contribute much less to the level widths. The relative constancy of the width ratios for these lines over so great an atomic number range supports the ratio method for handling width data for elements of vastly different atomic number.

IV. CONCLUSIONS

Since the width of a state is inversely proportional to the mean life of atoms in that state, a very probable Auger transition increases the level width. This results in a width increase for all lines which have this level as either an initial or final state. Because Auger transition probabilities frequently vary rapidly as a function of atomic number, one should expect rapid changes in line width. This prediction is verified. In this paper it has been shown that the width variation of $L\beta_3$ for elements $73 \leq Z \leq 92$ can be accounted for in terms of the Auger transitions $M_{\mathrm{III}} \rightarrow M_{\mathrm{V}}N_{\mathrm{IV,V}}$. Also the anomalous width behavior of $M\zeta$ reported by Kiessig has been shown to be associated with the rapid changes in the Auger probabilities $N_{\mathrm{II}} \rightarrow N_{\mathrm{III}}O_{\mathrm{II,III}}$, $N_{\mathrm{II}} \rightarrow N_{\mathrm{IV,V}}N_{\mathrm{IV,V}}$, and $N_{\mathrm{III}} \rightarrow N_{\mathrm{IV,V}}N_{\mathrm{IV,V}}$ for elements $38 \leq Z \leq 58$. Certain other irregular width variations have been shown to be explicable in terms of Auger transitions. More data on line widths would doubtless reveal other cases in which line widths are increased by Auger transitions.

The author wishes to thank Dr. F. R. Hirsh, Jr., for his encouragement and suggestions.

PHYSICAL REVIEW VOLUME 65, NUMBERS 5 AND 6 MARCH 1 AND 15, 1944

On Apertures of Transmission Type Electron Microscopes Using Magnetic Lenses

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(Received January 5, 1944)

The numerical aperture of light microscopes is larger than that of electron microscopes at their present state of development by a factor of 100 to 1000. This is due to the fact that electron lenses have higher aberrations than the highly corrected glass lens systems used in modern light microscopes. In the design of an electron microscope care has to be taken therefore that the proper numerical aperture should be used to give the optimum resolution. The majority of transmission type electron microscopes use at present magnetic lenses. For magnetic lenses of the axial field distribution $H(z)=H_0/[1+(z/a)^2]$ optimum conditions for size and location of apertures will be stated. The behavior of the condenser lens-objective lens system with respect to the angular aperture of the illuminating electron beam also will be discussed.

I. THE MAGNETIC LENS OF THE AXIAL FIELD DISTRIBUTION $H(z)=H_0/[1+(z/a)^2]$

THE small values of numerical apertures of electron microscopes can be achieved by allowing only such electrons to take part in the image formation whose paths do not pass through the lenses too far away from the optical axis. These paths are called paraxial. They can be described mathematically as the solutions $r=r(z)$ of the so-called paraxial ray differential equation:

$$\frac{d^2r(z)}{dz^2}+\frac{e}{8mV}H^2(z)r(z)=0, \tag{1}$$

where $r=r(z)$ signifies the distance of the electron from the axis at the point z—the axial coordinate; $H(z)$ is the axial field distribution; e/m is the specific charge of the electron, and V is the accelerating potential for the electrons.

It has been shown[1] that for an axial field distribution, very closely approximated in practical magnetic lenses,

$$H(z)=H_0/[1+(z/a)^2]. \tag{2}$$

The solution of Eq. (1) is given by

$$\begin{aligned} r(z)=a(1+(z/a)^2)^{\frac{1}{2}} \\ \times\{C_1 \sin [(1+K^2)^{\frac{1}{2}} \operatorname{arc\,cot} (z/a)] \\ +C_2 \cos [(1+K^2)^{\frac{1}{2}} \operatorname{arc\,cot} (z/a)]\}, \end{aligned} \tag{3}$$

[1] Walter Glaser, Zeits. f. Physik **117**, 285 (1941).

where H_0 is the maximum field strength on the axis, $z=\pm a$ is that coordinate where $H(z)=H(\pm a)=\frac{1}{2}H_0$ ($2a$ is termed the half-width of the field curve), and

$$K^2=eH_0{}^2a^2/8mV \tag{4}$$

is a parameter characterizing the lens strength. C_1 and C_2 are two arbitrary constants of integration, which can be determined so that the path satisfies any initial or boundary conditions.

