On Supersonic Waves in Cylindrical Tubes and the Theory of the Acoustic Interferometer

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A theory of the Pierce acoustic interferometer is given which takes into account the non-uniformity of the acoustic field. This leads to an interpretation of the satellite peaks observed in the experiments of Pielemeier and others, and to an improved formula for the determination of the absorption coefficient from the observational data.

1. INTRODUCTION

THE successful application of the Pierce acoustic interferometer to the determination of supersonic sound velocities is well known. Many attempts have been made to apply it to the measurements of absorption coefficients as well, but the accuracy obtained has appeared exceedingly low. L. Mandelstam has concluded that the most likely cause for this is the non-uniformity of the sound field, for which no correction has been made. Ordinary methods of reduction of the experimental data have assumed that the quartz plate used as a radiator vibrates like a piston, giving rise to a purely plane wave. However, real plates do not come up to this ideal, usually exhibiting a rather irregular form of vibration.

The present paper deals with the theory of the interferometer, taking into account this more complex pattern of the acoustic field, and describes a new method of reduction of the experimental data which is believed to be more accurate and general in application than those previously employed.

Before attacking the general theory, it may be desirable to give a brief description of the principle of the interferometer and of the manner in which it is employed. A cylindrical metal tube of arbitrary sectional form is closed at one end by a quartz plate driven by an electrical sine wave generator, and at the other by a movable piston used as a reflector. The whole volume of gas in the tube is set into vibration, in virtue of which there is an acoustic reaction on the sound source. The observational result is a reaction curve in which some quantity measuring the acoustic output of the source is plotted as a function of the distance between the emitter and reflector plates. The reduction of the observations consists of the deduction of the ultrasonic velocity and the absorption coefficient from the form of this curve. In order to study this latter problem we must find an analytical expression for the acoustic reaction on the emitter in terms of the velocity of sound on the one hand, and in terms of parameters describing the form of the vessel and the character of the vibration of the quartz plate on the other.

2. GENERAL THEORY

A rigorous solution of this theoretical problem would be extremely difficult. In our analysis we shall consider the various types of waves which may be generated in such a system, and their effect on the output of the source. This mathematical theory is similar in principle to the theory of electrical wave guides.

If, at a given frequency, the absorption may be considered as small and the transverse dimensions of the cylinder as large, we may follow Kirchhoff[1] in neglecting viscosity and heat losses on the surface of the cylinder, and consider it as infinitely rigid, perfectly smooth, and absolutely thermonegative. By virtue of these conditions, we assume the existence of a velocity potential satisfying the equation

$$\nabla^2\varphi+k_0^2\varphi=0 \qquad (1)$$

throughout the interior of the cylinder. Here $k_0=\beta_0-i\alpha_0$ is a complex wave factor for the unconfined medium ($\beta_0=\omega/v_0$), and the time factor $e^{i\omega t}$ is understood.

With the z axis parallel to the generators of the

[1] G. Kirchhoff, Pogg. Ann. **132**, 177 (1868).

cylinder, the reflecting surface in the plane $z=0$, and the radiating surface in the plane $z=L$, the boundary conditions take the form

$$\partial\varphi/\partial n=0, \quad \text{on the side walls of the cylinder,} \quad (2)$$

$$\partial\varphi/\partial z=0, \quad \text{in the plane } z=0, \quad (3)$$

$$\partial\varphi/\partial z=F(x, y), \quad \text{in the plane } z=L, \quad (4)$$

where $F(x, y)$ gives the normal component of the velocity on the surface of the radiator.

Following Rayleigh's method[2] we look for particular solutions of Eq. (1) in the form of progressive plane waves propagated in the z direction, by setting

$$\varphi=\psi(x, y)e^{\pm imz}, \quad (5)$$

where ψ satisfies the condition

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+(k_0^2-m^2)\psi=0 \quad (6)$$

with $f=k_0^2-m^2$, and where it will be assumed that f is real.

