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The Electrical Oscillations of a Prolate Spheroid. Paper III

The Antenna Problem

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The straight wire antenna is approximated by a perfectly conducting prolate spheroid of eccentricity very close to unity. (1) The first three harmonics of the axially symmetrical free oscillations are discussed, with calculation of the wave-length λ and the logarithmic decrement δ. (2) The tangential electric field component of a plane wave with **E** parallel to the axis of the spheroid is expanded in the prolate spheroidal wave functions of Paper II. (3) The oscillations forced by such a plane wave are discussed for frequencies in the neighborhood of resonance with reference to current distribution, radiation resistance, and scattered field.

1. FREE OSCILLATIONS

WE shall discuss briefly the axially symmetrical free oscillations of a perfectly conducting prolate spheroid so thin that the squares and higher powers of the value t_0 of $t\equiv(\eta^2-1)^{\frac{1}{2}}$ at the surface[1] can be neglected as compared with unity. Under these circumstances t_0 represents effectively the ratio of the minor semi-axis b of the spheroid to the major semi-axis a. We are concerned with the wave functions (II-15), in which v_{l1} stands for the function r_{l1} representing a diverging wave as given by (II-32).

The sole boundary condition is $E_\xi=0$ for $t=t_0$, that is,

$$\frac{d}{dt}\{tr_{l1}(\eta)\}=0, \quad t=t_0. \tag{III-1}$$

a. Fundamental or First Harmonic

For $l=1$, Eq. (III-1) becomes

$$b_{11}=(4/3^2)i\epsilon^3 l_{11}a_{11}, \tag{III-2}$$

where

$$l_{11}\equiv\left[\log\frac{(1+t_0^2)^{\frac{1}{2}}+1}{(1+t_0^2)^{\frac{1}{2}}-1}-2-\frac{1}{5}\epsilon^2+\frac{9}{5^3\cdot 7}\epsilon^4-\frac{886}{3^4\cdot 5^5\cdot 7}\epsilon^6+\cdots\right]^{-1}$$

if we neglect squares and higher powers of t_0.

For the limiting case $t_0=0$ of an infinitely thin antenna $l_{11}=0$ and hence $b_{11}=0$. Hence, as shown in II, $\epsilon=\pi/2$. Consequently, if λ is the wave-length and δ the logarithmic decrement of the free oscillations,

$$\lambda=4a, \quad \delta=0. \tag{III-3}$$

To discuss antennas of finite thickness we put $t_0\equiv(10)^{-n}$ and make $\epsilon=\pi/2$ everywhere except in the factor of b_{11} which vanishes for this value of ϵ. This approximation yields results accurate to within one percent for $n\geq 3$ and to within two or three percent for $n=2$. Then we find from (III-2) that

$$\epsilon=\frac{\pi}{2}\left[1-\frac{0.1685i}{n-0.228}\right].$$

[1] See Paper II for definitions of the symbols appearing in this paper. References to Paper II will be indicated by II and to Paper I by I.

In terms of the wave-length and the logarithmic decrement

$$\epsilon=\frac{\pi}{2}\frac{4a}{\lambda}\left[1-i\frac{\delta}{2\pi}\right]. \tag{III-4}$$

Therefore

$$\lambda=4a, \quad \delta=1.059/(n-0.228). \tag{III-5}$$

b. Second Harmonic

For $l=2$ we have the boundary condition

$$b_{21}=\frac{4}{3^3\cdot 5^2}i\epsilon^5 l_{21}a_{21}, \tag{III-6}$$

where

$$l_{21}\equiv\left[\log\frac{(1+t_0^2)^{\frac{1}{2}}+1}{(1+t_0^2)^{\frac{1}{2}}-1}-3+\frac{1}{3\cdot 7}\epsilon^2-\frac{95}{3^3\cdot 7^3}\epsilon^4+\cdots\right]^{-1}.$$