II. ANGULAR APERTURE. THE CONDENSER LENS-OBJECTIVE LENS SYSTEM

The numerical aperture is defined by

$$N=n\cdot\sin\alpha,$$

where n is the index of refraction and α is the so-called angular aperture, that is, the largest angle formed by light rays which pass the center of the object. In electron microscopes $n=1$ and $\sin\alpha\approx\alpha$ for the values $\alpha=10^{-3}$ to 10^{-2} radian.

Borries and Ruska[2] calculated the angular aperture of a lens system, consisting of a condenser lens and an objective lens, for various lens strength of the condenser lens (Fig. 1a). Their results are given by the following formulas and graphs (Fig. 1b).

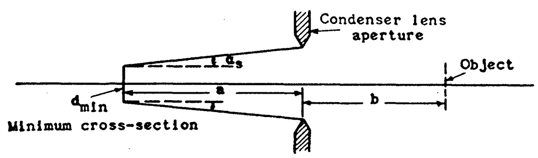


FIG. 1a. Condenser lens. Notation for Eq. (5).

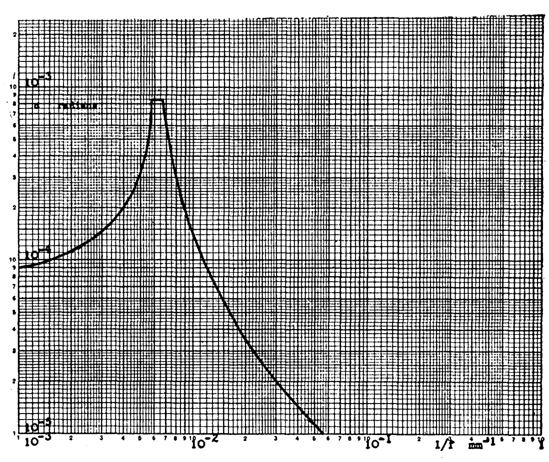


FIG. 1b. Angular aperture in function of the refracting power of the condenser lens according to B. v. Borries and E. Ruska.

$$\alpha_c=\frac{d_{\min}/2}{a+b-(a\cdot b/f_c)} \quad \text{for} \quad 0<\frac{1}{f_c}<\left(\frac{a+b}{a\cdot b}-\frac{d_{\min}}{2a^2\alpha_s}\right),$$

$$\alpha_c=\alpha_{c\ \max}=\frac{a}{b}\alpha_s \quad \text{for} \quad \left(\frac{a+b}{a\cdot b}-\frac{d_{\min}}{2a^2\alpha_s}\right)<\frac{1}{f_c}<\left(\frac{a+b}{a\cdot b}+\frac{d_{\min}}{2a^2\alpha_s}\right), \tag{5}$$

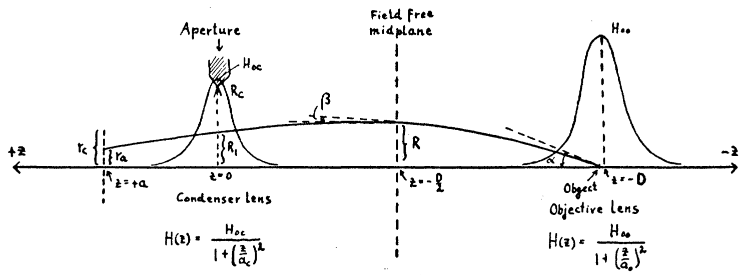
$$\alpha_c=\frac{d_{\min}/2}{(a\cdot b/f_c)-(a+b)} \quad \text{for} \quad \left(\frac{a+b}{a\cdot b}+\frac{d_{\min}}{2a^2\alpha_s}\right)<\frac{1}{f_c}<\infty,$$

where $d_{\min}$ is the diameter of the minimum cross section of the electron beam in front of the cathode (the virtual source), a is the distance of this minimum from the condenser lens, and b is the distance of the object from the same lens. f_c is the focal length of this lens and α_s is the angle given in Fig. 1a.

