On Applications of the λ-Limiting Process to the Theory of the Meson Field

W. PAULI
The Institute for Advanced Study, Princeton, New Jersey
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The λ-limiting process introduced by Wentzel and Dirac for the interaction of an electron with an electromagnetic field is applied to the interaction of a heavy particle (nucleon) with the meson field. One obtains in this way a more precise and relativistically invariant form of the theory which, in its classical interpretation, avoids all divergences of point sources. It is shown that the pseudoscalar and vector theories with a coupling constant f of the dimension of length have in this form the character of a weak coupling theory with no stable isobars existing for $\mu f \ll 1$. The higher approximations for the interactions between nucleons are investigated for the example of a Rosenfeld-Møller mixture with the results that they are finite as long as the nucleons are treated to be at rest and small in comparison with the f^2 approximation for distances of the nucleons from each other sufficiently larger than f.

1. INTRODUCTION

THE previous form of the theory of interaction of a meson field with heavy particles (protons and neutrons, which are both called also "nucleons") suffered from the deficiency of a fundamental character in that the range of validity of the perturbation theory with respect to the coupling constant of this interaction could not be stated without introducing new assumptions about the size of the nucleons. Since the higher approximations are divergent for point sources, one tried first to assume extended sources with a certain finite radius a. It turned out, however, that for the pseudoscalar and vector theories which alone give the right spin dependence of the nuclear forces, the perturbation theory never holds for values of a and of the coupling constant f with the dimension of length[1] which are required by experience. On the contrary, for sources small in comparison with the Compton wave-length μ^{-1} of the meson and for $a \ll f$, the different approximation of strong coupling has to be applied. These strong coupling theories, however, had consequences in contradiction to experiment.[2] Because of the existence of stable isobars with higher values of spin and charge, the highly charged nuclei should be unstable; moreover, the magnetic moment of the proton and the free neutron should be just equal apart from their sign and should nearly cancel in the deuteron in contradiction to the experimental facts.

One had the impression that the results of the perturbation theory are actually valid in a wider range than would follow from the model of the extended source. In the present paper[3] it is shown that this is understandable and can be formulated more precisely with the help of the so-called "λ-limiting process" which was applied in the case of the electromagnetic field by Wentzel and Dirac in order to avoid the classical singularities of the self-energy of a point electron[4] in a relativistically invariant way. In Section 2 it is shown that this limiting process can be considered as the result of a natural generalization of the model of an extended source with respect to the reality conditions of the field variables in the momentum space, and that the main physical content of this formalism, which has the further advantage of preserving the relativistic invariance of the final results,[5] can be characterized as making the constants of inertia of the degrees of freedom of the nucleon's spin and the isotopic spin equal to zero. These latter constants were

[1] In the following we use the natural unit where one puts $\hbar = c = 1$.

[2] W. Pauli and S. Kusaka, Phys. Rev. **63**, 400 (1943), where other literature is referred to.

[3] A short abstract of the contents of this paper was given in Phys. Rev. **63**, 221A (1943), No. 25.

[4] Compare the literature quoted in the author's report in Rev. Mod. Phys. **15**, 175 (1943).

[5] In the present paper we do not give an explicit proof for this relativistic invariance, which can be done for the Hamiltonians of the type used in Section 3 with the help of Dirac's formalism with several time coordinates in a way similar to Dirac's proof for the case of electrodynamics. Compare also J. M. Jauch, Phys. Rev. **63**, 334 (1943) where the application of the λ-limiting process to the problem of the magnetic moment of the nucleons is treated.

defined by Bhabha[6] in his classical theory of neutral vector mesons, and he proposed also to assume their values equal to zero for the actual particles of nature.[7] The discussion of the classical model shows that no stable excited states of the nucleons exist in this theory if $\mu f \ll 1$, a condition which certainly is fulfilled in nature and which characterizes the theory here discussed as a weak coupling theory in a more precise form.

The Hamiltonians for the interaction between nucleons and meson field given in Section 4 do not contain anything new and are only preparation for the investigation of the interaction between different nucleons in higher approximations treated in the concluding Section 5. We use here the important method of successive canonical transformations which is due to Stueckelberg and Patry,[8] and we are particularly interested in the special mixture of a pseudoscalar and a vector meson field introduced by Rosenfeld and Møller[9] which seems to be in better agreement with the empirical form of the nuclear forces than other assumptions about the meson field. While Rosenfeld and Møller themselves merely guessed that the higher approximations for the nuclear forces would be relatively small if the distance r between the nucleons is larger than the coupling constant f mentioned above, Stueckelberg[10] showed that the higher approximations diverge for all distances r if the model of a point source is used because these higher approximations contain terms proportional to a^{-1}, where a is the radius of the nucleon. This result is in agreement with the later investigations from the standpoint of the strong coupling[11] theories in which the stable isobars of the nucleon play an essential role.

In Section 4 we resume Stueckelberg's investigation of the Rosenfeld-Møller mixture in order to apply the λ-limiting process to this problem. For this purpose the use of the momentum space instead of the ordinary space in the description of the field variables is convenient, and it also simplifies the calculation itself. The result is the re-establishment of Rosenfeld and Møller's original condition $r \gg f$ for the validity of the perturbation theory at least as far as the order of magnitude is concerned. It is true that the numerical factors in the higher approximations tend to diminish this range of validity of the f^2 approximation, but it has to be remembered that the other point of view, which is used in the above calculations and according to which the nucleons are considered to be at rest, restricts the validity of the results to $r \gg M^{-1}$, where M^{-1} is the Compton wave-length of the proton, which is practically of the same order of magnitude as f. It is therefore questionable whether the exact form of the higher approximations with respect to f^2 has any significance, and we merely stress their order of magnitude and the circumstance that the λ-limiting process is sufficient to make them all convergent so long as the nucleons can be assumed to stay at rest.

[6] H. J. Bhabha, Proc. Roy. Soc. **A178**, 314 (1941).

[7] It is an interesting question whether the λ-limiting process can be generalized relativistically invariant in such a way that this constant is given an arbitrary value different from zero and that divergences of theory are still avoided. My attempts to find such a generalization have not been successful.

[8] E. C. G. Stueckelberg and J. F. C. Patry, Helv. Phys. Acta **13**, 167 (1940).

[9] C. Møller and L. Rosenfeld, Kgl. Danske Vid. Sels. Math.-Fys. Med. **17**, No. 8 (1940).

[10] E. C. G. Stueckelberg, Helv. Phys. Acta **13**, 347 (1940).

[11] R. Serber and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

2. THE λ-LIMITING PROCESS AS A GENERALIZATION OF THE EXTENDED SOURCE MODEL

Choosing as the simplest example the interaction of neutral pseudoscalar mesons with a heavy particle (nucleon) at rest in the origin of the coordinate system, we start with the well-known Hamiltonian

$$H = \tfrac{1}{2}\int \{\pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\} d\mathbf{x} + (4\pi)^{\frac{1}{2}} f \int U(x)\boldsymbol{\sigma}\cdot\nabla\varphi d\mathbf{x}, \quad (1)$$

where the real quantities $\pi(x)$, $\varphi(x)$ are canonically conjugate pseudoscalar fields; natural units with $\hbar = c = 1$ are used, μ is the rest mass of the meson in these units, f the coupling constant with the dimension of length, $\frac{1}{2}\boldsymbol{\sigma}$ the spin of the nucleon, and $U(x)$ the source function normalized according to

$$\int U(x) dV = 1. \quad (2)$$

If brackets mean the Poisson symbols in the classical interpretation, and the commutators multiplied by the imaginary unit i in the quantum mechanical interpretation, we have in both cases

$$[\varphi(x), \varphi(x')] = [\pi(x), \pi(x')] = 0, \quad [\pi(x), \varphi(x')] = \delta(x-x'); \tag{3}$$

$$[\sigma_i, \sigma_j] = -2\sigma_k, \quad i, j, k \text{ cyclic permutation of } 1, 2, 3, \tag{4}$$

and for the time derivative $\dot{F}$ of every observable F holds the equation of motion

$$\dot{F} = [H, F]. \tag{5}$$

In quantum theory σ_1, σ_2, σ_3 are the well-known spin matrices, while in the classical theory we can assume $\boldsymbol{\sigma}$ to be a unit vector.