For a thin antenna this gives approximately

$$\epsilon=\pi\left[1-\frac{0.107i}{n-0.5}\right],$$

from which it follows that

$$\lambda=2a, \quad \delta=0.67/(n-0.5). \tag{III-7}$$

c. Third Harmonic

For $l=3$ the boundary condition becomes

$$b_{31}=\frac{32}{3^3\cdot 5^4\cdot 7^2}i\epsilon^7 l_{31}a_{31}, \tag{III-8}$$

where

$$l_{31}\equiv\left[\log\frac{(1+t_0^2)^{\frac{1}{2}}+1}{(1+t_0^2)^{\frac{1}{2}}-1}-\frac{11}{3}+\frac{1}{2\cdot 3\cdot 5}\epsilon^2-\frac{59}{2\cdot 3^4\cdot 5^3\cdot 11}\epsilon^4+\cdots\right]^{-1}.$$

As the series for b_{31} converges too slowly for $\epsilon=3\pi/2$, we have used (II-37). For a thin antenna we get, approximately,

$$\epsilon=\frac{3\pi}{2}\left[1-\frac{0.086i}{n-0.4}\right]$$

from which it follows that

$$\lambda=(4a/3), \quad \delta=0.54/(n-0.4). \tag{III-9}$$

The series given in II have not been carried far enough to calculate even rough values of the logarithmic decrement for higher harmonics than the third. However, the wave-length for a thin antenna is given by $\lambda=4a/l$ for the lth harmonic, however great the integer l may be.

2. INCIDENT WAVE

Preparatory to the investigation of the electrical oscillations of a perfectly conducting prolate spheroid forced by a plane electromagnetic wave of angular frequency ω with the electric vector parallel to the long axis of the spheroid, it is necessary to expand the field components of the incident wave in the prolate spheroidal wave-functions specified by Eqs. (II-15) to (II-18). Evidently $v_{lm}(\eta)=p_{lm}(\eta)$ since $q_{lm}(\eta)$ becomes infinite at the origin.

Designating unit vectors in the directions of increasing ξ, η, ϕ by $\boldsymbol{\xi}_1$, $\mathbf{n}_1$, $\boldsymbol{\phi}_1$, respectively, and putting $s\equiv(1-\xi^2)^{\frac{1}{2}}$, $t\equiv(\eta^2-1)^{\frac{1}{2}}$, $\epsilon\equiv(\kappa\mu)^{\frac{1}{2}}(\omega f/c)=2\pi f/\lambda$, as before, we find the electric and magnetic field intensities in the incident plane wave are

$$\mathbf{E}^i=\left[\boldsymbol{\xi}_1\frac{\eta s}{(\eta^2-\xi^2)^{\frac{1}{2}}}+\mathbf{n}_1\frac{\xi t}{(\eta^2-\xi^2)^{\frac{1}{2}}}\right]e^{i\epsilon st\sin\phi}, \tag{III-10}$$

$$\mathbf{H}^i=\left[-\boldsymbol{\xi}_1\frac{\xi t\cos\phi}{(\eta^2-\xi^2)^{\frac{1}{2}}}+\mathbf{n}_1\frac{\eta s\cos\phi}{(\eta^2-\xi^2)^{\frac{1}{2}}}-\boldsymbol{\phi}_1\sin\phi\right]e^{i\epsilon st\sin\phi}, \tag{III-11}$$

for a wave of unit amplitude. As usual we have omitted the time factor $e^{-i\omega\tau}$.