In these calculations, of purely light optical character, it was assumed that no other lens was located between the condenser lens and the object. This, however, is no longer true if the object is located so that it is well within the field of the objective lens. Glaser[1] has shown that for the condition of minimum spherical aberration this location is imperative. In this case a part of the field of the objective lens lies between the object and the condenser lens and has a focusing effect on the incoming illuminating electron beam, changing the angular aperture as given by the condenser lens strength alone.

Assuming for both lenses a field form given by Eq. (2) it is possible to recalculate the value of the angular aperture, taking into account the focusing action of the objective lens fragment. It is necessary, however, to assume a field free plane midway between the objective lens and the condenser lens—an assumption normally realized in electron microscopes (Fig. 2).

[2] B. v. Borries and E. Ruska, Zeits. f. tech. Physik **20**, 225 (1939).



Notation:

$\left.\begin{matrix} H_{00} \\ H_{0c} \end{matrix}\right\}$ maximum field strengths of the lenses.

$\left.\begin{matrix} 2a_0 \\ 2a_c \end{matrix}\right\}$ half-widths of the field strength curves.

$$\left.\begin{matrix} K_{00}^2 = \dfrac{eH_{00}^2 a_0^2}{8mV} \\ K_{0c}^2 = \dfrac{eH_{0c}^2 a_c^2}{8mV} \end{matrix}\right\} \text{ or } \left\{\begin{matrix} \omega_0 = (1+K_{00}^2)^{\frac{1}{2}} \\ \omega_c = (1+K_{0c}^2)^{\frac{1}{2}} \end{matrix}\right\} \text{ parameters characterizing the strengths of the lenses.}$$

FIG. 2. Electron path through the condenser lens and objective lens.

The values of ω realized in practice are $1 \leqq \omega \leqq 2$; r_c is the radius of the cross-over disk of electrons in front of the cathode of the electron microscope; R_c is the radius of the condenser lens aperture; R, R_1, r_a are ordinates of the electron path at the abscissas $z=-D/2$, $z=0$, $z=+a$, respectively.

The method for arriving at a solution for the angular aperture α consists of the following steps. Two solutions of the paraxial ray equation, one for each lens, are taken and matched at the field free midplane so that their distances from the optical axis are equal and their slopes are identical. The total path has to pass the center of the object and must also go through a point on the circular disk of the electron cross-over. Among all such paths the one is taken which gives the largest angle α for given lens strengths and position coordinates. This $\alpha_{\max}$ is then the angle determining the angular aperture.

Glaser[1] has shown that for minimum spherical aberration the object would have to be very near the center of the objective lens. For the sake of simplicity it is assumed here that the object is at the center of the lens.

Rewriting the formulas of Borries and Ruska in terms of quantities used here, we find the angle $\alpha_{\max}$ is given by

$$\tan \alpha_{\max} = \frac{-r_a \omega_c}{a_c\{[1+(a/a_c)^2][1+(D/a_c)^2]\}^{\frac{1}{2}} \sin \{\omega_c[\operatorname{arc\,cot}(a/a_c) - \pi + \operatorname{arc\,cot}(D/a_c)]\}}. \tag{6}$$

By varying ω_c between 1 and 2 a variation of $\alpha_{\max}$ is obtained which is identical with that given by the previous formulas.

If ω_c becomes

$$\omega_c = \frac{\pi}{\operatorname{arc\,cot}(a/a_c) - \pi + \operatorname{arc\,cot}(D/a_c)}, \tag{7}$$

the cross-over would have its image at the object position, which means that rays leaving the center of the cross-over at any angle would pass through the center of the object at any angle up to 90°. This angle, however, cannot be reached because of the limiting effect of the physical aperture of the condenser lens (Fig. 3).