For the purpose of our generalization it is convenient to pass from the ordinary space to the momentum space with the help of the Fourier transformation

$$\varphi(x) = (2\pi)^{-\frac{3}{2}} \int q(k) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \quad \pi(x) = (2\pi)^{-\frac{3}{2}} \int p(k) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \tag{6a}$$

$$q(k) = (2\pi)^{-\frac{3}{2}} \int \varphi(x) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}, \quad p(k) = (2\pi)^{-\frac{3}{2}} \int \pi(x) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}. \tag{6b}$$

The rule (3) for the brackets is then equivalent to

$$[q(k), q(k')] = [p(k), p(k')] = 0, \quad [p(k), q(k')] = \delta(k-k'). \tag{7}$$

We further introduce the Fourier-transformed function $v(k)$ of $U(x)$ according to

$$U(x) = (2\pi)^{-3} \int v(k) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \quad v(k) = \int U(x) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}. \tag{8}$$

The different choice of the normalization factors in (8) is convenient because the condition (2) is then equivalent to the simple statement

$$v(0) = 1. \tag{9}$$

The Hamiltonian written in the momentum space is given by

$$H = \frac{1}{2} \int \{p(k)p(-k) + k_0^2 q(k)q(-k)\} d\mathbf{k} + \frac{if}{\pi\sqrt{2}} \int v(-k)\boldsymbol{\sigma}\cdot\mathbf{k} q(k) d\mathbf{k}, \tag{10}$$

where

$$k_0 = +(k^2+\mu^2)^{\frac{1}{2}}. \tag{11}$$

In order to fulfill the conditions that $U(x)$, $\pi(x)$, $\varphi(x)$ have to be real, the quantities $p(k)$, $q(k)$, $v(k)$ have to fulfill the reality conditions

$$q(-k) = q^*(k), \quad p(-k) = p^*(k), \quad v(-k) = v^*(k), \tag{12}$$

where a star denotes the conjugate complex. It is, however, remarkable that the reality of the Hamiltonian (and also of the total momentum) holds already, if the weaker conditions,

$$\begin{aligned} v(k)q(-k) &= v^*(-k)q^*(k), \\ v(k)p(k) &= v^*(-k)p^*(-k), \\ p(k)p(-k) &= p^*(k)p^*(-k), \\ q(k)q(-k) &= q^*(k)q^*(-k), \\ v(k)v(-k) &= v^*(k)v^*(-k), \end{aligned} \tag{13}$$

are fulfilled. The latter conditions are simplified by using the new variables

$$\tilde{q}(k) = q(k)v(-k), \quad \tilde{p}(k) = p(k)v(k) \tag{14}$$

which satisfy reality conditions analogous to (13),

$$\tilde{q}(-k) = \tilde{q}^*(k), \quad \tilde{p}(-k) = \tilde{p}^*(k)$$

and by using the real quantity

$$G(k) = G^*(k) = v(k)v(-k). \tag{14a}$$

The Hamiltonian in the new variables is

$$H = \frac{1}{2} \int [G(k)]^{-1} \{\tilde{p}(k)\tilde{p}(-k) + k_0^2 \tilde{q}(k)\tilde{q}(-k)\} d\mathbf{k} + \frac{if}{\pi\sqrt{2}} \int \boldsymbol{\sigma}\cdot\mathbf{k}\tilde{q}(k) d\mathbf{k}. \tag{15}$$

The value of the brackets is

$$[\tilde{p}(k), \tilde{q}(k')] = G(k)\delta(k-k'). \tag{16}$$

The new form of the Hamiltonian makes it evident that models with different source functions $v(k)$ belonging to the same function $G(k)$ are equivalent, and that only the latter function has a physical meaning. According to (9) one has always

$$G(0) = 1, \tag{17}$$

while the particular case $G(k) = \text{const.} = 1$ corresponds to a point source.

Whereas according to the stronger reality conditions (13) of the usual model of an extended source, $G(k)$ is necessarily positive, the renouncement of the reality of the field variables in the ordinary space has led us to the weaker condition that $G(k)$ has to be real only.

It has to be shown that the particular choice

$$G(k) = \cos(\lambda_0 k_0 - \boldsymbol{\lambda}\cdot\mathbf{k}) \tag{18}$$

has the remarkable property that an originally Lorentz invariant theory remains so if λ_0, $\boldsymbol{\lambda}$ is simultaneously transformed as a four vector. Finally, one has to carry through the so-called "λ-limiting process," namely, $(\lambda_0, \boldsymbol{\lambda}) \rightarrow 0$, which eliminates the particular choice of the λ vector from the final results and which makes all divergences of the classical model disappear provided that the four vector λ_0, $\boldsymbol{\lambda}$ is always timelike, that is, if

$$\lambda_0^2 > \boldsymbol{\lambda}^2. \tag{19}$$

As long as one has only to consider a single coordinate system, it is permissible to put $\boldsymbol{\lambda} = 0$; hence

$$G(k) = \cos\lambda_0 k_0 = \cos\{\lambda_0(k^2+\mu^2)^{\frac{1}{2}}\}. \tag{18a}$$

The λ-limiting process avoids the (classical) divergences of the point-source model without introducing in the final results a finite extension of the source and without destroying the relativistic invariance of a theory.

In earlier papers we defined for the model of an extended source the reciprocal of its radius a by

$$a^{-1} = \int\int \frac{U(x)U(x')}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}d\mathbf{x}'$$

which in the momentum space can be written

$$a^{-1} = \frac{1}{2\pi^2}\int G(k)\frac{d\mathbf{k}}{k^2}. \tag{20}$$

Maintaining this definition of a^{-1} also in the case of a non-positive $G(k)$ without any longer connecting a with the dimension of the source, we find for the particular choice (20) for $G(k)$[12]

$$a^{-1} = \frac{2}{\pi}\int_0^\infty \cos\{\lambda_0(k^2+\mu^2)^{\frac{1}{2}}\}dk = \mu J_1(\lambda_0\mu). \tag{21}$$

It is remarkable that the limiting process applied to the quantity a^{-1} gives zero:

$$\lim_{\lambda_0\rightarrow 0} a^{-1} = 0; \tag{21a}$$

while for the original model of a finite source, the transition to a point source, namely, $G(k)\rightarrow 1$ means always $a^{-1}\rightarrow\infty$. Some physical consequences of the result (21a) are given in the next section.