Expanding the exponential as a power series we find for the two components of the electric field of the incident plane wave:

$$E_\xi{}^i=\frac{\eta}{(\eta^2-\xi^2)^{\frac{1}{2}}}\left[s-\frac{\epsilon^2t^2}{2^2}s^3+\frac{\epsilon^4t^4}{2^2\cdot4^2}s^5-\frac{\epsilon^6t^6}{2^2\cdot4^2\cdot6^2}s^7+\cdots+i\epsilon\sin\phi\left\{ts^2-\frac{\epsilon^2t^3}{2\cdot2^2}s^4+\cdots\right\}\right.$$
$$\left.+\frac{\epsilon^2}{2^2}\cos2\phi\left\{t^2s^3-\frac{\epsilon^2t^4}{3\cdot2^2}s^5+\cdots\right\}+i\frac{\epsilon^3}{2^3\cdot3}\sin3\phi\{t^3s^4-\cdots\}+\cdots\right],$$

$$E_\eta{}^i=\frac{\xi}{(\eta^2-\xi^2)^{\frac{1}{2}}}\left[t-\frac{\epsilon^2t^3}{2^2}s^2+\frac{\epsilon^4t^5}{2^2\cdot4^2}s^4-\frac{\epsilon^6t^7}{2^2\cdot4^2\cdot6^2}s^6+\cdots+i\epsilon\sin\phi\left\{t^2s-\frac{\epsilon^2t^4}{2\cdot2^2}s^3+\cdots\right\}\right.$$
$$\left.+\frac{\epsilon^2}{2^2}\cos2\phi\left\{t^3s^2-\frac{\epsilon^2t^5}{3\cdot2^2}s^4+\cdots\right\}+i\frac{\epsilon^3}{2^3\cdot3}\sin3\phi\{t^4s^3-\cdots\}+\cdots\right].$$

Since the boundary conditions involve $E_\xi{}^i$ only, we need concern ourselves with this component alone. In order to obtain the coefficients with the desired accuracy, it is necessary to calculate the first few terms of $u_{71}(\xi)$ and $p_{71}(\eta)$ in addition to the functions given in Table II of II.

We find for the desired expansion

$$E_\xi{}^i=\frac{1}{(\eta^2-\xi^2)^{\frac{1}{2}}}\left\{\frac{\eta}{2t}\left[M'_{10}u_{11}(\xi)\frac{d}{dt}\{tp_{11}(\eta)\}+\frac{2}{5^2}\epsilon^2M'_{30}u_{31}(\xi)\frac{d}{dt}\{tp_{31}(\eta)\}+\frac{3}{7^2\cdot9^2}\epsilon^4M'_{50}u_{51}(\xi)\frac{d}{dt}\{tp_{51}(\eta)\}+\cdots\right]\right.$$
$$+i\epsilon\sin\phi\left[\frac{\eta}{s}M''_{11}u_{11}(\xi)p_{11}(\eta)+\frac{2}{5^2}\frac{\eta}{s}\epsilon^2M''_{31}u_{31}(\xi)p_{31}(\eta)+\cdots-\frac{\xi}{s}M'_{21}u_{21}(\xi)p_{21}(\eta)\right.$$
$$\left.-\frac{2}{7^2}\frac{\xi}{s}\epsilon^2M'_{41}u_{41}(\xi)p_{41}(\eta)+\cdots\right]-\frac{\epsilon^2}{2^2}\cos2\phi\left[\frac{\eta}{s}M''_{22}u_{22}(\xi)p_{22}(\eta)+\cdots\right.$$
$$\left.\left.-\frac{\xi}{s}M'_{32}u_{32}(\xi)p_{32}(\eta)-\cdots\right]+\cdots\right\}, \tag{III-12}$$

where

$$M'_{10}\equiv1-\frac{2}{5^2}\epsilon^2+\frac{93}{5^4\cdot7^2}\epsilon^4-\frac{446}{3^4\cdot5^5\cdot7^2}\epsilon^6+\cdots,$$

$$M'_{30}\equiv1-\frac{32}{3^3\cdot5^2}\epsilon^2+\frac{742}{3^4\cdot5^3\cdot11^2}\epsilon^4+\cdots,$$

$$M'_{50}\equiv1-\frac{10}{3\cdot13^2}\epsilon^2+\cdots,$$

$$M''_{11}\equiv1-\frac{2}{5^2}\epsilon^2+\cdots,$$

$$M'_{21}\equiv1-\frac{2}{7^2}\epsilon^2+\cdots,$$

$$M''_{31}\equiv1+\cdots,\quad M'_{41}\equiv1+\cdots,\quad M''_{22}\equiv1+\cdots,\quad M'_{32}\equiv1+\cdots.$$