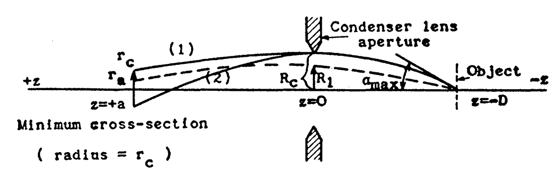


FIG. 3. Limiting effect of the physical aperture of the condenser lens.

There will be a value ω_{c1} which realizes path (1) and a greater value $\omega_{c2}(\omega_{c2}>\omega_{c1})$ which realizes path (2). In order to find the limiting ω_c it is necessary to find the distance R_1, at which any path passes the condenser lens center (where the physical aperture is assumed to be located). R_1 is given by

$$R_1=\frac{a_c}{\omega_c}\left[1+\left(\frac{D}{a_c}\right)^2\right]^{\frac{1}{2}}\tan\alpha\sin\left[\omega_c\left(\operatorname{arc\,cot}\frac{D}{a_c}-\frac{\pi}{2}\right)\right] \tag{8}$$

together with

$$r_c=\frac{a_c}{\omega_c}\left\{\left[1+\left(\frac{a}{a_c}\right)^2\right]\left[1+\left(\frac{D}{a_c}\right)^2\right]\right\}^{\frac{1}{2}}\tan\alpha\sin\left[\omega_c\left(\pi-\operatorname{arc\,cot}\frac{a}{a_c}-\operatorname{arc\,cot}\frac{D}{a_c}\right)\right]; \tag{9}$$

the ratio R_1/r_c can be plotted as a function of ω_c:

$$\frac{R_1}{r_c}=\frac{1}{[1+(a/a_c)^2]^{\frac{1}{2}}}\frac{\sin\{\omega_c[\operatorname{arc\,cot}(D/a_c)-(\pi/2)]\}}{\sin\{\omega_c[\pi-\operatorname{arc\,cot}(a/a_c)-\operatorname{arc\,cot}(D/a_c)]\}}. \tag{10}$$

ω_{c1} and ω_{c2} are the values for which $R_1/r_c=\pm R_c/r_c$. For all values of ω_c, $\omega_{c1}\leqq\omega_c\leqq\omega_{c2}$ the physical aperture becomes the limiting factor for the angular aperture.

Repeating now the same procedure for the case where the fragment of the objective lens is active in forming the angular aperture, we obtain the following expressions:

$$\tan\alpha_{\max}=\frac{(1/a_c)r_a\omega_c[1+(D/2a_c)^2]^{\frac{1}{2}}/[1+(a/a_c)^2]^{\frac{1}{2}}}{(K_1/a_c)\omega_c\cos\{\omega_c[\pi-\operatorname{arc\,cot}(a/a_c)-\operatorname{arc\,cot}(D/2a_c)]\}+\{(K_1D/2a_c^2)+K_2[1+(D/2a_c)^2]\}\sin\{\omega_c[\pi-\operatorname{arc\,cot}(a/a_c)-\operatorname{arc\,cot}(D/2a_c)]\}}, \tag{11}$$

where

$$\begin{aligned}K_1&=\frac{a_0}{\omega_0}\left[1+\left(\frac{D}{2a_c}\right)^2\right]^{\frac{1}{2}}\sin\left[\omega_0\left(\frac{\pi}{2}-\operatorname{arc\,cot}\frac{D}{2a_c}\right)\right],\\ K_2&=\frac{1}{\omega_0}\frac{1}{[1+(D/2a_c)^2]^{\frac{1}{2}}}\left\{\frac{D}{2a_0}\sin\left[\omega_0\left(\frac{\pi}{2}-\operatorname{arc\,cot}\frac{D}{2a_c}\right)\right]+\omega_0\cos\left[\omega_0\left(\frac{\pi}{2}-\operatorname{arc\,cot}\frac{D}{2a_0}\right)\right]\right\}.\end{aligned} \tag{12}$$

K_1 and K_2 depend on ω_0, the parameter of the objective lens, and the location of the object; these quantities are plotted in Fig. 4.