3. APPLICATION TO THE CLASSICAL THEORY OF PROTON ISOBARS (FREE GYRATION)

As is well known there exists in the classical interpretation of the theory, under certain conditions for the value of the coupling constant and for the source function, the possibility of a continuum of energy values for a heavy particle surrounded by its own meson field due to the presence of a free gyration of the field and the spin or isotopic spin of the heavy particle. This type of periodic motion was interpreted by Oppenheimer and Schwinger[13] as corresponding to discrete energy values in the quantum-theoretical treatment in which the total charge and the total angular momentum of the system have discrete values which are in suitable units integers and half-odd integers, respectively. They proved also that in the case of a strong coupling the treatment according to the correspondence principle agrees with the more rigorous quantum-mechanical treatment in the case of the charged scalar and the neutral pseudoscalar theory.[14] It

[12] Compare for the evaluation of the integral, J. M. Jauch, Phys. Rev. **63**, 334 (1943), Appendix.

[13] J. R. Oppenheimer and J. Schwinger, Phys. Rev. **60**, 150 (1942).

[14] The quantum-mechanical treatment in the case of the charged and symmetrical pseudoscalar theory was given by S. M. Dancoff and W. Pauli, Phys. Rev. **62**, 85 (1942).

is the purpose of the following computation to investigate the conditions for the existence of such excited states (isobars) of the heavy particle if the λ-limiting process is applied. It will be sufficient for it to give the classical treatment of the neutral pseudoscalar theory for the results in this case are typical also for more general cases.

We start with the equations of motion which one obtains according to the application of the general rule (5) to the Hamiltonian (15) if one inserts for F the quantities $\tilde{\varphi}(k)$, $\tilde{q}(k)$ and the components σ_1, σ_2, σ_3 of the spin of the heavy particle using the bracket expressions (4) and (16). In this way one obtains

$$\frac{\partial \tilde{q}(k)}{\partial t}=\tilde{p}(-k),$$

$$\frac{\partial \tilde{p}(k)}{\partial t}=-k_0^2\tilde{q}(-k)-\frac{if}{\pi\sqrt{2}}G(k)\boldsymbol{\sigma}\cdot\mathbf{k},$$

hence

$$\frac{\partial^2\tilde{q}(k)}{\partial t^2}+k_0^2\tilde{q}(k)=\frac{if}{\pi\sqrt{2}}G(k)\boldsymbol{\sigma}\cdot\mathbf{k}, \tag{22}$$

and

$$\dot{\boldsymbol{\sigma}}=-\frac{\sqrt{2}if}{\pi}\int[\boldsymbol{\sigma}\times\mathbf{k}]\tilde{q}(k)d\mathbf{k}. \tag{23}$$

We suppose a free gyration of the spin around the x_3 axis, according to

$$\sigma_1=(1-C^2)^{\frac{1}{2}}\cos\omega t,\quad \sigma_2=(1-C^2)^{\frac{1}{2}}\sin\omega t,\quad \sigma_3=C, \tag{24}$$

C being a constant between -1 and $+1$. Inserting this in (22) we obtain

$$\tilde{q}(k)=\frac{if}{\pi\sqrt{2}}G(k)\left\{\frac{(1-C^2)^{\frac{1}{2}}}{k^2+\mu^2-\omega^2}(k_1\cos\omega t+k_2\sin\omega t)+\frac{C}{k^2+\mu^2}k_3\right\}. \tag{25}$$

In the case

$$\omega^2<\mu^2 \tag{26}$$

the denominator in (25) is always different from zero and the field $\varphi(x)$ in the x space decreases to zero for large distances from the heavy particle. In the other case $\omega^2>\mu^2$ the field in large distances has the form of a spherical wave and describes free mesons not bounded by the heavy particle. In the following we shall therefore assume that the condition (26) holds.

By inserting the expression (25) for $\tilde{q}(k)$ in the right side of (23), the integrals can be simplified under the assumption that $G(k)$ is spherically symmetric; that means that it depends only on the absolute value k of the vector $\mathbf{k}$. In this case we obtain the following relation connecting the frequency ω with the values of C

$$\omega=\frac{4}{3\pi}f^2C\int_0^\infty G(k)\left\{\frac{1}{k^2+\mu^2-\omega^2}-\frac{1}{k^2+\mu^2}\right\}k^4dk. \tag{27}$$

Before we discuss this relation we write down the analogous expressions for the energy and for the angular momentum. The latter is given by

$$\mathbf{L}=-\pi\int[\mathbf{x}\times\boldsymbol{\nabla}\varphi]d\mathbf{x}+\tfrac{1}{2}\boldsymbol{\sigma},$$

or in the momentum space by

$$\mathbf{L}=\int[G(k)]^{-1}\tilde{p}(k)\left[\frac{\partial\tilde{q}(k)}{\partial\mathbf{k}}\times\mathbf{k}\right]d\mathbf{k}+\tfrac{1}{2}\boldsymbol{\sigma}. \tag{28}$$

It is in an integral of motion for a spherically symmetrical $G(k)$ a specialization which has already been mentioned. Inserting in the expression (15) for the Hamiltonian and in (28) for the angular momentum, the value (24) for $\boldsymbol{\sigma}$, (25) for $\tilde{q}(k)$ and $\dfrac{\partial\tilde{q}(-k)}{\partial t}$ for $p(k)$ one obtains for the energy

$$E=E_0-\frac{f^2}{3\pi}\int_0^\infty G(k)(1-C^2)\left\{\frac{1}{k^2+\mu^2-\omega^2}-\frac{1}{k^2+\mu^2}-\frac{2\omega^2}{(k^2+\mu^2-\omega^2)^2}\right\}k^4dk, \tag{29}$$

where

$$E_0=-\frac{f^2}{3\pi}\int_0^\infty G(k)\frac{k^4}{k^2+\mu^2}dk \tag{29a}$$

is the self-energy of the nucleon corresponding to the ordinary static solution ($C=1$, $\omega=0$). The corresponding expression of the angular momentum, which in our problem has the direction of the x_3 axis ($L_1=L_2=0$, $L_3=L$) is

$$L=\frac{2f^2}{3\pi}\omega(1-C^2)\int_0^\infty G(k)\frac{k^4}{(k^2+\mu^2-\omega^2)^2}dk+\tfrac{1}{2}C. \tag{30}$$

The integrals occurring in (27), (29), (30) may be evaluated in the following way. We split the integrand into one part without denominator, containing the integrals

$$N=\frac{2}{\pi}\int_0^\infty G(k)k^2dk,\quad a^{-1}=\frac{2}{\pi}\int_0^\infty G(k)dk, \tag{31}$$

and another part which converges for a point source $G(k)\rightarrow 1$. In the latter part we shall insert $G(k)=1$ which is allowed both for the λ-limiting process and for an extended source [$G(k)$ positive] with dimensions small in comparison with μ^{-1}, that is, $\mu a\ll 1$. We shall restrict ourselves to these two cases. It has to be emphasized, however, that in the former case for which our formulas will also hold, μa is in no way small. Using

$$\frac{2}{\pi}\int_0^\infty \frac{dk}{k^2+\mu^2-\omega^2}=(\mu^2-\omega^2)^{-\frac{1}{2}},$$

$$\frac{2}{\pi}\int_0^\infty \frac{dk}{(k^2+\mu^2-\omega^2)^2}=\tfrac{1}{2}(\mu^2-\omega^2)^{-\frac{3}{2}},$$

and its specializations for $\omega=0$, one finds with the help of the indicated method,

$$\frac{2}{\pi}\int_0^\infty G(k)\frac{k^4}{k^2+\mu^2}dk=N-\frac{\mu^2}{a}+\mu^3,$$

$$\frac{2}{\pi}\int_0^\infty G(k)\frac{k^4}{k^2+\mu^2-\omega^2}dk=N-\frac{\mu^2-\omega^2}{a}+(\mu^2-\omega^2)^{\frac{3}{2}},$$

$$\frac{2}{\pi}\int_0^\infty G(k)\frac{k^4}{(k^2+\mu^2-\omega^2)^2}dk=\frac{1}{a}-\frac{3}{2}(\mu^2-\omega^2)^{\frac{1}{2}}.$$