3. THIN RECEIVING ANTENNA

In this section we shall investigate the current produced in a very eccentric perfectly conducting prolate spheroid by an incident plane electromagnetic wave of angular frequency ω and unit amplitude, the electric field of which is parallel to the long axis of the spheroid. We shall be particularly interested in the frequency of resonance and frequencies close to resonance. Since the eccentricity of the spheroidal conductor under consideration is close to unity, the value t_0 of the parameter t at its surface is effectively equal to the ratio of the minor semi-axis b of the conducting spheroid to the major semi-axis a. We shall confine our discussion to conductors for which $t_0 < (10)^{-2}$, with particular attention to the limiting case attained as t_0 goes to zero. Therefore we can neglect terms of order t_0^2 as compared with terms of order unity.

Remembering that the first term in $p_{l1}(\eta)$ for l odd is t, it is clear, then, that we need retain only this first term in the part of the expression (III-12) for E_ξ^i which is independent of ϕ. As regards the part of this expression involving sin ϕ, terms in the first power of t appear, but this part of the expression determines only the distribution of the current in azimuth and has no effect on the total current through any cross section of the conductor. Therefore we shall neglect it. The same is true of the part of the expression involving cos 2ϕ, which, however, contains no terms of order less than t^2 and is negligible on that account. Similar considerations apply *a fortiori* to the remaining parts of the expression.

Hence we write as a sufficiently good approximation

$$E_\xi^i = \frac{\eta}{(\eta^2-\xi^2)^{\frac{1}{2}}}\left[M'_{10}u_{11}(\xi) + \frac{2}{5^2}\epsilon^2 M'_{30}u_{31}(\xi) + \frac{3}{7^2\cdot 9^2}\epsilon^4 M'_{50}u_{51}(\xi) + \cdots\right]. \quad \text{(III-13)}$$

The corresponding component of the electric field intensity in the scattered wave must be

$$E_\xi^r = \frac{\eta}{(\eta^2-\xi^2)^{\frac{1}{2}}}\left[N'_{10}u_{11}(\xi)\frac{1}{t}\frac{d}{dt}\{tr_{11}(\eta)\} + \epsilon^2 N'_{30}u_{31}(\xi)\frac{1}{t}\frac{d}{dt}\{tr_{31}(\eta)\} + \epsilon^4 N'_{50}u_{51}(\xi)\frac{1}{t}\frac{d}{dt}\{tr_{51}(\eta)\} + \cdots\right] \quad \text{(III-14)}$$

in order to satisfy the boundary condition $E_\xi^i + E_\xi^r = 0$ for $t = t_0$. Consequently, if terms in t^2 and higher powers are neglected,

$$N'_{10} = \frac{2}{3}\epsilon^2 l_{11} M'_{10} \frac{\frac{4}{3^2}\epsilon^3 l_{11}a_{11} - ib_{11}}{\left(\frac{4}{3^2}\epsilon^3 l_{11}a_{11}\right)^2 + b_{11}^2},$$

$$N'_{30} = -\frac{8}{3^2\cdot 5^4\cdot 7}\epsilon^4 l_{31} M'_{30} \frac{\frac{32}{3^3\cdot 5^4\cdot 7^2}\epsilon^7 l_{31}a_{31} - ib_{31}}{\left(\frac{32}{3^3\cdot 5^4\cdot 7^2}\epsilon^7 l_{31}a_{31}\right)^2 + b_{31}^2},$$

