the present measurements on the L_{III} state of silver with those of Webster, Hansen, and Duveneck[3] on the K state of the same element. Two facts are clearly shown:

(1) The cross sections for excitation to the K state and the L_{III} state are both greatest for cathode rays whose kinetic energies are about 3.5 times the ionization energy of the state in question.

(2) When $U>3.5$, the decrease in ionization cross section with increasing cathode-ray energy is more rapid for the L_{III} state than for the K state.

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Finite Self-Energies in Radiation Theory. Part III

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The invariant field theory of Part II is interpreted, in agreement with F. Bopp, as Maxwell's theory with a linear differential relation between the fields E, B and D, H involving a new constant k which measures the reciprocal radius of the electron. The former "mesonic field" of minimum frequency $\nu_0=kc/2\pi$ represents polarization of the vacuum. The electron is a singularity in the D, H field whereas E, B remain finite. Instead of obeying dynamical equations of motion, the electron moves under the condition that the Lorentz force vanishes identically on the singularity, so that no work is done on the particle. All energy is located in the field. In this respect the theory is unitary. Electromagnetic and inert mass are identical. In contrast to Dirac's classical electron which is subject to advanced and retarded potentials and displays self-acceleration, the field theory works with retarded potentials only, and self-acceleration is avoided. Stable equilibrium between electrons and radiation is granted by spontaneous and induced transitions, similar to Einstein's derivation of Planck's radiation formula. In spite of displaying a magnetic moment the electron does not have magnetic self-energy, so that its radius is the ordinary electrostatic radius $1/k=\epsilon^2/2mc^2$. In contrast to Born-Infeld's non-linear theory, our field equations allow a Fourier representation as a basis for the quantum theory of Part IV.

1. CLASSICAL FIELD THEORY

THE modification of electrostatics proposed in Part I and its electromagnetic continuation discussed in Part II[1] rest on the assumption that vacuum is polarizable, as described by linear differential relations between the vectors E, B and D, H (for details see Section 6):

$$D=E-k^{-2}\Box E,\quad H=B-k^{-2}\Box B,\tag{1}$$

where $\Box$ is the Laplace operator in x, y, z, ict. The constant k of dimension $[l^{-1}]$ determines the minimum frequency $\nu_0=kc/2\pi$ of waves of polarization or "meson waves." k also plays the role of the reciprocal electronic radius, although the charges ϵ are condensed in mathematical points only. The simplicity and naturalness of our approach are demonstrated by the fact that the same modification of Maxwell's theory has been proposed independently and simultaneously by F. Bopp.[2] Whereas we began in Part I with a Fourier representation of the field of a point particle with finite self-energy, Bopp started from a formal generalization of the Lagrangian function of the field $E, B=f_{\alpha\beta}$, namely,

$$L=-(1/16)\{(f_{\alpha\beta})^2+k^{-2}(\partial f_{\alpha\beta}/\partial x_\gamma)^2\}+J_\alpha\varphi_\alpha\tag{2}$$

where J is the 4 current and φ is the 4 potential.

The relation to other field theories (Maxwell, Born-Infeld) become obvious if the vectors E, H of Part II are called E, B, and the vectors E'', H'' are called D, H. Our "meson field" $E'=D-H$

[1] A. Landé, Phys. Rev. **60**, 121 (1941). A. Landé and L. H. Thomas **60**, 541 (1941).

[2] F. Bopp, Ann. d. Physik **38**, 345 (1940).

TABLE I. Notation.

Our former	$E, H; V, A$	$E'', H''; V'', A''$	$E', H'; V', A'$
Bopp	$E, B; \varphi, a$	$D, H; \phi, A$	$F, G; V/k, U/k$

and $H' = H - B$ then represents the electric and magnetic polarization of the vacuum (see Table I). In contrast to Born-Infeld, our relation between D, H and E, B is *linear* so that it is easy to obtain a Fourier representation (not discussed by Bopp) with subsequent quantization. Infinities are avoided automatically without resorting to arbitrary cutting-off procedures.

The new field theory is unitary insofar as all energy is supposed to be located in the field, without additional "mechanical" energy of particles. Instead of introducing mechanical equations of motion for the electrons, we postulate that the field E and B (which is responsible for Lorentz force and work on the particles) shall vanish at all times on the point singularities themselves. As a consequence of this condition we were able in Part II to derive a quasimechanical equation of motion:

$$m(\ddot{x})_0 = \text{external force} + (2\epsilon^2/3c^3)(d\ddot{x}/dt)_0 \quad (3)$$

valid at a particular time ($t=0$). m is an abbreviation for $k\epsilon^2/2c^2$ and represents the electromagnetic mass of the surrounding field.

The simplest solution of the field equations is an electrostatic field of spherical symmetry surrounding a point charge ϵ (II, Section 4):

$$D = \epsilon/r^2, \quad E = \epsilon/r^2 - \epsilon(1/r^2 + k/r)\exp(-kr).$$

In contrast to D, the field E remains finite at $r=0$, namely, $E = \epsilon k^2/2$. However, similar to Born-Infeld, the field component E_x jumps from the finite value $+|E|$ to $-|E|$ when passing through the singularity, with the average value $E_x = 0$ on the singularity itself. This discontinuity preserves the individuality of the particle. The electric field energy of the particle at rest has value $W = k\epsilon^2/2 = mc^2$.