Equations (27), (29), (30) get in this way their final form

$$\omega=\frac{2}{3}f^2C\left\{\frac{\omega^2}{a}-[\mu^3-(\mu^2-\omega^2)^{\frac{3}{2}}]\right\}, \tag{32}$$

$$E=E_0+\frac{f^2}{6}(1-C^2)\times\left\{\frac{\omega^2}{a}+\mu^3-(\mu^2-\omega^2)^{\frac{3}{2}}-3\omega^2(\mu^2-\omega^2)^{\frac{1}{2}}\right\}, \tag{33}$$

$$E_0=-\frac{f^2}{6}\left(N-\frac{\mu^2}{a}+\mu^3\right), \tag{33a}$$

$$L=\frac{1}{3}f^2\omega(1-C^2)\left\{\frac{1}{a}-\frac{3}{2}(\mu^2-\omega^2)^{\frac{1}{2}}\right\}+\frac{1}{2}C. \tag{34}$$

For periodic motions the energy of which depends only on the amount of the resulting angular momentum, the frequency ω is given by the general mechanical relation

$$\omega=dE/dL. \tag{35}$$

Introducing the auxiliary function of the frequency

$$W=\omega L-(E-E_0), \tag{36}$$

this relation is equivalent to

$$dW/d\omega=L. \tag{37}$$

The latter form can be checked easily in our case. We obtain first

$$W=\frac{1}{2}\omega C+\frac{f^2}{6}(1-C^2)\left\{\frac{\omega^2}{a}-\mu^3+(\mu^2-\omega^2)^{\frac{3}{2}}\right\}, \tag{38}$$

and we note that according to (32) this can also be written

$$W=\frac{1}{2}\omega\left(C+\frac{1-C^2}{2C}\right)=\frac{\omega}{4C}. \tag{38a}$$

From (38) one derives

$$dW=dC\frac{1}{2}\left\{\omega-\frac{2}{3}f^2C\left[\frac{\omega^2}{a}-\mu^3+(\mu^2-\omega^2)^{\frac{3}{2}}\right]\right\}$$
$$+d\omega\left\{\frac{1}{2}C+\frac{1}{3}f^2\omega(1-C^2)\left[\frac{1}{a}-\frac{3}{2}(\mu^2-\omega^2)^{\frac{1}{2}}\right]\right\}.$$

The factor of dC vanishes according to (32), whereas the factor of $d\omega$ is equal to L according to (34); hence the relation (37) is proved.

The terms proportional to $1/a$ in the condition (32) for the frequency can easily be interpreted. By inserting (25) in (24) it follows that the terms proportional to $1/a$ in the equation of motion for $\boldsymbol{\sigma}$ are

$$\dot{\boldsymbol{\sigma}}=-\tfrac{2}{3}f^2a^{-1}[\boldsymbol{\sigma},\ddot{\boldsymbol{\sigma}}]+\cdots. \tag{39}$$

On account of the linear connection between $\tilde{q}(k)$ and $\boldsymbol{\sigma}$ [Eq. (22)] Eq. (39) holds generally, and not only for periodic motions as would appear from its derivation. Bhabha[15] in his classical treatment of the neutral vector mesons called the expression on the right-hand side of (39) the mechanical spin inertia term. The result $a^{-1}=0$ of the λ-limiting process means therefore that this process makes the spin inertia disappear; hence this process is in accordance with Bhabha's conjecture that for the real particles of nature the inertia terms will always be zero.

We add the remark that for $G(k)=\cos\lambda_0 k_0$, not only a^{-1} does tend to zero together with λ_0 but also N defined in (31), and according to (33a) the self-energy of the nucleon then becomes $E_0=-\frac{1}{6}f^2\mu^3$ which is small in comparison with the proton mass for the actual value of the coupling constant $(f\mu)^2\sim\frac{1}{10}$.

We now turn to the discussion of whether Eq. (32) for ω has a solution for $-1\leqq C\leqq 1$ and $\omega<\mu$. We carry through the discussion for two cases. First we assume a small extended source corresponding to a positive $G(k)$ and to $\mu a\ll 1$. In this case it is sufficient to retain the terms proportional to a^{-1}, and we obtain for the frequency

$$\omega=\frac{3}{2f^2}\frac{a}{C}$$

which can be fulfilled in the intervals in question for

$$(f\mu)^2>\tfrac{3}{2}(\mu a). \tag{40}$$

With the same approximation we obtain from (33), (34), (38):

$$W=\tfrac{1}{2}\omega L=E-E_0,\quad L=1/2C,$$
$$\omega=\frac{3a}{f^2}L,\quad E-E_0=\frac{3a}{2f^2}L^2. \tag{41}$$

The condition $\omega<\mu$ implies an upper bound for L. This is the old result of Oppenheimer and Schwinger which in the case of a strong coupling $(f\mu)^2\gg\mu a$ agrees with the quantum mechanical result if we put $L^2=l(l+1)$ with a half-odd integer l which is not too large.

The second case we want to discuss is the result of the λ-limiting process with $a^{-1}=0$. Putting $x=\omega/\mu$ Eq. (32) has here the form

$$-1=\tfrac{2}{3}(f\mu)^2C\frac{1-(1-x^2)^{\frac{3}{2}}}{x},$$

which has solutions for $0\leqq x\leqq 1$ and $-1\leqq C\leqq 1$ only if

$$\gamma\cdot\tfrac{2}{3}(f\mu)^2>1, \tag{42}$$

where γ is the maximum of the function $[1-(1-x^2)^{\frac{3}{2}}]x^{-1}$ in the interval $0\leqq x\leqq 1$ which is only slightly larger than 1. This is not fulfilled in nature, where $(f\mu)^2\sim\frac{1}{10}$. Therefore in the theory based on the λ-limiting process there does not exist a free gyration corresponding to a stable excited state of the nucleon with the actual value of the coupling constant, a result stated in the introduction.

The theory of the scattering of a free meson can be treated in a similar way with this classical model. We do not give, however, this calculation in this paper because it contains nothing new in comparison with the analogous computation of Bhabha.[16] The quantum theory of the meson scattering, especially the comparison of the results of the theory based on the λ-limiting process with the theory of Heitler and Peng,[17] needs further investigation.

4. THE HAMILTONIAN FOR A MIXED PSEUDOSCALAR AND VECTOR FIELD IN INTERACTION WITH SEVERAL NUCLEONS

We must now generalize our Hamiltonian by including the isotopic spin, assuming the presence of several heavy particles and considering a mixture of a pseudoscalar meson and a vector meson. In the following we describe the nucleons, not with the help of a wave field in ordinary space and second quantization, but with the configuration space. The coordinates of the nucleons are denoted by $\mathbf{z}_A$ where capital Roman indices enumerate the different nucleons and run from 1 to N if N nucleons are present. As in the last section we introduce in the interaction energy

[15] See reference 6. In our units Bhabha's constant K is connected with our a^{-1} by $K/I=\frac{2}{3}f^2a^{-1}$.