$$N'_{50} = \frac{16}{3^8\cdot 5^2\cdot 7^4\cdot 11}\epsilon^6 l_{51} M'_{50} \frac{\frac{256}{3^9\cdot 5^3\cdot 7^4\cdot 11^2}\epsilon^{11} l_{51}a_{51} - ib_{51}}{\left(\frac{256}{3^9\cdot 5^3\cdot 7^4\cdot 11^2}\epsilon^{11} l_{51}a_{51}\right)^2 + b_{51}^2},$$

where l_{11} and l_{31} are defined in Section 1 of this paper, and

$$l_{51} \equiv \left[\log\frac{(1+t_0^2)^{\frac{1}{2}}+1}{(1+t_0^2)^{\frac{1}{2}}-1} - \frac{137}{2\cdot 3\cdot 5} + \frac{1}{5\cdot 13}\epsilon^2 + \cdots\right]^{-1}.$$

The magnetic field strength in the scattered wave corresponding to (III-14) consists of the single component

$$H_{\phi}{}^{r}=-i\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}\epsilon[N'_{10}u_{11}(\xi)r_{11}(\eta)+\epsilon^2N'_{30}u_{31}(\xi)r_{31}(\eta)+\epsilon^4N'_{50}u_{51}(\xi)r_{51}(\eta)+\cdots], \qquad \text{(III-15)}$$

and the current is

$$I=[2\pi cfstH_{\phi}{}^{r}]_{t=t_0}.$$

This gives

$$I=2\pi\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}cf\epsilon\left[\frac{M'_{10}l_{11}}{a_{11}}\frac{\frac{4}{3^2}\epsilon^3l_{11}a_{11}-ib_{11}}{\left(\frac{4}{3^2}\epsilon^3l_{11}a_{11}\right)^2+b_{11}^2}\{su_{11}(\xi)\}+\frac{\epsilon^2}{3\cdot5^2}\frac{M'_{30}l_{31}}{a_{31}}\frac{\frac{32}{3^3\cdot5^4\cdot7^2}\epsilon^7l_{31}a_{31}-ib_{31}}{\left(\frac{32}{3^3\cdot5^4\cdot7^2}\epsilon^7l_{31}a_{31}\right)^2+b_{31}^2}\{su_{31}(\xi)\}\right.$$

$$\left.+\frac{\epsilon^4}{5\cdot7^2\cdot9^2}\frac{M'_{50}l_{51}}{a_{51}}\frac{\frac{256}{3^9\cdot5^3\cdot7^4\cdot11^2}\epsilon^{11}l_{51}a_{51}-ib_{51}}{\left(\frac{256}{3^9\cdot5^3\cdot7^4\cdot11^2}\epsilon^{11}l_{51}a_{51}\right)^2+b_{51}^2}\{su_{51}(\xi)\}+\cdots\right]e^{-i\omega\tau} \qquad \text{(III-16)}$$

with inclusion of the time factor.

We consider first the limiting case of an antenna of length $2a=2f$ and zero thickness. Then $l_{11}=l_{31}=l_{51}=0$, and the current is zero except for those wave-lengths which make one of the coefficients b_{11}, b_{31}, b_{51} vanish. It was shown in II that b_{11} vanishes for $\epsilon=\pi/2$, b_{31} vanishes for $\epsilon=3\pi/2$, and b_{51} vanishes for $\epsilon=5\pi/2$. Hence, as $\epsilon\equiv2\pi f/\lambda$ and $a=f$ when $t_0=0$, it follows that the first three resonances occur at wave-lengths equal to twice, two-thirds, and two-fifths the length of the antenna. The resonance is infinitely sharp but not infinitely high.

a. First Resonance

We shall discuss the first resonance in some detail, as it is by far the most important. First consider an infinitely thin antenna. Making use of Eqs. (II-34) and (II-38), we find for the current at first resonance

$$\begin{aligned}I&=\left[2\pi c\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}a\right]M'_{10}(\epsilon_0)(1-\xi^2)^{\frac{1}{2}}u_{11}(\xi)\cos\omega\tau\\&=\left[2\pi c\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}a\right]1.045\cos\left(\frac{\pi}{2}\xi\right)\cos\omega\tau,\end{aligned} \qquad \text{(III-17)}$$

in exact agreement with the result of the approximate method employed in I.