Two point charges ϵ_1 and ϵ_2 at distance r have field energy

$$W = k\epsilon_1^2/2 + k\epsilon_2^2/2 + (\epsilon_1\epsilon_2/r)\cdot[1 - \exp(-kr)]. \quad (4)$$

Two opposite charges $+\epsilon$ and $-\epsilon$ therefore yield

$$\begin{aligned} W &= mc^2 + mc^2 - \epsilon^2/r \text{ for large } r, \\ W &= mc^2 kr \text{ for small } r. \end{aligned} \quad (5)$$

Hence, when r decreases from ∞ to 0, field energy of value $2mc^2$ is released. It thus turns out that transformation of mutual electrostatic into radiation energy with neutralization of $+\epsilon$ and $-\epsilon$ (annihilation) is possible already in the classical domain.

Another significant feature of our theory concerns electrons vibrating with high frequencies $\nu > \nu_0 = kc/2\pi$. The energy emitted in this case is reduced by a factor (II, Section 6)

$$1 - (1 - \nu_0^2/\nu^2)^{\frac{1}{2}} \quad (6)$$

as compared with the normal Maxwell-Lorentz radiation. For $\nu \gg \nu_0$ the reduction factor approaches $\nu_0^2/2\nu^2$, so that energy is emitted at the rate ν^2 rather than ν^4 per second, or ν rather than ν^3 per period. This is of importance for the quantum theory of stationary states which always presumes that the energy emission during a single period is small compared with the energy of the state itself.

Our theory in its classical form does not distinguish between self-field and external field, except for the special case of uniform motion. On the other hand, the quantum theory of radiation is dualistic. Here one considers electrons in uniform motion with definite self-energies $W = W_0(1-\beta^2)^{-\frac{1}{2}}$ surrounded by a "pure field" with constant Fourier amplitudes. Transitions between stationary states result from mutual perturbations between pure field and particles. The difficulties of the quantum theory of radiation arise primarily from the task of describing the interaction between external field and mechanical particles in a dualistic fashion, although the classical theory of the electron in the field is unitary. The difficulty is solved in Part IV by considering as zero approximation particles without rest mass which therefore travel with the velocity of light. When the interaction with the field is introduced, stationary states of the particles with any velocity smaller than c become possible. These states may be obtained from states with zero velocity by Lorentz transformations, and the rest mass now has a definite value. Our special type of semiunitary field theory in which the particles retain their individuality as singularities although all energy is located in the field, offers a satisfactory background from the classical and quantum points of view.

2. THE PROBLEM OF SELF-ACCELERATION

For vanishing external field Eq. (3) reads

$$d\ddot{x}/d(ct)=a\ddot{x} \quad \text{with} \quad a=3mc^2/2\epsilon^2. \qquad (7)$$

The solution of this non-relativistic equation is

$$dx/d(ct)=\exp(act)+\beta \qquad (7')$$

where β is the velocity ratio v/c at $t=-\infty$. A free electron thus should be able to accelerate itself from any initial velocity v at $t=-\infty$ to larger and larger velocities, the beginning phase of the process being described by (7′). Relativistic corrections are needed for higher velocities only. Dirac[3] has emphasized the fundamental importance of self-acceleration in his relativistic theory of the classical point electron. There he assumes that the force on the point charge is determined by the difference of retarded and advanced potentials, in order to get rid of the infinite self-energy. As a consequence he obtains self-acceleration. However, the whole idea of a free particle acquiring larger and larger velocities on its own accord is so non-physical that any theory yielding self-acceleration might be rejected almost *a priori*. Yet, self-acceleration is an inevitable consequence of any theory in which the electron is subject to a differential equation of motion of finite order because the equation of motion at the beginning of the process for small velocities is always approximated by (7) with the solution (7′). It is a decisive advantage of the present field theory that the singularity does not obey a differential equation of motion. The force on the electron at t depends on the whole path before t rather than on the motion in the immediate neighborhood of t.

In II, Section 4 we discussed an electron moving with coordinate

$$x=ut+\tfrac{1}{2}ft+\tfrac{1}{6}gt^3 \qquad (7'')$$

during a short time t near $t=0$ and with small coefficients u, f, and g whose meaning is $u=(\dot{x})_0$, $f=(\ddot{x})_0$, $g=(d\ddot{x}/dt)_0$ at $t=0$. The self-force was found to be

$$\begin{aligned}\text{self-force}&=-(\epsilon^2k/2c^2)\cdot f+(2\epsilon^2/3c^3)\cdot g\\ &=-m(\ddot{x})_0+(2\epsilon^2/3c^3)(d\ddot{x}/dt)_0\end{aligned}$$

where m is an abbreviation for $(\epsilon^2k/2c^2)$ representing the finite mass of the point charge ϵ. Our requirement of a vanishing total force means vanishing self-force for a free electron, so that the latter at $t=0$ satisfies the equation

$$m(\ddot{x})_0=(2\epsilon^2/3c^3)(d\ddot{x}/dt)_0.$$

This equation does not permit conclusions about self-acceleration before or after $t=0$. Whether self-acceleration is possible depends on the question whether the self-field vanishes during an accelerated motion. Before answering this question of dynamics we must first find the correct kinematic description of a relativistically accelerated motion.