In order to determine the range of ω_c values for which the physical aperture of the condenser lens becomes the limiting factor for the angular aperture α the distance R_1 must be determined.

The equation giving R_1 in this case is

$$\begin{aligned}R_1=\frac{a_c}{\omega_c}\frac{1}{[1+(D/2a_c)^2]^{\frac{1}{2}}}K_1&\left\{\frac{\omega_c}{a_c}\cos\left[\omega_c\left(\frac{\pi}{2}-\operatorname{arc\,cot}\frac{D}{2a_c}\right)\right]\right.\\ &\left.+\left[\frac{D}{2a_c^2}+\frac{K_2}{K_1}\left(1+\left(\frac{D}{2a_c}\right)^2\right)\right]\sin\left[\omega_c\left(\frac{\pi}{2}-\operatorname{arc\,cot}\frac{D}{2a_c}\right)\right]\right\},\end{aligned} \tag{13}$$

and from Eq. (11)

$$\begin{aligned}r_c=\frac{a_c}{\omega_c}\left[\frac{1+(a/a_c)^2}{1+(D/2a_c)^2}\right]^{\frac{1}{2}}K_1&\left\{\frac{\omega_c}{a_c}\cos\left[\omega_c\left(\pi-\operatorname{arc\,cot}\frac{a}{a_c}-\operatorname{arc\,cot}\frac{D}{2a_c}\right)\right]\right.\\ &\left.+\left[\frac{D}{2a_c^2}+\frac{K_2}{K_1}\left(1+\left(\frac{D}{2a_c}\right)^2\right)\right]\sin\left[\omega_c\left(\pi-\operatorname{arc\,cot}\frac{a}{a_c}-\operatorname{arc\,cot}\frac{D}{2a_c}\right)\right]\right\}\tan\alpha.\end{aligned} \tag{14}$$

Forming again R_1/r_c and plotting this ratio as a function of ω, we find the two values ω_c, for which $R_1/r_c = \pm R_c/r_c$. These are called ω_{c1}, ω_{c2}. For all values ω for which $\omega_{c1} \leqq \omega_c \leqq \omega_{c2}$ the angular aperture is determined by the physical condenser lens aperture.

Furthermore, there will be one value $\omega = \omega^*$ for which the denominator in Eq. (11) becomes zero (or $R_1/r_c = +\infty$). For this value ω^* the cross-over of the electrons in front of the cathode is focused on the object. This means that all rays starting at the center of the cross-over will pass through the center of the object. In Eq. (11) r_a and the denominator become zero; hence tan α is indeterminate, as it should be.

Figures 5 and 6 are obtained if the following values are assumed:

$D = 420$ mm, $a = 250$ mm, $a_c = 3$ mm, $a_0 = 1.5$ mm; $r_c = 0.05$ mm, $R_c = 0.4$ mm, hence $R_c/r_c = 8$; $\omega_0 = 1.5$, $1 \leqq \omega_c \leqq 2$.

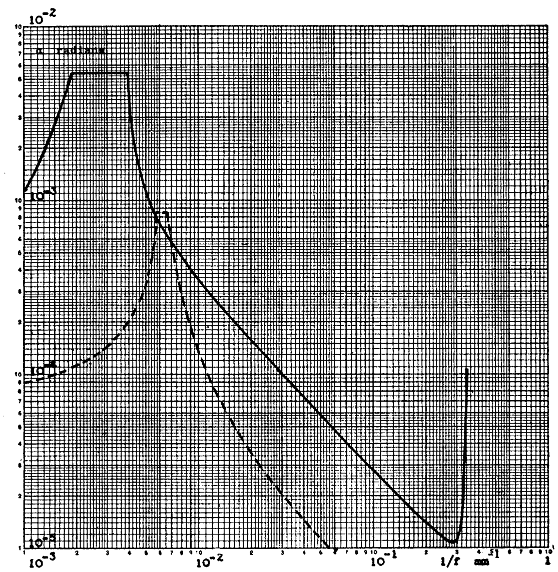


FIG. 5. Angular aperture of the incoming beam in function of the refracting power of the condenser lens.