Supposing this equation solved, with boundary condition (2), we find the types of waves which may be propagated along the cylinder without change of type, corresponding to a discrete set of values f_j, $j=0, 1, 2, \cdots$ for f, with associated values for the wave factor $m_j=\beta_j-i\alpha_j$. Under the conditions assumed, the different solutions of Eq. (6) will form an orthogonal set of functions when integrated over the cross section of the cylinder, and we shall suppose further that they have been normalized.

The types of waves permitted vary with the form of the bounding cylinder, as do their phase velocities and attenuation coefficients. The wave $f_0=0$ corresponds to an ordinary uniform plane wave; the greater being the value of f_j, the more complicated the wave pattern becomes.

Separating the real and imaginary parts of $m_j^2=k_0^2-f_j$, we obtain

$$2\beta_j^2=(\beta_0^2-\alpha_0^2-f_j)+[(\beta_0^2-\alpha_0^2-f_j)^2+4\alpha_0^2\beta_0^2]^{\frac{1}{2}}, \quad (7)$$

$$2\alpha_j^2=-(\beta_0^2-\alpha_0^2-f_j)+[(\beta_0^2-\alpha_0^2-f_j)^2+4\alpha_0^2\beta_0^2]^{\frac{1}{2}}. \quad (8)$$

[2] Lord Rayleigh, *Theory of Sound*, Vol. 2, Section 301.

The typical form of dependence of β_j and α_j on the value of f_j is shown in Fig. 1. The waves for which f_j is of the order of β_0^2 have such large attenuation that they practically disappear within a few wave-lengths, while if $\alpha_0=0$, waves for which $f_j\geqslant\beta_0^2$ are not propagated at all. This type of attenuation is not due to adsorption in the medium, but to interference effects of primary waves from the source and their reflected waves from the sides of the cylinder. It is this

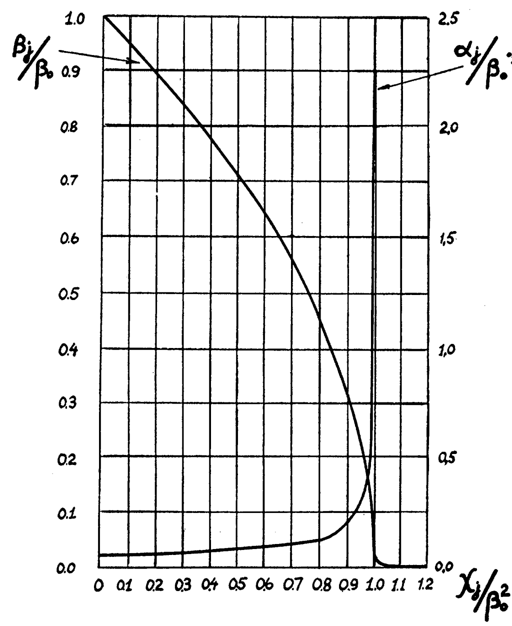

FIG. 1. Attenuation and phase velocity of corrugated waves in tubes for $\alpha_0/\beta_0=6\times10^{-4}$.

effect which constitutes the basis for the present theory.

The general solution of Eq. (1) is obtained as a linear superposition of all of these special waves

$$\varphi=\sum \psi_j(x, y)[c_j' \exp(im_jz)+c_j'' \exp(-im_jz)]. \quad (9)$$

From the form of this solution, boundary condition (2) is satisfied automatically. Condition (3) will be satisfied by taking $c_j'=c_j''=c_j$.