Let us assume tentatively that the motion of a certain free particle were actually of the form (7′) at least for large negative t and for small values of the initial velocity β. Let us ask also: What is the correct relativistic continuation of this accelerated motion? The answer may be found by relativistic transformations without resorting to dynamics. Since the electron is free, the acceleration at any time must depend only on the proper time s measured on the particle itself. For large negative t, however, t and s are identical provided that $\beta\ll 1$. Instead of (7′) we therefore may write relativistically

$$\begin{aligned}&dx/d(cs)=\exp(acs)+\beta,\\ &d(ct)/d(cs)=1 \text{ for } s=-\infty.\end{aligned} \qquad (8)$$

With reference to a Lorentz system xt in which the electron is at rest at the beginning of the process we also have

$$\begin{aligned}&dx_0/d(cs)=\exp(acs),\\ &d(ct_0)/d(cs)=1 \text{ for } s=-\infty.\end{aligned} \qquad (8')$$

Generalizing these equations tentatively for all s values we write

$$\begin{aligned}dx/d(cs)&=S(e^{acs}+b), & d(ct)/d(cs)&=C(e^{acs}+b),\\ dx_0/d(cs)&=S(e^{acs}), & d(ct_0)/d(cs)&=C(e^{acs}),\end{aligned}$$

where $b(\beta)$ is a function of β which coincides with β for small β, whereas $S(z)$ is a function of z which coincides with z for small z like a sine function, and $C(z)\rightarrow 1$ for small z like a cosine function. We now write down relativistic transformation formulae from xt to x_0t_0

$$\begin{aligned}dx&=[dx_0+d(ct_0)\cdot\beta](1-\beta^2)^{-\frac{1}{2}},\\ d(ct)&=[dx_0\cdot\beta+d(ct_0)](1-\beta^2)^{-\frac{1}{2}}.\end{aligned}$$

[3] P. A. M. Dirac, Proc. Roy. Soc. **167**, 148 (1938).

Dividing by $d(cs)$ and writing z for e^{acs} we obtain

$$S(z+b)=[S(z)+C(z)\cdot\beta]\cdot(1-\beta^2)^{-\frac{1}{2}},$$
$$C(z+b)=[S(z)\beta+C(z)]\cdot(1-\beta^2)^{-\frac{1}{2}}.$$

These equations represent rotations in Minkowski space. Indeed, if we write

$$(1-\beta^2)^{-\frac{1}{2}}=\cos(ib)=\mathrm{Cosh}\,(b),\quad \beta(1-\beta^2)^{-\frac{1}{2}}=-i\sin(ib)=\mathrm{Sinh}\,(b),\qquad \mathrm{Tanh}\,b=\beta,$$

we have

$$S(z+b)=S(z)\,\mathrm{Cosh}\,(b)+C(z)\,\mathrm{Sinh}\,(b),$$
$$C(z+b)=S(z)\,\mathrm{Sinh}\,(b)+C(z)\,\mathrm{Cosh}\,(b).$$

These formulae are valid only if the functions S and C are Sinh and Cosh. The correct relativistic continuation of (7′) in the xt system, therefore, is

$$dx/d(cs)=\mathrm{Sinh}\,(e^{acs}+b),\quad d(ct)/d(cs)=\mathrm{Cosh}\,(e^{acs}+b),\qquad b=\mathrm{Tanh}^{-1}\beta,\quad (9)$$

where $\beta=v/c$ describes the original constant velocity of the electron in the xt system at $t=-\infty$, and s is the proper time. Equation (9) represents the only accelerated motion possible for a free electron with (7′) as non-relativistic approximation for $t\approx-\infty$. As a consequence we obtain the following invariant description of self-acceleration:

$$d^2x_r/d(cs)^2=a\cdot\exp(acs),\qquad (9')$$

where x_r is the Lorentz system in which the electron is momentarily at rest.

Next let us calculate the self-force according to the field theory if the electron is relativistically accelerated according to (9). In order to simplify the mathematical problem let us discuss only the initial part of the process between $t=-\infty$ and $t=(1/ac)\cdot\ln C$ where C is a small positive number. During this initial phase we may use the approximation (7′). Shifting the zero point of the time scale we may consider the motion

$$d\xi/d(c\tau)=C\exp(ac\tau),\quad \xi=(C/a)[\exp(ac\tau)-1]\qquad (9'')$$

during $-\infty<\tau<0$. The zero point of the ξ axis coincides with the position of the electron at $\tau=0$. Furthermore, instead of discussing the self-force during the whole time $\tau<0$, we calculate it at $\tau=0$ itself. In the original time scale this is the time $t=(1/ac)\ln C$ where C is any small number, so that what applies to $t=0$ also applies to a continuity of times t in the original time scale.

In order to find the self-force of the motion (9″) at $\tau=0$ we apply the method of retarded potentials (Part II, Section 4) and proceed in three steps.

(1) If the exponential function in (9″) is expanded into a series we have

$$\xi=(C/a)[ac\tau+\tfrac{1}{2}(ac\tau)^2+\tfrac{1}{6}(ac\tau)^2].$$

If we omit all higher terms, the result was already found in II, Section 4:

$$\text{Self-force}=-(\epsilon^2k/2c^2)(\ddot{\xi})_0+(2\epsilon^2/3c^3)(d\ddot{\xi}/d\tau)_0$$
$$=-\epsilon^2a^2C[(k/2a)-(2/3)].$$

The self-force vanishes for $a=0$ (uniform motion) and also for $a=3k/4$ (self-accelerated motion), that is, for $a=3mc^2/2\epsilon^2$ which is Dirac's value.