[16] H. J. Bhabha, reference 6, p. 333 ff. One has to identify his constant $g_2{}^2/I$ with our f^2 and his constant β with our a^{-1}. The λ-limiting process makes $\beta=0$. The total scattering coefficient of the meson by the nucleon is identical with Bhabha's expression (82), p. 337, if one substitutes for $\sin^2\theta$ its average $\frac{2}{3}$ and includes a factor 2 which is due to the difference between the pseudoscalar theory here considered and the vector theory treated by Bhabha.

[17] W. Heitler and H. W. Peng, Proc. Camb. Phil. Soc. **38**, 296 (1942); W. Heitler, Proc. Camb. Phil. Soc. **37**, 291 (1941).

of the nucleons with the meson field first a general source function $U(x)$ which degenerates into a δ function for a point source and which will later be specialized according to the λ-limiting process which re-establishes the relativistic invariance of the theory.

The total Hamiltonian of the system consists of three parts H_S, H_V, H_M which describe the pseudoscalar mesons, the vector mesons, and the nucleons, respectively. For the sake of simplicity we neglect the interaction of the longitudinal vector mesons with the nucleons, and retain only the interaction of the transverse vector mesons and the pseudoscalar mesons. If we denote by $\boldsymbol{\alpha}$, β the Dirac matrices and put as usual

$$\boldsymbol{\gamma}=-i\beta\boldsymbol{\alpha};\quad s_1=-i\alpha_2\alpha_3,\ \cdots;\qquad \gamma_5=-i\alpha_1\alpha_2\alpha_3; \tag{43}$$

and denote with τ_α ($\alpha=1, 2, 3$) the components of the isotopic spin, one has according to Kemmer[18] and Bhabha[19] for the "symmetric" theory

$$\begin{aligned}H_S=\tfrac{1}{2}\int\textstyle\sum_\alpha\{\pi_\alpha{}^2+(\nabla\varphi_\alpha)^2+\mu^2\varphi_\alpha{}^2\}d\mathbf{x}\\ +(4\pi)^{\frac{1}{2}}f_S\int\sum_{A,\,\alpha}U(x-z_A)\tau_\alpha{}^A\\ \times\{\mathbf{s}^A\cdot\nabla\varphi_\alpha-\gamma_5{}^A\pi_\alpha\}d\mathbf{x}\\ +2\pi f_S{}^2\int\sum_{A,\,B,\,\alpha}U(x-z_A)\\ \times U(x-z_B)\tau_\alpha{}^A\tau_\alpha{}^B\gamma_5{}^A\gamma_5{}^Bd\mathbf{x}+,\end{aligned} \tag{44}$$

$$\begin{aligned}H_V=\tfrac{1}{2}\int\sum_\alpha\left\{\boldsymbol{\pi}_\alpha{}^2+\frac{1}{\mu^2}(\nabla\cdot\boldsymbol{\pi}_\alpha)^2\right.\\ \left.+(\nabla\times\boldsymbol{\phi}_\alpha)^2+\mu^2\boldsymbol{\phi}_\alpha{}^2\right\}d\mathbf{x}\\ +(4\pi)^{\frac{1}{2}}f_V\int\sum_{A,\,\alpha}U(x-z_A)\tau_\alpha{}^A\\ \times\{\mathbf{s}^A\cdot(\nabla\times\boldsymbol{\phi}_\alpha)-\boldsymbol{\gamma}^A\cdot\boldsymbol{\pi}_\alpha\}d\mathbf{x}\\ +2\pi f_V{}^2\int\sum_{A,\,B,\,\alpha}U(x-z_A)U(x-z_B)\\ \times\tau_\alpha{}^A\tau_\alpha{}^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{x},\end{aligned} \tag{45}$$

$$H_M=\sum_A\left\{\boldsymbol{\alpha}^A\cdot\frac{1}{i}\frac{\partial}{\partial\mathbf{z}^A}+M\beta^A\right\}. \tag{46}$$

[18] N. Kemmer, Proc. Roy. Soc. **A166**, 127 (1938); we are interested in his cases (*c*) and (*d*) and in the interactions proportional to his constants f_b and g_d.

[19] H. J. Bhabha, Proc. Roy. Soc. **A166**, 501 (1938).

In the limiting case of a point source, the theory is relativistically invariant, φ being a pseudoscalar and $\boldsymbol{\phi}$ together with $\varphi_0=\mu^{-2}\nabla\cdot\boldsymbol{\pi}$ a four vector. π is canonically conjugate to φ and $\boldsymbol{\pi}$ canonically conjugate to $\boldsymbol{\phi}$ according to the bracket expressions (α, $\beta=1, 2, 3$; i, $k=1, 2, 3$)

$$\begin{aligned}[\pi_\alpha(x),\ \varphi_\beta(x')]=\delta_{\alpha\beta}\delta(x-x'),\\ [\pi_{\alpha i}(x),\ \varphi_{\beta k}(x')]=\delta_{\alpha\beta}\delta_{ik}\delta(x-x').\end{aligned} \tag{47}$$

Moreover, the components of the isotopic spin have the same commutation rules as the components of the ordinary spin, namely,

$$[\tau_1,\ \tau_2]=-2\tau_3,\ \cdots, \tag{48}$$

and the brackets of all quantities belonging to different nucleons are zero. With the help of these relations one obtains the equation of motion of all quantities by application of the general rule (5).

As was shown by Kemmer, the terms of the Hamiltonian quadratic in the coupling constants are not unique, but there exist the two possibilities to choose for them

$$2\pi f_S{}^2\int\sum_{A,\,B,\,\alpha}U(x-z_A)U(x-z_B)\tau_\alpha{}^A\tau_\alpha{}^B\gamma_5{}^A\gamma_5{}^Bd\mathbf{x}$$

or

$$2\pi f_S{}^2\int\sum_{A,\,B,\,\alpha}U(x-z_A)U(x-z_B)\tau_\alpha{}^A\tau_\alpha{}^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{x}$$

for the pseudoscalar meson, and

$$2\pi f_V{}^2\int\sum_{A,\,B,\,\alpha}U(x-z_A)U(x-z_B)\tau_\alpha{}^A\tau_\alpha{}^B(\boldsymbol{\gamma}^A\cdot\boldsymbol{\gamma}^B)d\mathbf{x}$$

or

$$2\pi f_V{}^2\int\sum_{A,\,B,\,\alpha}U(x-z_A)U(x-z_B)\times\tau_\alpha{}^A\tau_\alpha{}^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{x}$$

for the vector meson. The differences between these two alternatives are relativistic invariants in the limit of point sources.[20] There is no *a priori* reason for one possibility or the other because both can be derived from a Lagrangian without explicit terms of the second order in the coupling constant with suitable independent variables.

[20] With the help of quantized wave functions $\psi(z)$ and $\psi^+=\psi^*\beta$, μ, $\nu=1,\ \cdots,\ 4$, $z_4=ix_0$, these invariants can be written $\sum_\mu\ [\psi^+i\gamma_5\gamma_\mu\psi]^2$ and $\sum_{\mu<\nu}[\psi^+i\gamma_\mu\gamma_\nu\psi]^2$, respectively.

The particular choice in the expressions (44) and (45) of these terms has only the *a posteriori* reason that it gives simpler results for the potential energy between two slowly moving nucleons, which in this case does not contain terms of the type $\delta(z_A - z_B)$. This situation, which has been noticed by different authors, does not seem entirely satisfactory.