Since the element of length $d\lambda_\xi$ corresponding to the increment $d\xi$ is

$$d\lambda_\xi=f\left(\frac{\eta_0{}^2-\xi^2}{1-\xi^2}\right)^{\frac{1}{2}}d\xi,$$

the applied electromotive force along $d\lambda_\xi$ is

$$E_\xi{}^i d\lambda_\xi=\frac{ad\xi}{(1-\xi^2)^{\frac{1}{2}}}\left[M'_{10}u_{11}(\xi)+\frac{2}{5^2}\epsilon^2M'_{30}u_{31}(\xi)+\frac{3}{7^2\cdot9^2}\epsilon^4M'_{50}u_{51}(\xi)+\cdots\right]\cos\omega\tau.$$

Remembering that the functions $u_{l1}(\xi)$ are orthogonal, we find that the mean time rate of absorption of energy is

$$\pi c\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}a^2[M'_{10}(\epsilon_0)]^2\int_{-1}^{1}[u_{11}(\xi)]^2d\xi.$$

The radiation resistance R is defined as the ratio of this quantity to one-half the square of the current amplitude at the center of the antenna. Hence,

$$R=\frac{1}{2\pi c}\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}}\frac{\int_{-1}^{1}[u_{11}(\xi)]^2d\xi}{[u_{11}(0)]^2}=\frac{1.2188}{2\pi c}\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}}. \qquad \text{(III-18)}$$

If R_p is the radiation resistance in ohms

$$R_p=4\pi c^2(10)^{-9}R=73.08\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}} \text{ ohms}, \qquad \text{(III-19)}$$

in exact agreement with the result of the approximate method employed in I.

Finally we obtain, with the aid of (II-34) and (II-38), the following simple expressions for the three non-vanishing field components in the wave scattered by an infinitely thin antenna at resonance:

$$E_\xi{}^r=-\frac{4\eta_0 M'_{10}(\epsilon_0)}{\pi\{(\eta^2-\xi^2)(1-\xi^2)\}^{\frac{1}{2}}}\cos\left(\frac{\pi}{2}\xi\right)\cos\left\{\frac{\pi}{2}(\eta-\eta_0)-\omega\tau\right\}, \qquad \text{(III-20)}$$

$$E_\eta{}^r=-\frac{4\eta_0 M'_{10}(\epsilon_0)}{\pi\{(\eta^2-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}}\sin\left(\frac{\pi}{2}\xi\right)\sin\left\{\frac{\pi}{2}(\eta-\eta_0)-\omega\tau\right\}, \qquad \text{(III-21)}$$

$$H_\phi{}^r=\frac{4\eta_0 M'_{10}(\epsilon_0)}{\pi\{(1-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}}\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}\cos\left(\frac{\pi}{2}\xi\right)\cos\left\{\frac{\pi}{2}(\eta-\eta_0)-\omega\tau\right\}, \qquad \text{(III-22)}$$

where $M'_{10}(\epsilon_0)=0.8206$ and, for the limiting case of an infinitely thin antenna, $\eta_0=1$. These expressions are valid at resonance for all $\eta>1$.