(2) If the next expansion term $(ac\tau)^4/24$ is added to the series, the self-force becomes infinite for every value of a except $a=0$.

(3) If the complete exponential function $\exp(ac\tau)$ is used the self-force becomes *finite* again due to the fact that successive expansion terms contribute infinite terms of opposite signs. The resulting self-force according to the method II, Section 4 is

$$-e^2C\cdot\int_0^\infty J_1(kR)\{(aR^{-1}e^{-aR})+(a^{-1}R^{-3}e^{-aR})-(R^{-3}a^{-1})+(R^{-2}e^{-aR})\}d(kR)$$

or after integration:

$$\text{Self-force}=-\epsilon^2a^2C\left\{\frac{(a^2+k^2)^{\frac{1}{2}}-a}{a}-\frac{(a^2+k^2)^{\frac{3}{2}}-k^3}{3a^3}+\frac{1}{3}\right\}.$$

If we had used the exact relativistic motion (9) rather than (9″), the bracket would contain an additional term linear in the small quantity C. For small but finite C the self-force vanishes only if the bracket is zero. The latter, however, does not vanish for $a=3k/4$ nor for any other value of a save $a=0$ (or $a\approx0$ if the small correction term linear in C is added). That is, our field theory does not allow self-acceleration in its beginning phase. Therefore, the whole process of self-acceleration is not compatible with the field theory. It will be noticed that the bracket in the last equation for the self-force would vanish for $a=3k/4$, if the positive root $(a^2+k^2)^{\frac{1}{2}}$ were replaced by the negative root. However, because of our application of retarded potentials the root appears with positive sign, and we arrive at the gratifying result that self-acceleration is impossible in our field theory.

3. NEGATIVE ENERGY DENSITY

It seems impossible to devise a field theory of charged particles without violating the requirement that the density of the field energy ought to be larger in the presence of a field than without field, that is, be positive definite in the presence of a field. As an example we mention Born-Infeld's field theory which leads to negative energy densities[4] wherever the vectors D and B have certain large values. (In our theory negative field energy is connected with large frequencies $\nu>\nu_0=kc/2\pi$ rather than with large field intensities). Dirac[3] in his classical theory of the electron simply assumes that an infinite amount of negative energy of unknown origin is condensed near the point electron so as to counterbalance somehow the infinite positive Maxwell energy. He thereby denies that the mass of the electron is of electromagnetic origin.

The negative energy density of the meson field in our theory seems to lead to certain difficulties. Consider an electron vibrating with a frequency ν larger than ν_0 for a limited time only. As shown in Part II, it then emits a group of Maxwell waves E, B and a group of meson waves of polarization $D-E$, $H-B$. The two groups travel with different group velocities and become separated at some distance from the source. The Maxwell group carries more energy than was emitted by the electron; the meson group with its negative energy re-establishes the energy balance. A resonator exposed to the Maxwell group alone then might transform the Maxwell energy into mechanical energy of the resonator, thereby reducing the energy remaining in the field to larger and larger negative values. This objection does not hold, however, in our unitary theory where all energy is located and conserved in the field, and the work on the resonating charged particle is zero. From the energetic point of view our position with respect to negative field energy is the same as that of an observer above sea level who discovers that weights may be dropped below zero level at the expense of other weights raised above sea level. Difficulties appear, however, if one tries to derive a statistical distribution of photons and mesons over positive and negative levels, at least if he wants to apply the same statistical methods which had been developed for positive levels.

In case of Boltzmann statistics the relative number of particles on two energy levels is

$$\bar{n}_1/\bar{n}_2=\exp\,(W_2-W_1/kT),$$

and remains unchanged if both energies are counted from a new zero point. Bose statistics, however, yield the following formula for the average number of particles of energy W:

$$\bar{n}=1/[\exp\,(W/kT)-1],$$

where W is an absolute energy value without an additional constant. $\bar{n}_1/\bar{n}_2$ depends on the absolute position of the zero energy. Furthermore, Bose's formula is meaningless for negative values of W unless a new interpretation of a "negative number of particles" is invented. However, one must remember that Bose particles are "particles" only in a very restricted sense, and that the success of Bose's procedure is more or less accidental. Bose showed that the same Planck radiation law which was derived from an equilibrium between radiation and radiating matter (Planck's and Einstein's derivations) could also be obtained from certain statistics applied to "particles without individuality." We cannot expect that a modified Planck law (asked for by modified field theories) should still be derivable from Bose's corpuscular method. This applies in particular to our case where "pure field" and "electronic field" are undistinguishable within the range $1/k$ of the electronic "radius." Radiation of wave-length $\lambda=2\pi/k$ may be treated in connection with electrons responsible for the radiation equilibrium, similar to Einstein's derivation of Planck's law, as follows.

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[4] M. Born, Proc. Roy. Soc. **143**, 410 (1934). See also W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, 1936), Eq. (5), p. 237.