The form of the Hamiltonians H_S and H_V as integrals over the momentum space is given by

$$H_S=\tfrac{1}{2}\int\sum_\alpha\{p_\alpha(k)p_\alpha(-k)+k_0^2q_\alpha(k)q_\alpha(-k)\}d\mathbf{k} +\frac{f_S}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\tau_\alpha^A \times\{i(\mathbf{s}^A\cdot\mathbf{k})q_\alpha(k)-\gamma_5^Ap_\alpha(k)\}d\mathbf{k} +\frac{f_S^2}{4\pi^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k) \times\tau_\alpha^A\tau_\alpha^B\gamma_5^A\gamma_5^B, \quad (49)$$

$$H_V=\tfrac{1}{2}\int\sum_\alpha\Big\{\mathbf{p}_\alpha(k)\cdot\mathbf{p}_\alpha(-k) +\frac{1}{\mu^2}(\mathbf{k}\cdot\mathbf{p}_\alpha(k))(\mathbf{k}\cdot\mathbf{p}_\alpha(-k)) +k_0^2\mathbf{q}_\alpha(k)\cdot\mathbf{q}_\alpha(-k) -(\mathbf{k}\cdot\mathbf{q}_\alpha(k))(\mathbf{k}\cdot\mathbf{q}_\alpha(-k))\Big\}d\mathbf{k} +\frac{f_V}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\tau_\alpha^A \times\{i\mathbf{s}^A\cdot[\mathbf{k}\times\mathbf{q}_\alpha(k)]-\boldsymbol{\gamma}^A\cdot\mathbf{p}_\alpha(k)\}d\mathbf{k} +\frac{f_V^2}{4\pi^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k) \times\tau_\alpha^A\tau_\alpha^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{k}. \quad (50)$$

The functions $p_\alpha(k)$, $q_\alpha(k)$ and $\mathbf{p}_\alpha(k)$, $\mathbf{q}_\alpha(k)$ are here defined as in Eq. (6), and they satisfy the canonical relation

$$[p_\alpha(k),q_\beta(k')]=\delta_{\alpha\beta}\delta(k-k'), \qquad [p_{\alpha i}(k),p_{\beta j}(k')]=\delta_{\alpha\beta}\delta_{ij}\delta(k-k'), \quad (51)$$

and reality conditions analogous to (13). The functions $v(k)$ and $G(k)$ are again given by (8) and (15). Of course it is possible to define new quantities analogous to Eq. (14) of the preceding section which make it evident that only $G(k)$ has a physical meaning. This will not, however, be necessary for the following.

The part of H_V which describes the free mesons can be simplified by the canonical substitution

$$\mathbf{p}'=\mathbf{p}-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p})+\frac{\mu}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}), \qquad \mathbf{q}'=\mathbf{q}-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q})-\frac{1}{\mu k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}), \quad (52)$$

with the inverse formulae

$$\mathbf{p}=\mathbf{p}'-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}')-\frac{\mu}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}'), \qquad \mathbf{q}=\mathbf{q}'-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}')-\frac{1}{\mu k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}'). \quad (52a)$$

One obtains in this way

$$H_V=\tfrac{1}{2}\int\sum_\alpha\{\mathbf{p}_\alpha'(k)\cdot\mathbf{p}_\alpha'(-k) +k_0^2\mathbf{q}_\alpha'(k)\cdot\mathbf{q}_\alpha'(-k)\}d\mathbf{k} +\frac{f_V}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A) \times\tau_\alpha^A\Big\{i\mathbf{s}^A\cdot[\mathbf{k}\cdot\mathbf{q}_\alpha'(k)] -\boldsymbol{\gamma}^A\cdot\Big[\mathbf{p}'(k)-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}'(k)) -\frac{\mu}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}'(k))\Big]\Big\}d\mathbf{k} +\frac{f_V^2}{4\pi^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k) \times\tau_\alpha^A\tau_\alpha^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)dk+\cdots. \quad (50')$$

In this paper we shall deal particularly with the nuclear forces derived from the so-called Rosenfeld-Møller mixture, which consists of a pseudoscalar and a vector meson with equal coupling constants. For the sake of simplicity we shall assume also that the rest masses of the two kinds of mesons are equal. This may not be so in reality, but the purpose of the calculations carried through in the next section is not to compute the exact form of the nuclear forces, but merely to show the conditions for which the higher approximations of the perturbation theory regarding the nuclear forces are relatively small. For this purpose it may be sufficient to choose a simpler model than the actual case. Further simplifications arise from the approximation where the mass of the nucleon is considered infinite in comparison with the meson mass, or in other words where the recoil energy of the nucleon in all intermediate states can be neglected. In this case the four-component wave function of the nucleon can be reduced to a two-component wave function in a representation of the Dirac matrices where β is diagonal, the two smaller components being negligible. All matrices which anticommute with β, as for instance γ_5 and $\boldsymbol{\gamma}$, can be neglected. The four-component spin matrix $\mathbf{s}$ can be replaced by the two-component spin matrix $\boldsymbol{\sigma}$, and β can be put equal to unity. The part H_M of the Hamiltonian is then unnecessary, and putting in our case

$$f_S=f_V=g/\mu, \tag{53}$$

where g is dimensionless, one has

$$\begin{aligned}H=H_S+H_V=\tfrac{1}{2}\int\sum_\alpha\{&p_\alpha(k)p_\alpha(-k)\\&+\mathbf{p}_\alpha'(k)\cdot\mathbf{p}_\alpha'(-k)+k_0^2[q_\alpha(k)q_\alpha(-k)\\&+\mathbf{q}_\alpha'(k)\cdot\mathbf{q}_\alpha'(-k)]\}d\mathbf{k}\\&+\frac{g}{\pi\sqrt{2}\mu}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\tau_\alpha{}^Ai\mathbf{s}^A\\&\cdot\{\mathbf{k}q_\alpha(k)+[\mathbf{k}\times\mathbf{q}_\alpha'(k)]\}d\mathbf{k}\\&+\frac{g^2}{4\pi^2\mu^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]\\&\times G(k)\tau_\alpha{}^A\tau_\alpha{}^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{k}+\cdots.\end{aligned}\tag{54}$$

As was shown by Stueckelberg,[21] the simple properties of this particular mixture appear more clearly if one applies the canonical substitution

$$\begin{aligned}\mathbf{q}''&=(i/\mu)\{[\mathbf{k}\times\mathbf{q}']+\mathbf{k}q\},\\\mathbf{p}''&=-i(\mu/k^2)\{[\mathbf{k}\times\mathbf{p}']+\mathbf{k}p\},\\q''&=-(i/\mu)(\mathbf{k}\cdot\mathbf{q}'),\\p''&=i(\mu/k^2)(\mathbf{k}\cdot\mathbf{p}'),\end{aligned}\tag{55}$$

with the inversion

$$\begin{aligned}\mathbf{q}'&=-i(\mu/k^2)\{[\mathbf{k}\times\mathbf{q}'']+\mathbf{k}q''\},\\\mathbf{p}'&=(i/\mu)\{[\mathbf{k}\times\mathbf{p}'']+\mathbf{k}p''\},\\q&=i(\mu/k^2)(\mathbf{k}\cdot\mathbf{q}''),\\p&=-(i/\mu)(\mathbf{k}\cdot\mathbf{p}'').\end{aligned}\tag{55a}$$

