Theory of the Continuous X-Ray Spectrum: Short Wave Limit

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An earlier result for the intensity distribution of continuous x-radiation is reduced to applicability at the short-wave limit; this is accomplished by passing to the limit of zero velocity for the scattered electron.

IN an earlier work[1] the results of the theory of the continuous x-ray spectrum—relativity and retardation of potential neglected—were given in a form reasonably convenient for numerical computation of intensity distributions. The results as given, however, are not directly applicable to the special case of the short wave (Duane-Hunt) limit—i.e., where the incident electron loses all its energy in the production of an x-ray quantum. Also, experience has shown that whenever one attempts to carry out the computations near the short wave limit—i.e., when the emergent electron speed is small compared with the incident speed—the numerical work is unwieldy to the point of uselessness, so that extrapolation to the short wave limit is not feasible. In the present work the difficulty is overcome by a reduction of the earlier results to direct applicability to the short wave limit.

ANALYTICAL TREATMENT

The absolute intensity of continuous x-radiation observed at an angle Θ (made with the direction of bombardment, taken as the x direction) is given by[2]

$$I_\nu(\Theta)=\frac{(8\pi^2e^2\hbar^2/m^2c^3)n_1^4}{[\exp(2\pi i n_1)-1][1-\exp(-2\pi i n_2)]}\times(L_x^2\sin^2\Theta+L_z^2\cos^2\Theta+L_y^2) \quad (1)$$

ergs per unit solid angle per unit frequency range per bombarding electron per atom-per-square-centimeter of target area. Here,

$in_{1,2}=(Ze^2/\hbar v_{1,2})$;
$v_{1,2}$ = initial, final electron velocities;
Z = atomic number of target element;

$$L_x^2=(-1/\xi_0)\left\{B^2\sum_{r=0}^{\infty}\frac{s_r w_0^{r+1}}{r+1}-\xi_0+\sum_{r=1}^{\infty}t_r[(1-\xi_0)w_0^r-rI_{r-1}]-B\sum_{r=0}^{\infty}I_r(u_r+u_r^*)\right\}; \quad (2)$$

$$L_y^2=L_z^2=[(1-n_1^2)/2\xi_0]\times\sum_{r=0}^{\infty}p_r\{(\xi_0-r-2)I_{r+1}+(1-\xi_0)w_0^{r+2}\}; \quad (3)$$

$$\begin{aligned}&\xi_0=-4n_1n_2/(n_1-n_2)^2;\\ &B=(n_1+n_2)/(n_1-n_2);\quad w_0=\xi_0/(\xi_0-1);\\ &s_r=\sum_{\nu=0}^{r}g_{1\nu}g^*_{1,r-\nu};\quad t_r=\sum_{\nu=0}^{r}g_{2\nu}g^*_{2,r-\nu};\\ &u_r=\sum_{\nu=0}^{r}g_{1\nu}g^*_{2,r-\nu};\quad p_r=\sum_{\nu=0}^{r}g_{3\nu}g^*_{3,r-\nu};\\ &I_r=\int_0^{w_0}[w^r/(1-w)]dw;\end{aligned} \quad (4)$$

the $g_{\sigma\nu}$ are given through the recursion formulae:

$$g_{2,\nu+1}=\frac{\nu^2-n_1n_2+\nu(n_1-n_2)}{(\nu+1)^2}g_{2\nu},$$

$$g_{1\nu}=(1+\nu/n_1)g_{2\nu},\quad g_{3\nu}=g_{1\nu}/(\nu+1);\quad g_{\sigma0}=1.$$

Close to the short wave limit, $in_2\gg in_1$, $in_2\gg 1$, whence important simplifications are found to occur. In this limiting case[3]

[1] R. Weinstock, Phys. Rev. **61**, 584 (1942).

[2] Here the y direction is taken perpendicular to both the direction of incident electron and direction of observation. As given here, the expression for $I_\nu(\Theta)$ differs somewhat in appearance from that given in reference 1: Use has been made of the fact that $(Z/ak_1)^2=-n_1^2$; also, $(\mathfrak{M}_x^2/|A|^2)$ has been replaced by $(-16\pi n_1^2/\xi_0^2)L_x^2$ (etc.).