Next we pass to the case of an antenna of small but finite thickness. As in Section 1 we put $t_0=(10)^{-n}$. If $n\geq 3$ we do not incur an error as great as one percent if we retain the expressions for the wave-length of resonance and for the radiation resistance at resonance which were found for the infinitely thin antenna, and even for $n=2$ the error is only about two percent. But now the resonance peak in the current is no longer infinitely sharp. To determine its form we shall use only the first term in the general expression (III-16) for the current, ignoring a small but almost negligible asymmetry due to succeeding terms. We find

$$I=\left[2\pi c\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}a\right]\frac{1.045\cos\left(\frac{\pi}{2}\xi\right)}{1+35.2(n-0.228)^2\left(\frac{\delta\lambda}{\lambda_r}\right)^2}\cos(\omega\tau+\phi) \qquad \text{(III-23)}$$

for the current, where $\lambda_r=4a$ is the wave-length of resonance and $\delta\lambda$ the deviation from λ_r. The lead ϕ of the current ahead of the electromotive-force is given by

$$\tan\phi=5.93(n-0.228)\frac{\delta\lambda}{\lambda_r}. \qquad \text{(III-24)}$$

These expressions are valid, of course, only for values of the ratio $\delta\lambda/\lambda_r$ small compared with unity. They represent quite typical resonance curves.

b. Second Resonance

In discussing the second resonance we shall consider only the case of an infinitely thin antenna, and, as the series involved converges more slowly in this region, we can do no more than calculate a rough value of the current amplitude at resonance. We find for the current at resonance

$$I = -\left[2\pi c\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}} a\right] 0.04 \cos\left(\frac{3\pi}{2}\xi\right) \cos \omega\tau. \tag{III-25}$$

The current is in phase with the impressed electromotive force in the two extreme thirds of the antenna, but out of phase in the middle third. As the current amplitude at the center of the antenna is only some 4 percent of that at first resonance, the second and higher order resonances are evidently of little importance as compared with the first resonance.

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Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition

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The partition function of a two-dimensional "ferromagnetic" with scalar "spins" (Ising model) is computed rigorously for the case of vanishing field. The eigenwert problem involved in the corresponding computation for a long strip crystal of finite width (n atoms), joined straight to itself around a cylinder, is solved by direct product decomposition; in the special case $n=\infty$ an integral replaces a sum. The choice of different interaction energies ($\pm J, \pm J'$) in the (0 1) and (1 0) directions does not complicate the problem. The two-way infinite crystal has an order-disorder transition at a temperature $T=T_c$ given by the condition

$$\sinh(2J/kT_c)\sinh(2J'/kT_c)=1.$$

The energy is a continuous function of T; but the specific heat becomes infinite as $-\log|T-T_c|$. For strips of finite width, the maximum of the specific heat increases linearly with $\log n$. The order-converting dual transformation invented by Kramers and Wannier effects a simple automorphism of the basis of the quaternion algebra which is natural to the problem in hand. In addition to the thermodynamic properties of the massive crystal, the free energy of a (0 1) boundary between areas of opposite order is computed; on this basis the mean ordered length of a strip crystal is

$$(\exp(2J/kT)\tanh(2J'/kT))^n.$$

INTRODUCTION

THE statistical theory of phase changes in solids and liquids involves formidable mathematical problems.

In dealing with transitions of the first order, computation of the partition functions of both phases by successive approximation may be adequate. In such cases it is to be expected that both functions will be analytic functions of the temperature, capable of extension beyond the transition point, so that good methods of approximating the functions may be expected to yield good results for their derivatives as well, and the heat of transition can be obtained from the difference of the latter. In this case, allowing the continuation of at least one phase into its metastable range, the heat of transition, the most appropriate measure of the discontinuity, may be considered to exist over a range of temperatures.

It is quite otherwise with the more subtle transitions which take place without the release of latent heat. These transitions are usually marked by the vanishing of a physical variable, often an asymmetry, which ceases to exist beyond the transition point. By definition, the strongest possible discontinuity involves the specific heat. Experimentally, several types are known. In the $\alpha-\beta$ quartz transition,[1] the specific heat becomes infinite as $(T_c-T)^{-\frac{1}{2}}$; this may be the rule for a great many structural transformations in crystals. On the other hand, supraconductors exhibit a clear-cut finite discontinuity of the specific heat, and the normal state can be continued at will below the transition

[1] H. Moser, Physik. Zeits. **37**, 737 (1936).