4. MODIFIED EINSTEIN EQUILIBRIUM

An electron vibrating with frequency $\nu>\nu_0=kc/2\pi$ emits negative meson energy at the ratio of $(1-\nu_0^2/\nu^2)^{\frac{1}{2}}$ to 1 in comparison to the normal Maxwell radiation, according to (6). This ratio has another significant meaning. The number of Jeans proper vibrations per unit cube and per wave-length interval $d\lambda$ is

$$dZ=(8\pi/\lambda^2)d(1/\lambda).$$

Since $c/\lambda=\nu$ for Maxwell waves, and

$$c/\lambda=(\nu^2-\nu_0^2)^{\frac{1}{2}}$$

for meson waves, we obtain the following numbers of levels within the same $d\nu$:

$$dZ''=8\pi\nu^2d\nu/c^3 \text{ for photons } h\nu \tag{10}$$
$$dZ'=(8\pi\nu^2d\nu/c^3)(1-\nu_0^2/\nu^2)^{\frac{1}{2}} \text{ for meaons } (-h\nu).$$

dZ' and dZ'' turn out to have the same ratio as the numbers of mesons and photons emitted "spontaneously" by the vibrating electron during a certain time. Hence, if a group of electrons has emitted just one photon onto every one of the dZ'' levels within $d\nu$, the same group of electrons has also emitted just one meson onto every one of the dZ' meson levels within $d\nu$. The total number of photons and mesons emitted is proportional to $(dZ''+dZ')$ whereas the total energy emitted is proportional to $h\nu\cdot(dZ''-dZ')$.

After this preparation let us discuss Einstein's derivation of Planck's radiation law, modified by our field theory. Two electronic energy levels W_a and $W_b<W_a$ may be occupied by N_a and N_b electrons, respectively. Electronic transitions between the two levels produce emissions and absorptions of quanta $+h\nu$ and $-h\nu$ to and from the Jeans radiation levels within a certain resonance interval $d\nu$ containing dZ'' photon levels and dZ' meson levels. An electronic transition from W_b to the higher level W_a corresponds to an increase of the electronic amplitude of vibration and to a decrease of the Maxwell field strength $|E''|$ (=absorption of photons) and to an increase of the meson field strength $|E'|$ (=emission of mesons). In terms of statistics this means that the probability of an electronic transition from W_b to W_a ought to be proportional to the difference of the numbers $n''dZ''$ of photons and the number $n'dZ'$ of mesons ready to be absorbed from the resonance interval $d\nu$:

$$P_{ba}=N_b[n''dZ''-n'dZ'].$$

Opposite transitions $a\rightarrow b$ induced by the radiation will be proportional to the same difference, whereas spontaneous transitions will be proportional to $1\cdot dZ''-1\cdot dZ'$ [see (6) and (10)] so that

$$P_{ab}=N_a[(n''+1)dZ''-(n'+1)dZ'].$$

In case of equilibrium we must have $P_{ab}=P_{ba}$ in the average, that is,

$$N_b/N_a=\frac{(\bar{n}''+1)dZ''-(\bar{n}'+1)dZ'}{\bar{n}''dZ''-\bar{n}'dZ'}.$$

The bars indicate average numbers of photons and mesons per level. If equilibrium shall apply to photon processes alone and to meson processes alone we must have

$$N_b/N_a=(\bar{n}''+1)/\bar{n}''=(\bar{n}'+1)/\bar{n}',$$

hence, $\bar{n}''=\bar{n}'$. That is, the average number of photons and mesons per Jeans level must be equal. If the electrons are subject to Boltzmann statistics $N=\text{const.}\exp(-E/kT)$ we obtain the following value for the average number of photons and mesons per level:

$$\bar{n}''=\bar{n}'=[\exp(h\nu/kT)-1]^{-1}. \tag{10'}$$

The total radiation energy within $d\nu$ has density

$$w_\nu d\nu=\bar{n}''h\nu dZ''+\bar{n}'(-h\nu)dZ'=(w_\nu''-w_\nu')d\nu$$
$$=\frac{8\pi h\nu^2d\nu}{c^3}\frac{1}{\exp(h\nu/kT)-1}\times[1-(1-\nu_0^2/\nu^2)^{\frac{1}{2}}]. \tag{10''}$$

For $\nu>\nu_0=kc/2\pi$ Planck's radiation formula is multiplied by the same factor (6) which already occurs in the classical energy emission of an electron for $\nu>\nu_0$. The result is obtained without applying statistics to the pure radiation.

5. STABILITY OF EQUILIBRIUM

In order to show that the equilibrium is stable consider small deviations from the average numbers

$$n''=\bar{n}''+\delta'',\quad n'=n'+\delta',$$
$$N_a=\bar{N}_a+\Delta,\quad N_b=\bar{N}_b-\Delta$$

which are related by the condition

$$\Delta = Z'\delta' - Z''\delta''$$

where Z' and Z'' are the number of radiation levels involved (formerly called dZ' and dZ''). Photonic processes lead to an increase of the number of photons, $d(n''Z'')/dt = P_{ab} - P_{ba}$ whereas $dZ' = 0$, that is,

$$d(n''Z'')/dt = (\bar{N}_a + \Delta)(\bar{n}'' + \delta'' + 1)Z'' - (\bar{N}_b - \Delta)(\bar{n}'' + \delta'')Z''.$$

The terms of zero order in the deviations cancel, and the second-order terms shall be neglected. This leaves the first-order terms:

$$d(\delta''Z'')/dt = -(\delta''Z'')(\bar{N}_b - \bar{N}_a) + \Delta \cdot (2\bar{n}'' + 1)Z''.$$

Mesonic processes yield similarly $d(n'Z')/dt = P_{ba} - P_{ab}$ and $dZ'' = 0$, that is, after removing second- and zero-order terms:

$$d(\delta'Z')/dt = -(\delta'Z')(\bar{N}_b - \bar{N}_a) + \Delta(2\bar{n}' + 1)Z'.$$