Moreover, the new functions fulfill the same reality conditions as the old ones. This substitution is convenient for the reason that the interaction energy depends only on $\mathbf{q}''$ but not on q''. Because of the relation

$$\begin{aligned}p_\alpha(k)p_\alpha(-k)&+\mathbf{p}_\alpha'(k)\cdot\mathbf{p}_\alpha'(-k)\\&+k_0^2[q_\alpha(k)q_\alpha(-k)+\mathbf{q}_\alpha'(k)\cdot\mathbf{q}_\alpha'(-k)]\\&=\frac{k^2}{\mu^2}[\mathbf{p}_\alpha''(k)\cdot\mathbf{p}_\alpha''(-k)+p_\alpha''(k)p_\alpha''(-k)]\\&+\frac{k_0^2\mu^2}{k^2}[\mathbf{q}_\alpha''(k)\cdot\mathbf{q}_\alpha''(-k)+q_\alpha''(k)q_\alpha''(-k)]\end{aligned}$$

the field described by the new scalars p_α'', q_α'' does not interact with the nucleon and can be split off. We do not write it down any longer, and we omit the double prime again in the following final result:

$$\begin{aligned}H=\frac{1}{2}\int\sum_\alpha\bigg\{&\frac{k^2}{\mu^2}\mathbf{p}_\alpha(k)\cdot\mathbf{p}_\alpha(-k)\\&+\frac{k_0^2\mu^2}{k^2}\mathbf{q}_\alpha(k)\cdot\mathbf{q}_\alpha(-k)\bigg\}d\mathbf{k}\\&+\frac{g}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\\&\times\tau_\alpha{}^A\boldsymbol{\sigma}^A\cdot\mathbf{q}_\alpha(k)d\mathbf{k}\\&+\frac{g^2}{4\pi^2\mu^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]\\&\times G(k)\tau_\alpha{}^A\tau_\alpha{}^B(\boldsymbol{\sigma}^A\cdot\boldsymbol{\sigma}^B).\end{aligned}\tag{56}$$

[21] E. C. C. Stueckelberg, see reference 10; the transformation is made there in the ordinary x space.

5. THE NUCLEAR FORCES BETWEEN NUCLEONS AT REST IN HIGHER APPROXIMATIONS FOR A ROSENFELD-MØLLER MIXTURE OF MESONS

The method of Stueckelberg to find the interaction energy between nucleons consists of the use of successive canonical transformations which eliminate step by step the field variables. The neglect of the motion (recoil energies in the intermediate states) of the nucleons is permitted as long as the distance between the nucleons is larger than their Compton wave-length M^{-1}.

Writing

$$H=H_0+gH_1+g^2H_2 \tag{57}$$

to express the dependence of the Hamiltonian on the coupling constant, we perform first a canonical transformation

$$H'=e^{iW}He^{-iW}=H+[W,H]+\frac{1}{2!}[W,[W,H]]+\cdots, \tag{58}$$

where H' is the Hamiltonian in the new variables. While the middle term $e^{iW}He^{-iW}$ holds only in quantum mechanics, the last form holds both in classical and quantum mechanics. We first assume W proportional to g according to $W=gW_1$ and arrange the result again in a power series of g

$$H'=H_0+g\{[W,H_0]+H_1\}+g^2\left\{\frac{1}{2!}[W_1,[W_1,H_0]]+[W_1,H_1]+H_2\right\}+g^3\left\{\frac{1}{3!}[W_1,[W_1,[W_1,H_0]]]+\frac{1}{2!}[W_1,[W_1,H_1]]+[W_1,H_2]\right\}+\cdots. \tag{59}$$

We choose W_1 in such a way that H_1 is just canceled, namely,

$$[W_1,H_0]=-H_1 \tag{60}$$

and obtain

$$H'=H_0+g^2\{\tfrac{1}{2}[W_1,H_1]+H_2\}+g^3\{\tfrac{1}{3}[W_1,[W_1,H_1]]+[W_1,H_2]\}+\cdots. \tag{61}$$

For the Hamiltonian (56) of the Rosenfeld-Møller mixture the condition (60) is fulfilled for

$$W_1=-\frac{1}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(k)\exp(-i\mathbf{k}\cdot\mathbf{z}_A)\times\frac{k^2}{k_0{}^2\mu^2}\tau_\alpha{}^A\boldsymbol{\sigma}^A\cdot\mathbf{p}_\alpha(k)d\mathbf{k}.$$

The term $\frac{1}{2}[W_1,H_1]$ which occurs in H_2' consists of two parts of which one is independent of the field variables while the other is bilinear in this variable. One finds

$$\frac{g^2}{2}[W_1,H_1]=-\frac{g^2}{4\pi^2}\sum_{A,B,\alpha,\beta,i,j}\left\{\tau_\alpha{}^A\tau_\alpha{}^B\boldsymbol{\sigma}^A\cdot\boldsymbol{\sigma}^B\times\int\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k)\frac{k^2}{k_0{}^2\mu^2}d\mathbf{k}+\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}\int\int v(k')v(-k)\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{z}_A]\times\frac{k'^2}{k_0'^2\mu^2}p_{\alpha i}(k')q_{\beta j}(k)d\mathbf{k}d\mathbf{k}'\right\}. \tag{62}$$

Here the coefficient $\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}$ is an abbreviation for

$$\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}=[\tau_\alpha{}^A\sigma_i{}^A,\tau_\beta{}^A\sigma_j{}^A]=-2\overset{A}{\sigma}_{ij}\delta_{\alpha\beta}-2\delta_{ij}\overset{A}{\tau}_{\alpha\beta} \tag{63}$$

(with the notation $\tau_{12}=-\tau_{21}\equiv\tau_3,\cdots;\sigma_{12}=-\sigma_{21}\equiv\sigma_3,\quad\cdots$). Because of the particular choice of H_2 a certain cancellation occurs between the last term in (62) and H_2 since

$$\frac{1}{\mu^2}-\frac{k^2}{k_0{}^2\mu^2}=\frac{1}{k_0{}^2}$$

with the consequence that in the limit of point sources, $G(k)\to 1$, no interaction of the type $\delta(z_A-z_B)$ appears in the result. We get finally for the part $H'_{2,0}$ which is independent of the field variables—generally in the symbol $H'_{m,n}$ the first index is equal to the power of g with which it is multiplied, and the second index denotes the degree in the field variables—the result

$$g^2H'_{2,0}=\frac{g^2}{4\pi^2}\sum_{A,B,\alpha}\tau_\alpha{}^A\tau_\alpha{}^B\boldsymbol{\sigma}^A\cdot\boldsymbol{\sigma}^B\int\frac{G(k)}{k^2+\mu^2}\times\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}.$$

For $A \neq B$ it is always allowable to put $G(k)=1$. If we use

$$\frac{1}{2\pi^2}\int \frac{\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]}{k^2+\mu^2}d\mathbf{k}=\frac{\exp(-\mu r_{AB})}{r_{AB}}, \quad (64)$$

$r_{AB}=|\mathbf{z}_A-\mathbf{z}_B|$ being the distance between the two nucleons and take into account that every pair A, B occurs twice in the sum $\sum_{A,B}$ we find for the potential between two different nucleons A, B as far as it is proportional to g^2, the well-known result

$$V_{AB}=g^2\sum_\alpha \tau_\alpha{}^A\tau_\alpha{}^B\boldsymbol{\sigma}^A\cdot\boldsymbol{\sigma}^B[\exp(-\mu r_{AB})/r_{AB}]. \quad (65)$$