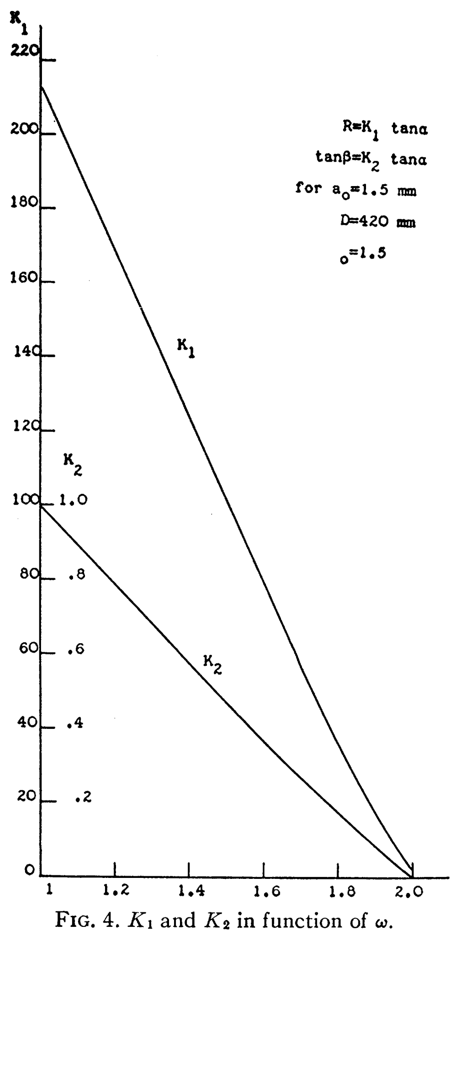


FIG. 4. K_1 and K_2 in function of ω.

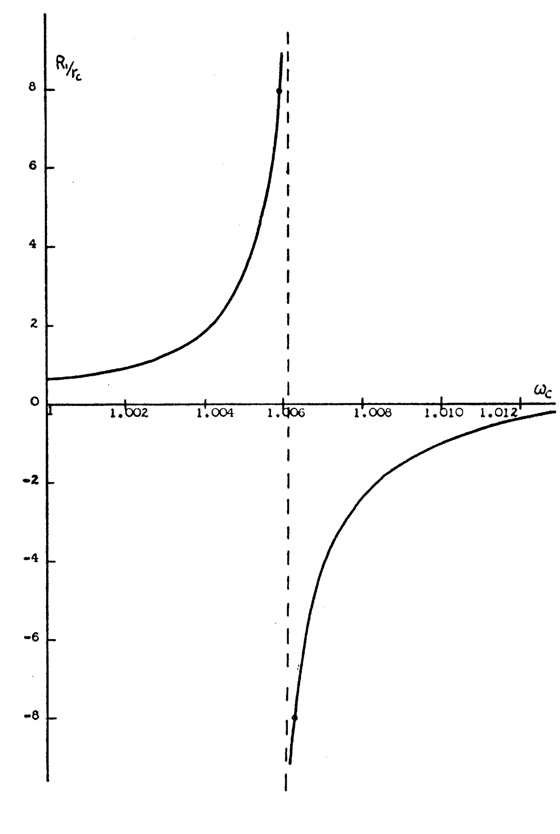


FIG. 6. R_1/r_c as a function of ω_c. [$\omega_{c1} = 1.0059$; $\omega^* = 1.0061$; $\omega_{c2} = 1.0063$.]