At last, the electronic number N_a increases with probability $P_{ba} - P_{ab}$, that is, after removing second- and zero-order terms:

$$d\Delta/dt = -\Delta\{(2\bar{n}'' + 1)Z'' - (2\bar{n}' + 1)Z'\} + (\bar{N}_b - \bar{N}_a)(\delta''Z'' - \delta'Z').$$

The last bracket is $-\Delta$. Writing C for the positive constant $\bar{N}_b - \bar{N}_a$, we thus obtain

$$(\alpha) \quad d\Delta/dt = -\Delta\{C + (2\bar{n}'' + 1)Z'' - (2\bar{n}' + 1)Z'\}$$

together with the former results

$$(\beta) \quad d\delta''/dt = -\delta''C + \Delta \cdot (2\bar{n}'' + 1),$$
$$(\gamma) \quad d\delta'/dt = -\delta'C + \Delta \cdot (2\bar{n}' + 1).$$

The factor of $-\Delta$ in (α) is positive since $C > 0$, and $\bar{n}'' = \bar{n}'$, as well as $Z'' > Z'$. Thus the absolute value of Δ will always decrease. The terms with Δ in (β) and (γ) will therefore shrink to zero after some time, and from thereon the absolute values of δ'' and δ' will always decrease. A small deviation from the average values $\bar{n}''$, $\bar{n}'$, $\bar{N}_a$, $\bar{N}_b$ thus tends to disappear automatically according to the transition probabilities accepted before, and the equilibrium turns out to be stable.

6. FIELD EQUATIONS

We now return to our former notation E, H for Maxwell's field E, B, and E'', H'' for Maxwell's D, H (Table I). Both fields shall be derived from potentials V, A and V'', A'', respectively:

$$E = -\nabla V - \dot{A}/c, \quad H = \nabla \times A; \tag{11}$$

$$E'' = -\nabla V'' - \dot{A}''/c; \quad H'' = \nabla \times A''. \tag{11'}$$

However, only the potential V'', A'' shall always obey the Lorentz condition:

$$(\nabla \cdot A) + \dot{V}/c = R, \tag{12}$$

$$(\nabla \cdot A'') + \dot{V}''/c = 0 \tag{12'}$$

where R on the right of (12) may be any scalar function of *xyzt*, to be restricted later. In our applications we usually consider the special case $R = 0$ only. As a consequence of (11), (12) one obtains field equations for E, H and E'', H'':

$$\nabla \times E + \dot{H}/c = 0, \quad (\nabla \cdot H) = 0; \tag{13}$$

$$\nabla \times E'' + \dot{H}''/c = 0, \quad (\nabla \cdot H'') = 0; \tag{13'}$$

$$\nabla \times H - \dot{E}/c = -\Box A + \nabla R, \quad (\nabla \cdot E) = -\Box V - \dot{R}/c; \tag{14}$$

$$\nabla \times H'' - \dot{E}''/c = -\Box A'', \quad (\nabla \cdot E'') = -\Box V''. \tag{14'}$$

The right-hand sides of (14′) represent 4π times the current j/c and density ρ of the true charge (which later on is supposed to be condensed to world lines only). The right-hand sides of (14) represent 4π times the free current and density. $\Box A$ and $\Box V$ have the dimensions of potentials divided by the square of a length. They may be written in the form

$$-\Box A = k^2 A', \quad -\Box V = k^2 V'; \tag{15}$$

$$-\Box A'' = 4\pi j/c, \quad -\Box V'' = 4\pi\rho; \tag{15'}$$

thereby defining a new "potential" A', V'. Hence $(k^2 A' + \nabla R)$ is 4π times the free current density, and $(k^2 V' - \dot{R}/c)$ is 4π the free charge density. As a consequence of (14) (14′) we obtain continuity equations for free and true charge:

$$(\nabla \cdot A') + \dot{V}'/c = -\Box R/k^2, \tag{16}$$

$$(\nabla \cdot j) + \dot{\rho} = 0. \tag{16'}$$

From the potentials A', V' we may also derive a new "field" E', H':

$$E' = -\nabla V' - \dot{A}'/c, \quad H' = \nabla \times A'. \tag{17}$$

(13) and (14′) are Maxwell's equations. It has been assumed that both fields $E, H(=E, B)$ and $E'', H''(=D, H)$ are derived from potentials, and that V'', A'' satisfy the Lorentz condition.

We now postulate a new relation between the two potentials, namely,

$$V'' = V - \Box V/k^2, \quad A'' = A - \Box A/k^2 \tag{18}$$

from which follows

$$E'' = E - \Box E/k^2, \quad H'' = H - \Box H/k^2 \tag{19}$$

or $D = E - \Box E/k^2$ and $H = B - \Box B/k^2$ in the usual notation. Eliminating $\Box A$ and $\Box V$ from (15) and (18) we obtain

$$A = A'' - A', \quad V = V'' - V'. \tag{20}$$

Hence (12) must be the difference of (12′) and (16). That is, the right-hand side R of (12) must satisfy the condition

$$-\Box R + k^2 R = 0. \tag{21}$$

From (20) we learn that

$$E = E'' - E', \quad H = H'' - H' \tag{22}$$

or $E = D - E'$ and $B = H - H'$ in the usual notation, which shows that the fields E' and H' determine an electric (P) and magnetic (M) polarization of the vacuum

$$E' = 4\pi P, \quad H' = -4\pi M. \tag{22'}$$

Equation (16) now reduces to

$$(\nabla \cdot A') + \dot{V}'/c = -R. \tag{23}$$

The field equations for E' and H' according to (17) and (23) read

$$\nabla \times E' + \dot{H}'/c = 0, \quad (\nabla \cdot H') = 0; \tag{24}$$

$$\nabla \times H' - \dot{E}'/c = -\Box A' - \nabla R, \quad (\nabla \cdot E') = -\Box V' + \dot{R}/c. \tag{24'}$$