We assume that, whatever the form of vibration of the source, $F(x, y)$ of condition (4) can be expanded in a series of the form

$$F(x, y)=\sum A_j\psi_j(x, y) \quad (10)$$

from which, restoring the time factor, our general

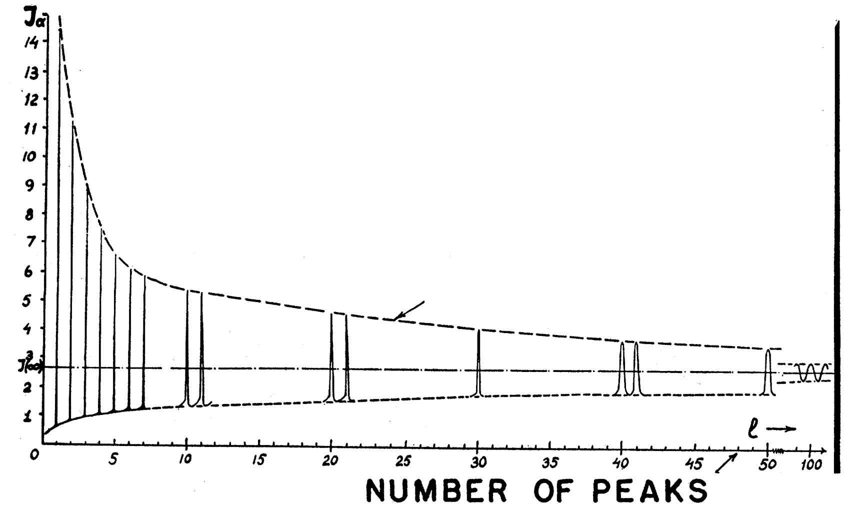

FIG. 2. The elementary reaction curve for air at a frequency of 100,000 cycles.

solution satisfying all conditions turns out to be

$$\varphi = \Re\left\{ e^{i\omega t} \sum \frac{A_j \psi_j}{im_j} \times \frac{\exp(im_j z) + \exp(-im_j z)}{\exp(im_j L) - \exp(-im_j L)} \right\}, \quad (11)$$

where the symbols $\Re$ and $\Im$ will be used to indicate the real and imaginary parts, respectively.

The acoustic output of the radiator is

$$\frac{dw}{dt} = \int\int \rho \left(\frac{\partial\varphi}{\partial t}\right)_{z=L} \cdot \left(\frac{\partial\varphi}{\partial z}\right)_{z=L} dS \quad (12)$$

in which ρ is the density of the medium. The integral is to be extended over the surface of the radiator.

By substitution of Eq. (11) in (12), taking advantage of the orthonormal character of the ψ's, and performing a time average in the usual fashion, we get

$$\left\langle \frac{dw}{dt} \right\rangle_{\text{Av}} = \frac{1}{2}\rho\omega \sum \frac{A_j^2}{\beta_j^2 + \alpha_j^2} \times [\beta_j \Im(\cot m_j L) + \alpha_j \Re(\cot m_j L)]. \quad (13)$$

To a sufficient approximation, the terms containing the factors $\alpha_j \Re(\cot m_j L)$ can be neglected, as they are small for both small and large values of α_j, since for large values of α_j

$$\Re(\cot m_j L) \sim \exp(-2\alpha_j L).$$

In the remaining expression we need consider only the terms for which $|\beta_j| \gg |\alpha_j|$ in which we can put

$$\omega\beta_j/(\beta_j^2 + \alpha_j^2) \approx \omega/\beta_j = v_j.$$

If the variation in the output of the sound generator over a resonance curve is small, we may assume that the experimentally measured plate current of the power tube which drives the oscillator is a linear function of the acoustic power output, and we can take as a measure of the observed curve the remaining quantity

$$\left\langle \frac{dw}{dt} \right\rangle_{\text{Av}} \cong I(L) = \sum \rho v_j A_j^2 \Im(\cot m_j L). \quad (14)$$

3. DISCUSSION

The simplest case is that in which the radiator vibrates like a piston, as is usually assumed, so that only A_0 differs from zero. This reduces $I(L)$ to the form

$$I(L) = \rho v_0 A_0^2 [1 + \exp(-2\alpha_0 L) \cdot \cos 2\beta_0 L] \quad (15)$$

for large values of L. The reaction curve for the ordinary theory is shown in Fig. 2. According to Eq. (14) the complete reaction curve is made up of a superposition of curves of this type.[3–5]

In order to explain the usual form of observed

[3] W. H. Pielemeier, Phys. Rev. **34**, 1187 (1929).
[4] J. S. Hubbard, Phys. Rev. **38**, 1011 (1931).
[5] L. Beliavskaya, Bull. Acad. Sci. U.S.S.R., No. 7 (1935).