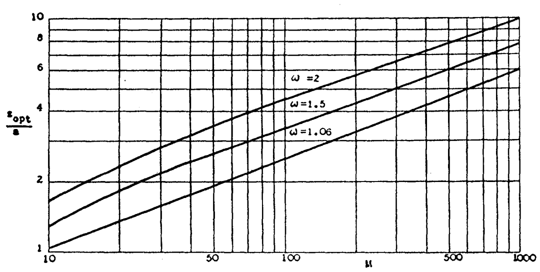


FIG. 7. Location of the aperture on the optical axis in function of the magnification and the lens parameters a and ω.

For the sake of comparison, Fig. 5 shows α_{max} as calculated by means of the indicated methods and as calculated for the same numerical values by the method of v. Borries and Ruska (dotted line).

Contrary to previous assumptions it is shown that the angular aperture cannot be decreased below any prescribed limit by increasing the refracting power of the condenser lens. There exists a definite minimum for α, as observed in electron microscopes, because of the effect of refracting properties of the objective lens fragment.

The formulas derived so far allow the determination of the angular aperture of the incoming electron beam. After passing the specimen the electrons are scattered and diffracted causing the angular aperture to be increased. In order to limit the angular aperture to values prescribed by limitations of the resolving power physical apertures must be used. These apertures are usually located at the center of the objective lens. It is obvious, however, that the diameter of the aperture must become smaller and smaller the nearer the object is to the center of the lens, if it has to achieve the same ray limiting action.

Knowing the exact solution for the electron path it is possible to determine an optimum position and size for such an aperture which achieves the same ray limiting effect. Let

$$y(z)=r(z)/a \quad \text{and} \quad x=z/a; \tag{15}$$

then Eq. (3) can be written as

$$y(z)=(1+x^2)^{\frac{1}{2}}\{C_1 \sin[\omega \operatorname{arc\,cot} x] + C_2 \cos[\omega \operatorname{arc\,cot} x]\}. \tag{16}$$

If the constants of integration C_1 and C_2 are determined so that the path goes through a point $x=x_0$, the x coordinate of the object position, and has there a slope $y'(x_0)=\tan\alpha$, Eq. (3) becomes:

$$y(x)=\frac{1}{\omega}[(1+x^2)(1+x_0^2)]^{\frac{1}{2}} \sin[\omega(\operatorname{arc\,cot} x_0 - \operatorname{arc\,cot} x)] \tan\alpha \tag{17}$$

or

$$y(x)=\frac{\sin[\omega(\operatorname{arc\,cot} x_0-\operatorname{arc\,cot} x)]}{\sin[\operatorname{arc\,cot} x_0]\sin[\operatorname{arc\,cot} x]}\cdot\frac{\tan\alpha}{\omega}. \tag{18}$$

The slope of this path at any point x is given by

$$y'(x_0)=\frac{\tan\alpha}{\omega}\left[\frac{1+x_0^2}{1+x^2}\right]^{\frac{1}{2}} \times\{x\sin[\omega(\operatorname{arc\,cot} x_0-\operatorname{arc\,cot} x)] +\omega\cos[\omega(\operatorname{arc\,cot} x_0-\operatorname{arc\,cot} x)]\}. \tag{19}$$

Let x_m be the x coordinate of the point of maximum deviation of the electron from the axis. Then x_m is given by the transcendental equation:

$$\operatorname{arc\,cot} x_0=\operatorname{arc\,cot} x_m+\frac{\pi}{\omega}-\frac{1}{\omega}\operatorname{arc\,cot}\frac{x_m}{\omega}. \tag{20}$$

The next step is to combine Eq. (8) with the formula for the magnification of the lens.[1]

$$M=\frac{1}{x_0-\cot(\pi/\omega)}\cdot\frac{1}{\sin(\pi/\omega)}. \tag{21}$$

M is then given as a function of x_m and the lens parameter

$$M=\frac{\sin\{\operatorname{arc\,cot} x_m+(\pi/\omega)-(1/\omega)\operatorname{arc\,cot}(x_m/\omega)\}}{\sin\{(1/\omega)\operatorname{arc\,cot}(x_m/\omega)-\operatorname{arc\,cot} x_m\}}. \tag{22}$$

For large values of M ($M>100$) Eq. (22) becomes:

$$M=3\frac{\sin(\pi/\omega)}{(1-\omega^2)}x_m^3. \tag{23}$$

The position coordinate of the maximum path deviation from the optical axis is plotted in Fig. 7 as a function of the magnification for several lens parameters. The value $\omega=1.06$ corresponds to a weak lens; ($\omega=1$ characterizes a lens of zero refracting power!) and $\omega=2$ to a strong lens. (It has been shown by Glaser[1] that for values $\omega \geqq 2$ an object has two images.)