Subtraction of (15) from (15′) yields, because of (20),

$$-\Box A' + k^2 A' = 4\pi j/c, \quad -\Box V' + k^2 V' = 4\pi\rho. \tag{25}$$

Equation (25) together with (15′)

$$-\Box A'' = 4\pi j/c, \quad -\Box V'' = 4\pi\rho \tag{25'}$$

shows that the true charge is the common source of the two otherwise independent fields E'', H'' and E', H'. True charge shall be condensed on singular world lines only.

The absolute values of the potentials A', V' have a physical meaning without additional constants since they occur in the field equations (14) whose right-hand sides read

$$k^2 A' + \nabla R \quad \text{and} \quad k^2 V' - \dot{R}/c. \tag{26}$$

They represent 4π times the free current and free charge density.

Since the fields $E = E'' - E'$ and $H = H'' - H'$ determine the Lorentz force and work, the energy-momentum tensor T must be the difference $T = T'' - T'$. In particular, the density of energy and flux are

$$w = w'' - w', \quad S = S'' - S', \tag{27}$$

where

$$w' = (1/4\pi)\{\tfrac{1}{2}(E'^2 + H'^2 + k^2 A'^2 + k^2 V'^2) + R\dot{V}'/c - \dot{R}V'/c + \tfrac{1}{2}R^2\}, \tag{28}$$

$$S' = (c/4\pi)\{[E' \times H'] + k^2 V' A' + V' \nabla R + \dot{A}' R/c\},$$

and

$$w'' = (1/4\pi)\tfrac{1}{2}(E''^2 + H''^2), \quad S'' = (c/4\pi)[E'' \times H'']. \tag{28'}$$

Equation (28) contains the right-hand side R of (12) and is more general than the expressions for w' and S' of Part II where we considered the case of $R = 0$ only.

Meson waves in vacuum may have longitudinal field components. Indeed, consider the special case

$$A_x' = a_x' \sin(2\pi\nu t + 2\pi x/\lambda), \quad A_y' = 0, \quad A_z' = 0, \quad V' = v' \cdot \sin(2\pi\nu t + 2\pi x/\lambda).$$

The amplitudes a_x' and V' are not restricted by (12). The field is

$$E_x' = -(a_x'\nu/c + v'/\lambda)2\pi \cos(2\pi\nu t + 2\pi x/\lambda),$$
$$E_y' = E_z' = H_x' = H_y' = H_z' = 0,$$

representing a longitudinal electric wave. In case of Maxwell waves we would have $a_x'' = -v''$, and $E_x'' = 0$. Maxwell waves do not have longitudinal components for two reasons: first, because $\lambda = c/\nu$, and second, because of the

Lorentz condition (12′). Meson waves do not have to satisfy the Lorentz condition, and their relation between λ and ν is

$$(\nu/c)^2=1/\lambda^2+(kc/2\pi)^2. \qquad (29)$$

7. ELECTRIC POLE AND MAGNETIC DIPOLE

In Part II we discussed the potential and field of an electric point charge ϵ. The field equations allow the following solution:

$$V''=\epsilon/r, \quad V'=(\epsilon/r)\exp(-kr), \qquad (30)$$
$$E_r''=\epsilon/r^2, \quad E_r'=\epsilon(1/r^2+k/r)\exp(-kr).$$

At large distance this is the ordinary Coulomb field of a point charge ϵ. In a similar way we now discuss the field of a magnetic dipole of moment μ. The field equations for $R=0$ allow the solution:

$$A_x''=0, \quad A_y''=\mu\frac{\partial}{\partial z}\left(\frac{1}{r}\right), \quad A_z''=-\mu\frac{\partial}{\partial y}\left(\frac{1}{r}\right)$$

$$A_x'=0, \quad A_y'=\mu\frac{\partial}{\partial z}(e^{-kr}/r),$$

$$A_z'=-\mu\frac{\partial}{\partial y}(e^{-kr}/r) \qquad (31)$$

with the magnetic fields

$$H_x''=\mu\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right), \quad H_y''=\mu\frac{3xy}{r^5},$$

$$H_z''=\mu\frac{3xz}{r^5},$$

$$H_x'=\mu\left\{\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right)+k\left(\frac{3x^2}{r^4}-\frac{1}{r^2}\right)+k^2\left(\frac{x^2}{r^3}-\frac{1}{r}\right)\right\}e^{-kr}, \qquad (31')$$

$$H_y'=\mu\left\{\frac{3xy}{r^5}+k\frac{3xy}{r^4}+k^2\frac{xy}{r^3}\right\}e^{-kr},$$

$$H_z'=\mu\left\{\frac{3xz}{r^5}+k\frac{3xy}{r^4}+k^2\frac{xz}{r^3}\right\}e^{-kr}.$$

For large r this is the ordinary Coulomb field of a dipole of moment μ parallel x. Although the field $H=H''-H'$ does not vanish for $r=0$, the Lorentz force $\mu\partial H/\partial x$ vanishes for $r=0$. The classical field theory remains unitary if we postulate as before that the electron moves so that the total Lorentz force and work on the singularity are always zero.