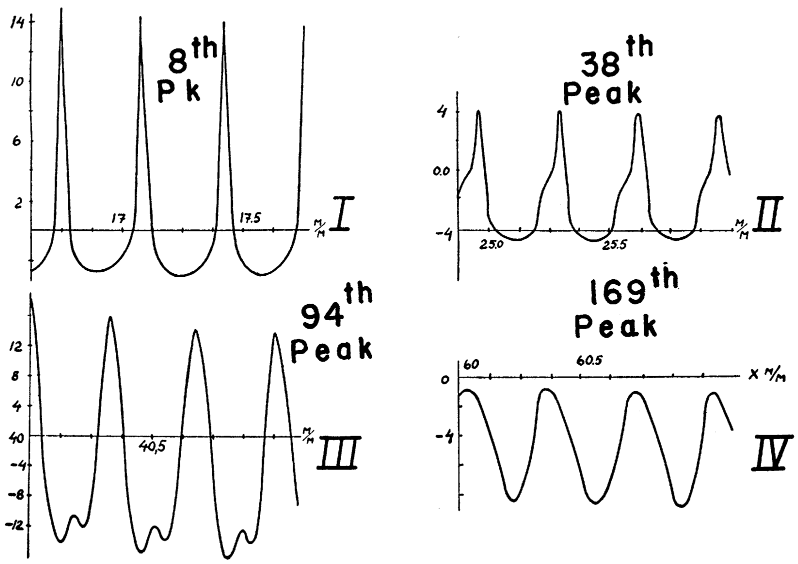

FIG. 3. The satellites after Pielemeier.

reaction curve, it is necessary to assume that in most cases the acoustic field set up consists of a group of waves forming a narrow wave packet. The presence of absorption diminishes the resolving power of the interferometer, so that the fine structure remains unresolved.

In particular cases, in which there are two dominant groups of waves, satellites may be observed, which gives us an explanation of the subsidiary peaks which have been observed by many authors.[3, 6–8] Figure 3 represents such a curve taken from the work of Pielemeier.[3] The measurement of the intervals indicates that the major peaks correspond to waves of high index f_j, while the minor peaks correspond to the nearly plane wave. This is peculiar to Y-cut crystals; for X-cut crystals the approximately uniform plane wave predominates.

Since only the peaks without satellites are used in the experimental work on absorption coefficients, we confine our attention to them. According to our theory, we consider them to be made up of a group of waves forming a single wave packet. We shall now show that the apparent absorption coefficient of such a wave packet arises from two independent causes: (a) the true absorption of the medium, and (b) an apparent absorption arising from the mutual interference effects of the waves comprising the packet. The separation of the two effects will lead us to our improved formula for the reduction of the experimental data.

For large values of $\alpha_0 L$, when the dominating part of the acoustic reaction on the source may be attributed to the first reflected wave, expression (14) takes the form

$$I(L)=\sum \theta_j[1+\exp(-2\alpha_j L)\cos(2\beta_j L)] \quad (16)$$

with

$$\theta_j=\rho v_j A_j^2.$$

Replacing the line spectrum by a continuous wave group, we get

$$I(L)\backsimeq\int_{\beta_0}^{0}\theta(\beta)[1+e^{-2\alpha L}\cos(2\beta L)]d\beta.$$

If we transfer the origin to the center of the packet β_c, and assume the whole spectrum to be confined within the interval $\beta_c\pm b$, we get

$$I(L)\triangleq C'(L)$$

$$\times\exp(-2\alpha_c L)\cos(2\beta_c L+\varphi)+\text{const.}, \quad (17)$$

[6] E. Klein and W. D. Hershberger, Phys. Rev. **37**, 760 (1931).
[7] R. Alleman, Phys. Rev. **55**, 87 (1939).
[8] W. H. Pielemeier, J. Acous. Soc. Am. **9**, 600 (1938).