For $A=B$ we obtain the self-energy the value of which in the limit $G(k)\rightarrow 1$ is given by

$$V_{AA}=\frac{9}{2}g^2\left(\frac{1}{a}-\mu\right), \quad (66)$$

if, according to quantum theory, $\sum_\alpha \tau_\alpha{}^2=(\boldsymbol{\sigma}^A)^2=3$ is inserted. For the case of the λ-limiting process we have to put $a^{-1}=0$; hence

$$V_{AA}=-\frac{9}{2}g^2\mu. \quad (66a)$$

In the discussion of terms of higher order in g we are mostly interested in the terms linear in the field variables, the lowest order of which is the third. These terms are according to (61), (62) given by

$$g^3H'_{3,1}=-\frac{g^3}{(2\pi^2)^{\frac{3}{2}}}\sum_{A,B,\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\times\int\frac{G(k)}{k_0{}^2}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}\times\int\exp(-i\mathbf{k}'\cdot\mathbf{z}_A)\frac{k'^2}{k_0{}'^2\mu^2}v(k')p_{\alpha i}(k')d\mathbf{k}',$$

where

$$\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\equiv\sum_{\beta,j}\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}\tau_\beta{}^B\sigma_j{}^B. \quad (67)$$

In the case $A=B$ one has to hermitize the right side by taking the arithmetical mean of the two orders of the factors $\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}$ and $\tau_\beta{}^A\sigma_j{}^A$. One finds in this way for $A=B$

$$\begin{bmatrix}AA\\ \alpha i\end{bmatrix}=0, \quad (67a)$$

for $A\neq B$

$$\begin{bmatrix}AB\\ \alpha i\end{bmatrix}=2\{(\sigma_k{}^A\sigma_l{}^B-\sigma_l{}^A\sigma_k{}^B)\tau_\alpha{}^B+\sigma_i{}^B(\tau_\beta{}^A\tau_\gamma{}^B-\tau_\gamma{}^A\tau_\beta{}^B)\} \quad (67b)$$

where (i, j, k) and (α, β, γ) are cyclic permutations of (1, 2, 3).

These new linear terms in the field one has now to transform away by a new canonical transformation of the type (58) with the help of a $W=g^3W_3$ which satisfies the condition

$$[W_3, H_0]=-H'_{3,1}$$

and is given by

$$W_3=-\frac{1}{(2\pi^2)^{\frac{3}{2}}}\sum_{A,B,\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\times\int\frac{G(k)}{k_0{}^2}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}\times\int\exp(i\mathbf{k}'\cdot\mathbf{z}_A)\frac{k'^2}{k_0{}'^2\mu^2}v(-k')q_{\alpha i}(k')d\mathbf{k}'. \quad (68)$$

This transformation gives rise to a part $H''_{6,0}$ of the new Hamiltonian which is contained in $H_6''=\frac{1}{2}[W_3, H'_{3,1}]$. One finds[22]

$$g^6H''_{6,0}=\tfrac{1}{2}g^6\sum_{A,B,C,D,\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\begin{bmatrix}CD\\ \alpha i\end{bmatrix}F_{CA}f_{AB}f_{CD} \quad (69)$$

with the definitions

$$f_{AB}=\frac{1}{2\pi^2}\int\frac{G(k)}{k_0{}^2}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}, \quad (70)$$

$$F_{AB}=-\frac{1}{2\pi^2}\int G(k)\frac{k^2}{k_0{}^4\mu^2}\times\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}. \quad (71)$$

For the limiting case of a point source the integral f_{AB} was evaluated above as

$$f_{AB}=\exp(-\mu r_{AB})/r_{AB} \text{ for } A\neq B,\qquad f_{AA}=a^{-1}-\mu. \quad (72)$$

[22] Compare E. C. C. Stueckelberg, reference 10, formulas (2.10) and (2.11).

In a similar way one finds

$$F_{AB}=\frac{1}{2\mu}\left(1-\frac{2}{\mu r_{AB}}\right)\exp\left(-\mu r_{AB}\right) \text{ for } A\neq B,$$
$$F_{AA}=\frac{1}{2\mu}\left(3-\frac{2}{\mu a}\right). \quad (73)$$

The result (69) means in general a three- and four-body interaction besides the two-body interaction. In the case of only two particles present the expression (69) gives an additional potential energy given by

$$V''_{AB}=\frac{g^6}{2\mu}\left[c_{AB}\frac{3\exp\left(-2\mu r_{AB}\right)}{r^2_{AB}}+d_{AB}\left(1-\frac{2}{\mu r_{AB}}\right)\frac{\exp\left(-3\mu r_{AB}\right)}{r^2_{AB}}\right], \quad (74)$$

where $A\neq B$ and

$$c_{AB}=\sum_{\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}, \quad d_{AB}=\sum_{\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\begin{bmatrix}BA\\ \alpha i\end{bmatrix}, \quad (75)$$

and where we put again $a^{-1}=0$ according to the λ-limiting process.

The evaluation of c_{AB} and d_{AB} with the help of the expression (67) gives, with convenient abbreviations, $s_{AB}\equiv(\boldsymbol{\sigma}_A\cdot\boldsymbol{\sigma}_B)$, $\tau_{AB}\equiv\sum_\alpha \tau_\alpha{}^A\tau_\alpha{}^B$:

$$c_{AB}=8\{3(3-s_{AB})+3(3-\tau_{AB})+4s_{AB}\tau_{AB}\}, \quad (76)$$

$$d_{AB}=-8\{(3-s_{AB})\tau_{AB}+s_{AB}(3-\tau_{AB})-4s_{AB}\tau_{AB}\}.^{23} \quad (77)$$

[23] E. C. C. Stueckelberg, see reference 10, Eq. (3.12) gives in both c_{AB} and d_{AB} an additional term $-64(3-s_{AB})(3-\tau_{AB})$ which I have been unable to confirm and which seems to be erroneous. This additional term causes also an overestimation of the numerical value of V''_{AB} by Stueckelberg.

For the ground state of the deuteron one has $\tau_{AB}=-3$, $s_{AB}=1$, $c_{AB}=3.32$, $d_{AB}=-3.32$, and V''/V small for $3(2g)^4/(\mu r)^2\ll 1$ as was indicated in the introduction.

While in the classical theory the only terms of the final Hamiltonian which give rise to an interaction between nucleons are those which are independent of the meson field, it is different in quantum theory. In this theory it is not possible to put simultaneously $p_\alpha(k)$ and $q_\alpha(k)$ equal to zero, and terms in the final Hamiltonian which are either bilinear in the $p_\alpha(k)$ or bilinear in the $q_\alpha(k)$ give in general a finite contribution to the nucleon interaction due to the zero-point energy of the field oscillators. There exist such terms of the order g^4 which are generated by a g^2W_2 which eliminates $g^2H'_{2,2}$ given by the first term of (62), and which are computed by different authors. It is interesting that this interaction energy of the order g^4 disappears again according to the new theory of Dirac, where mesons of negative energy are introduced in the intermediate states. This theory brings forth a greater similarity with the classical theory than the older form of the quantum theory, and particularly all effects due to the zero-point fluctuations of the field oscillators are canceled in the new theory. It would be possible to check this new theory if experimental tests for or against the fourth-order terms of the interaction energy were available.

On the other hand this new hypothesis of negative-energy mesons was not necessary in order to make the theory convergent in the approximation where the heavy particles are treated non-relativistically. In this approximation the convergence could be achieved with the λ-limiting process alone.