In order to find the energy of the magnetic field we use (27), (28) for the energy density $w=w''-w'$ with function $R\equiv 0$. Integration over space yields

$$W_{\text{mag}}=\mu^2k^3\tfrac{1}{9}.$$

The field energy of the point dipole is **finite** due to magnetic polarization of the vacuum.

If the electron really had a magnetic self-energy like this, the common electric and magnetic radius $1/k$ at which Coulomb's law breaks down would be determined by the equation

$$mc^2=\tfrac{1}{2}\epsilon^2k+\tfrac{1}{9}\mu^2k^3.$$

Substituting the quantum values

$$\mu=\epsilon\hbar/2mc, \quad \epsilon^2=\alpha\hbar c, \quad \alpha=1/137, \qquad (32)$$

this would yield the following equation for $\delta=2mc^2/\epsilon^2k$:

$$\frac{\delta}{2}=\tfrac{1}{2}+\tfrac{1}{9}(\alpha\delta)^{-2}$$

solved by $\delta=16.47$; that is, $1/k=16.47\ (\epsilon^2/2mc^2)$. The radius would be more than sixteen times the electric radius, and the magnetic energy more than fifteen times the electric energy.

However, results obtained from substituting quantum values into classical formulae do not mean very much. Take the example of the ratio of spin M to magnetic moment μ which according to quantum theory and experience is $\mu/M=\epsilon/mc$ corresponding to a g factor 2. In the classical field theory the spin is due to the radial electric field of the pole ϵ and the longitudinal field of the dipole μ which together produce angular momentum about the dipole axis, proportional to the product $\epsilon\mu$. In case of a point pole and point dipole with surrounding Coulomb fields, M becomes infinite. If instead we take a "model" in which the field is present outside the radius $1/k$ only, M becomes $2\epsilon\mu k/3c$. In our invariant field theory in which Maxwell and meson terms are subtracted the spin happens to be **zero**. It is not surprising that the classical theory cannot yield the correct spin, because the angular moment rests on the exact knowledge of perpendicular components of E and H simultaneously which cannot be measured without uncertainty. The actual value $M=\frac{1}{2}\hbar$ is just halfway between the field value $M=0$ and the "normal" value $M=\hbar$.

We learn, however, from these considerations that the ratio of magnetic moment to spin and also to magnetic energy varies widely with the special theory under consideration. The "model" with fields outside $1/k$ yields

$$W_{\text{el}}=\tfrac{1}{2}\epsilon^2k,\quad W_{\text{mag}}=\tfrac{1}{3}\epsilon^2k^3,\quad M=\tfrac{2}{3}\epsilon\mu k/c,$$
$$1/k=23.3\epsilon^2/2mc^2.$$

Quantum theory and experience, however, yield (see Part IV):

$$W_{\text{el}}=\tfrac{1}{2}\epsilon^2k,\quad W_{\text{mag}}=0,\quad M=\tfrac{1}{2}\hbar,$$
$$1/k=\epsilon^2/2mc. \qquad (33)$$

The result that the magnetic energy and magnetic mass are zero in spite of the presence of a magnetic moment μ is suggested already by the simple formula $\mu=\epsilon\hbar/mc$; if m in this formula should depend on μ implicitly, the ratio between electric and magnetic energy would have complicated values [like 16.47 (see above)]. Also, the radius $1/k$ would be much too large as soon as the magnetic energy plays any considerable part in the self-energy [even with so small a factor as $\frac{1}{9}$ (see above)]. Already the discussion of various classical models and field theories shows that it is well possible already on a classical basis to construct a dipole moment whose surrounding magnetic energy happens to vanish.

The satisfactory result of Part IV that the electron does not have magnetic self-energy arising from its magnetic moment, and that its radius is the electrostatic radius $1/k=\epsilon^2/2mc^2$, is explained in the following way. In the first place, the only sources of the field according to the field equations (25), (25′) are the electric charges, and there are no retarded potentials from magnetic poles or dipoles. Turning to the Fourier representation of Part II, Section 7, the Hamiltonian consists of terms referring to the particles alone, to the field alone, and perturbation terms. In our unitary theory the terms referring to the particles alone vanish [first two terms in II, (26)]. The part of the "mutual" energy which is proportional to ϵ^2 may be interpreted as electromagnetic "self-energy." It arises from scalar and vector potentials, but for a particle at rest the contribution of the scalar potential [last term in first line of II, (26)] is of opposite sign and half as large as the contribution of the vector potential [second line in II, (26)]; the resultant energy is $-\frac{1}{2}k\epsilon^2+k\epsilon^2=\frac{1}{2}k\epsilon^2$. If the particle is in an external field one also obtains mutual energy terms proportional to ϵ, namely, for a particle at rest $\epsilon\cdot(V'-V'')_{\text{ext}}$ and also, if Dirac's quantum method is applied, the scalar product $(\mu\cdot H''-H')$ where μ stands for $\epsilon\hbar/2mc$ as though the electron had a dipole moment μ. There are no terms, however, proportional to μ^2 which could be interpreted as magnetic self-energy. In spite of displaying spin and magnetic moment μ, the electron at rest has only electrostatic self-energy, and the mass is $m=\frac{1}{2}\epsilon^2k/c^2$.