An aperture placed at $x=x_m$ on the optical axis can have a considerably greater size than one placed at the center of lens and yet have the same ray limiting effect. The size of the aperture for any angular aperture α (as determined for a permissible spherical aberration) can be determined by the use of the formula:

$$\frac{\text{radius of aperture}}{\tan\alpha}=\frac{r_A}{\tan\alpha}=a\left[\frac{(1+x_m^2)(1+x_0^2)}{(\omega^2+x_m^2)}\right]^{\frac{1}{2}}. \quad (24)$$

$2r_A/a\tan\alpha$ is plotted in Fig. 8 as a function of the magnification for various values of ω.

The ordinate of the path at the center of the lens is given by

$$r(0)=\frac{a}{\omega}(1+x_0^2)^{\frac{1}{2}}\sin\left[\omega\left(\operatorname{arc}\cot x_0-\frac{\pi}{\omega}\right)\right]\tan\alpha. \quad (25)$$

The ratio of $2r_A/2r(0)=d_A/d_0$ is plotted in Fig. 9 as a function of the magnification for various values of ω. It shows the gain in aperture size for an aperture at the optimum position as compared with the customary aperture design at the center of the lens. The ratio is given by

$$\frac{r_A}{r(0)}=\omega\left[\frac{1+x_m^2}{\omega^2+x_m^2}\right]^{\frac{1}{2}}\frac{1}{\sin\{\omega[\operatorname{arc}\cot x_0-(\pi/2)]\}}. \quad (26)$$

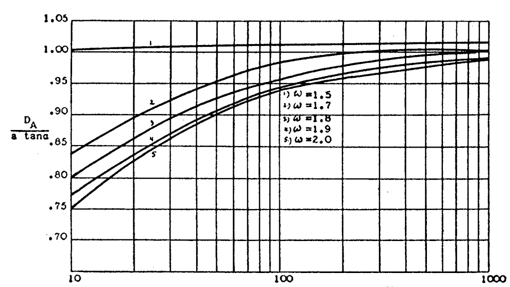


FIG. 8. Reduced aperture size in function of the magnification for various lens strength parameters.

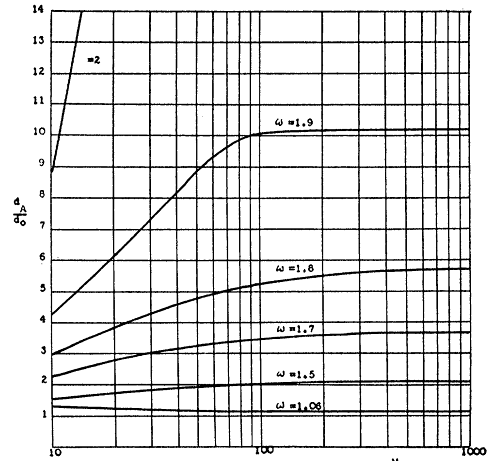


FIG. 9. Ratio of aperture sizes of two equivalent apertures, one located at optimum position (diameter d_A) and the other located at the center of the lens (diameter d_0).

The stronger the lens the more unfavorable is the aperture position at the center of the lens.

Because of the increased size, the difficulties in machining the apertures are reduced. Another advantage results from the location of the aperture outside the lens inasmuch as the removal for cleaning purposes is facilitated. Furthermore, the exact location of the aperture on the axis is not too critical since the path is parallel to the optical axis at the optimum position and hence changes its slope less rapidly than at other positions.