where

$$C'(L)\cos\varphi=\int_{-b}^{+b}\theta_s(\xi)\cdot\cos 2L\xi\cdot d\xi,$$
$$C'(L)\sin\varphi=\int_{-b}^{+b}\theta_a(\xi)\cdot\sin 2L\xi\cdot d\xi, \tag{18}$$

$$\theta_s(\xi)=\tfrac{1}{2}[\theta(\xi)+\theta(-\xi)],$$
$$\theta_a(\xi)=\tfrac{1}{2}[\theta(\xi)-\theta(-\xi)]. \tag{19}$$

These equations may be inverted in the form

$$\theta_s(\xi)=\cos\varphi\cdot\int_0^{\infty}C'(L)\cdot\cos 2\xi L\cdot dL,$$
$$\theta_a(\xi)=\sin\varphi\cdot\int_0^{\infty}C'(L)\cdot\sin 2\xi L\cdot dL. \tag{20}$$

It is evident that the relation between the form of the spectrum $\theta(\xi)$ and that of the reaction curve envelope $C'(L)$ is similar to that connecting the optical spectrum and the visibility curve of the Michelson interferometer.

Since a large variety of forms of the function $\theta(\xi)$ may produce effectively the same enveloping function $C'(L)$, it is not surprising that observed reaction curves may be approximated rather well by an over-all exponential decay curve of the form e^{-2aL}.

4. REDUCTION OF THE EXPERIMENTAL DATA

It is customary, in the reduction of experimental observations, to choose those reaction curves which may be represented with sufficient accuracy for large L by an expression of the form

$$\exp(-2\alpha_e L)\cdot\cos(2\beta_e L+\varphi). \tag{21}$$

By comparison with the theoretical curve (15) for a uniform plane wave, it is normally assumed that (a) the experimental decay coefficient α_e is equal to α_0 for the uniform wave, and (b) $\beta_e=\beta_0$, so that the wave-length of the uniform wave is equal to twice the distance between adjacent maxima.

By comparison with our more complete formula (17), we see that the dependence on L is determined in part also by the variation of the envelope function $C'(L)$. If we suppose that the over-all variation of the reaction curve is exponential in character, we must put

$$C'(L)=C_0\cdot e^{-2aL} \tag{22}$$

so that the apparent decay constant becomes

$$\alpha_e=\alpha_c+a. \tag{23}$$

The absorption coefficient α_c arises from direct absorption in the medium, while the absorption represented by the coefficient a is due to interference effects within the wave packet, the resultant absorption being due to the combination of the two effects.

Substituting (22) in (20) we find

$$\theta_s(\xi)=\cos\varphi\cdot\frac{a}{a^2+\xi^2}$$

so that

$$a=\Delta\xi$$

is the half-width of the symmetrical part of the packet.

Now we can express $\Delta\xi$ in terms of the width Δf, for according to (7) and (8)

$$2\beta_c\Delta\xi=\Delta f$$

so that to a first approximation

$$a=\Delta f/8\beta_c\cong\Delta f/8\beta_0. \tag{24}$$

The value of Δf is characteristic of the form of the vibrations of the source and the geometry of the interferometer, but does not depend on the properties of the medium in it.

According to Eq. (8), if $f_c/\beta_c^2<\frac{1}{2}$, we have

$$\alpha_c\cong\alpha_0[1+(f_c/2\beta_0^2)], \tag{25}$$

so that, if the width of the symmetrical part of the packet Δf and the position of its center f_c are known, the coefficient of attenuation α_0 referred to the supposed homogeneous plane wave may be deduced from the apparent absorption coefficient by means of the formula

$$\alpha_0=\frac{\alpha_e-(\Delta f/8\beta_0)}{1+(f_c/2\beta_0^2)}, \tag{26}$$

while the corresponding wave-length is given by the formula

$$\lambda_0=\lambda_e/[1+(f_c/2\beta_0^2)] \tag{27}$$

in terms of the apparent wave-length λ_e.

