugates, where $f(\mathbf{r}_s,t_s)$ and $\tilde{f}(\mathbf{r}_s,t_s)$ are chosen so as to make the C's commute with the operators (6.1). A short calculation shows that

$$\begin{aligned}f(\mathbf{r}_s,t)&=\frac{i}{2(2\pi)^{\frac{3}{2}}}\sum_s\frac{\epsilon_s}{k}\times\exp[-i(\mathbf{k}\cdot\mathbf{r}_s-kx_{0s})],\\ \tilde{f}(\mathbf{r}_s,t)&=-\frac{i}{2(2\pi)^{\frac{3}{2}}}\sum_s\frac{\epsilon_s}{\tilde{k}}\times\exp[-i(\mathbf{k}\cdot\mathbf{r}_s-\tilde{k}x_{0s})].\end{aligned} \tag{6.3}$$

Then, the modified auxiliary conditions are given by:

$$C(\mathbf{k})\Psi=0,\qquad \tilde{C}(\mathbf{k})\Psi=0,\qquad C^*(\mathbf{k})\Psi=0,\quad\text{and}\quad \tilde{C}^*(\mathbf{k})\Psi=0. \tag{6.4}$$

Transforming (6.4) to coordinate space, we obtain

$$C(\mathbf{r},x_0)=\operatorname{div}\mathbf{A}+\frac{\partial\varphi}{\partial x_0}+\frac{1}{a}B(\mathbf{r},x_0)-\frac{1}{(2\pi)^3}\sum_s\epsilon_s\int\left\{\frac{\sin k(x_0-x_{0s})}{k}-\frac{\sin\tilde{k}(x_0-x_{0s})}{\tilde{k}}\right\}\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_s)]dk. \tag{6.5}$$

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[10] E. C. G. Stückelberg, Helv. Phys. Acta **11**, 225–299 (1938).

[11] Dirac, Fock, and Podolsky, Physik. Zeits. Sowjetunion **2**, 473 (1932). Hereafter referred to as DFP.

[12] For the definitions of these quantities see reference 11, Eqs. (1) to (11), inclusive.

The field equations can be obtained by procedure similar to that used in DFP. From the defining equations, GE Eq. (3.3), or

$$\mathbf{E}=-\text{grad}\,\varphi-(1/c)(\partial\mathbf{A}/\partial t)$$
and
$$\mathbf{H}=\text{curl}\,\mathbf{A}, \tag{6.6}$$

it follows immediately that

$$\text{curl}\,\mathbf{E}+(1/c)(\partial\mathbf{H}/\partial t)=\mathbf{0}\quad\text{and}\quad\text{div}\,\mathbf{H}=0, \tag{6.7}$$

so that these remain as quantum mechanical equations. The other two equations can only be derived with the aid of Eq. (6.5).

Recalling Eq. (2.16), and observing that the first and the second terms under the integral sign of Eq. (6.5) satisfy Maxwell's and Yukawa's wave equations, respectively, we obtain from Eq. (6.6)

$$(1-a^2\Box)\left(\text{curl}\,\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\Psi=\left[\text{grad}\sum_s\frac{\epsilon_s}{4\pi}\Delta(X-X_s)\right]\Psi, \tag{6.8}$$

where X is the four-vector (x_1, x_2, x_3, x_0) and $\Delta(X)$ the invariant delta function defined in Eq. (4.8). Similarly

$$(1-a^2\Box)\,\text{div}\,\mathbf{E}\Psi=-\frac{1}{c}\left[\frac{\partial}{\partial t}\sum_s\frac{\epsilon_s}{4\pi}\Delta(X-X_s)\right]\Psi. \tag{6.9}$$

Now putting the various times equal to each other, we get for the equations corresponding to DFP Eqs. (45), (46), and (47) the generalized results:

$$(1-a^2\Box)\left(\text{div}\,\mathbf{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}\right)\Psi=0, \tag{6.10}$$

$$(1-a^2\Box)\left(\text{curl}\,\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right)\Psi=0, \tag{6.11}$$

and

$$(1-a^2\Box)\,\text{div}\,\mathbf{E}\Psi=-\sum_s\frac{\epsilon_s}{4\pi}\left[\frac{1}{c}\frac{\partial}{\partial t}\Delta(X-X_s)\right]_{t_s=t}\Psi. \tag{6.12}$$

The argument leading from DFP Eqs. (48) to (49) has to be modified slightly in view of the fact that the momenta conjugate to A_i are not $-(1-a^2\Box)E_i$, but $(1/c^2)(1-a^2\Box)\dot{A}_i$. Thus

$$[R_s, (1-a^2\Box)\mathbf{E}]=-\epsilon_s\Big[\varphi(\mathbf{r}_s, t_s)-\alpha_s\cdot\mathbf{A}(\mathbf{r}_s, t_s),\ (1-a^2\Box)\nabla\varphi(\mathbf{r}, t)+\frac{1}{c}(1-a^2\Box)\dot{\mathbf{A}}(\mathbf{r}, t)\Big]=ic\hbar\epsilon_s\boldsymbol{\alpha}_s\delta(\mathbf{r}-\mathbf{r}_s). \tag{6.13}$$

Since

$$\left[\frac{1}{c}\frac{\partial}{\partial X_0}\Delta(X)\right]_{X_0=0}=-4\pi\delta(|\mathbf{R}|),$$

Eqs. (6.11) and (6.12) become

$$(1-a^2\Box)\left(\text{curl}\,\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial T}\right)\Psi=\left(\sum_s\epsilon_s\boldsymbol{\alpha}_s\delta(\mathbf{r}-\mathbf{r}_s)\right)\Psi \tag{6.14}$$

and

$$(1-a^2\Box)\,\text{div}\,\mathbf{E}\Psi=\left(\sum_s\epsilon_s\delta(\mathbf{r}-\mathbf{r}_s)\right)\Psi. \tag{6.15}$$

These are natural generalizations of the equations of ordinary quantum electrodynamics.