Since many authors reduce their results to a calculation of $\alpha_0\lambda_0^2$, we have the corresponding formula

$$\alpha_0\lambda_0^2 = \frac{[\alpha_e - (\Delta f/8\beta_0)]\cdot\lambda_e^2}{[1+(f_c/2\beta_0^2)]}. \quad (28)$$

As far as can be judged, the corrections involving Δf and f_c are appreciable, and with an ideal uniform plane wave the ultrasonic interferometer would give a much less rapid attenuation than that usually measured.

The determination of Δf and f_c for a given apparatus may be carried out by observation of the reaction curve of a gas with known values of v_0 and α_0. The coefficient a obtained by the method of Lubny-Herzyk[9] may be used in (24) to determine Δf. With Δf known, f_c may be determined by measuring α_e and then using (25) with $\alpha_c = \alpha_0 - a$.

In conclusion I should like to express my deep gratitude to L. Mandelstam, under whose guidance this work has been done, and to K. Theordortchik for his exceptional attention to the present investigation.

[9] E. Pumper, J. Phys. U.S.S.R. **1**, No. 5–6 (1939).

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Ambiguities in the X-Ray Analysis of Crystal Structures*

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The intensities of the x-ray reflections from a given crystal of a known substance depend on the vector distances between the atoms and not directly on the coordinates of the atoms. The problem of uniqueness in the x-ray analysis of a crystal structure thus depends on the uniqueness of the determination of the arrangement of a periodic set of points in space by its vector distance set. In this paper a large number of cases is presented in which two, and sometimes three or four, non-congruent sets of points are homometric, i.e., have the same vector distance set. Although the investigation is largely based on a discussion of one-dimensional cyclotomic sets, i.e., those in which the points divide the period on the line into rational fractions, it is shown that there are many families of non-cyclotomic pairs and multiplets and that each of these families has its counterpart in two and three dimensions. The significance of these results for practical crystal analysis is discussed.

INTRODUCTION

It is well known that the F factors of x-ray analysis are completely determined in magnitude and in phase by the coordinates of the atoms in the crystal and by the f factors of the individual atoms. The absolute values of the F factors are, however, given by

$$|F(h_i)|^2 = \sum_{s,\,t=1}^{N}\sum f_s f_t \exp 2\pi i \mathbf{h}\cdot(\mathbf{r}_t - \mathbf{r}_s), \quad (1)$$

in which f_s and $\mathbf{r}_s$, respectively, are the f factor and the coordinate vector for the sth atom and

$$\mathbf{h} = \sum h_i \mathbf{b}_i. \quad (1a)$$

In this latter equation the quantities h_i are the Miller indices for the plane in question and the vectors $\mathbf{b}_i$ are the reciprocal lattice vectors. The intensities in the x-ray diffraction pattern thus depend only on the vector distances between atoms and on the f factors, since in Eq. (1) the coordinate vectors $\mathbf{r}_s$ do not enter alone, but only in the form of the differences $\mathbf{r}_t - \mathbf{r}_s$.

We shall not concern ourselves here with the problem of determining the distances from the values of $|F(h_i)|^2$ since this has been discussed in detail elsewhere.[1] We shall investigate the problem of the determination of the coordinate

* A preliminary report of this work was presented at the Ann Arbor Meeting of the American Society for X-Ray and Electron Diffraction in June, 1943.

[1] A. L. Patterson, Zeits. f. Krist. **90**, 517 (1935); D. Harker, J. Chem. Phys. **4**, 381 (1936); M. Avrami, Zeits. f. Krist. **100**, 381 (1939); S. H. Yu, Nature **149**, 638 (1942) and **150**, 151 (1942).