[3] The limiting values of rI_{r-1} and I_{r+1} are taken to an extra power of w_0, as is required by the occurrence of cancellations among the lower power terms when substitution is made into (2) and (3).

$$\xi_0=-4n_1/n_2, \quad w_0=4n_1/n_2,$$
$$B=-1, \quad I_r=w_0^{r+1}/(r+1),$$
$$rI_{r-1}=w_0^r+rw_0^{r+1}/(r+1),$$
$$I_{r+1}=[w_0^{r+2}/(r+2)]+[w_0^{r+3}/(r+3)].$$

The first recursion formula for the $g_{\sigma\nu}$ becomes

$$g_{2,\nu+1}=\frac{-n_2(n_1+\nu)}{(\nu+1)^2}g_{2\nu},$$

while the others remain unaltered. With these changes, we have

$$L_x^2=-\frac{w_0}{\xi_0}\sum_{r=0}^{\infty}\frac{w_0^r(s_r+t_r+u_r+u_r^*)}{r+1}$$
$$=\sum_{r=0}^{\infty}\frac{w_0^r}{r+1}\sum_{\nu=0}^{r}(g_{1\nu}+g_{2\nu})(g^*_{1,r-\nu}+g^*_{2,r-\nu}),$$

from (2) and (4);

$$L_x^2=\sum_{r=0}^{\infty}\frac{\sum_{\nu=0}^{r}(f_{1\nu}+f_{2\nu})(f^*_{1,r-\nu}+f^*_{2,r-\nu})}{r+1}. \tag{5}$$

Also, by virtue of the above simplifications, (3) becomes

$$L_y^2=L_z^2=(n_2^2w_0^3/2\xi_0)\sum_{r=0}^{\infty}\frac{p_rw_0^r}{(r+2)(r+3)}$$
$$=-8n_1^2\sum_{r=0}^{\infty}\frac{\sum_{\nu=0}^{r}f_{3\nu}f^*_{3,r-\nu}}{(r+2)(r+3)}, \tag{6}$$

where, for $\sigma=1,\ 2,\ 3,\ f_{\sigma\nu}=w_0^\nu g_{\sigma\nu}=(4n_1/n_2)^\nu g_{\sigma\nu}$. The recursion relations among the $f_{\sigma\nu}$ are thus found to be

$$f_{2,\nu+1}=\frac{-4n_1(n_1+\nu)}{(\nu+1)^2}f_{2\nu}, \quad f_{1\nu}=(1+\nu/n_1)f_{2\nu},$$
$$f_{3\nu}=f_{1\nu}/(\nu+1); \quad f_{\sigma 0}=1.$$

Finally, (1) reduces to

$$I_\nu(\Theta)=\frac{(8\pi^2e^2\hbar^2/m^2c^3)n_1^4}{\exp(2\pi in_1)-1}$$
$$\times(L_x^2\sin^2\Theta+L_z^2\cos^2\Theta+L_y^2)$$

for the absolute intensity at the short wave limit [in the units given for (1)]. Now the $L^2_{x,y,z}$ are given by (5) and (6).

In terms of the tube voltage V we have, since in our non-relativistic approximation $\frac{1}{2}mv_1^2=eV/300$,

$$in_1=Ze^{\frac{3}{2}}(150m)^{\frac{1}{2}}/\hbar V^{\frac{1}{2}}=[Ze^2m^{\frac{1}{2}}/(4\pi\hbar^3c)^{\frac{1}{2}}]\lambda_0^{\frac{1}{2}}, \tag{7}$$

where λ_0 is the minimum wave-length for given V, consistent with energy conservation. (The latter expression holds at the short wave limit only, since only there does $\frac{1}{2}mv_1^2=2\pi\hbar c/\lambda_0$.) Since for a given angle of observation $I_\nu(\Theta)$ depends upon n_1 alone, (7) reveals that pairs of values of Z, λ_0 satisfying $Z\lambda_0^{\frac{1}{2}}=$constant should yield the same absolute intensity at the limit.

NUMERICAL CHECK

Just as in the case of computations carried out far from the short wave limit, a numerical check is desirable to effect an estimate of the error involved in the breaking off of the expansions (5) and (6) after a finite number of terms. The desired check is here also provided by the ingeniously attained result of Sommerfeld and Maue[4]—a relatively simple expression for the sum $L^2=L_x^2+L_y^2+L_z^2$. Although the check itself converges no more rapidly than do (5) and (6), the simplicity of their result renders the numerical calculation of L^2 much easier than the individual $L^2_{x,y,z}$ to the same number of terms.

According to Sommerfeld-Maue,

$$L^2=-\frac{\xi_0}{2n_1^2}\frac{d}{d\xi_0}|\mathfrak{F}|^2,$$

where $\mathfrak{F}=F(-n_1,\ -n_2,\ 1,\ \xi_0)$, the hypergeometric function of indicated parameters. That is, $\mathfrak{F}$ is expressible as a power series $\sum_{\nu=0}^{\infty}a_\nu\xi_0^\nu$ whose coefficients are defined through

$$a_{\nu+1}=\frac{(-n_1+\nu)(-n_2+\nu)}{(\nu+1)^2}a_\nu, \quad a_0=1.$$

Now, $|\mathfrak{F}|^2=\sum_{r=0}^{\infty}\sum_{\nu=0}^{r}a_\nu a^*_{r-\nu}\xi_0^r$; we find, therefore, upon differentiation,

$$L^2=(-1/2n_1^2)\sum_{r=0}^{\infty}r\sum_{\nu=0}^{r}a_\nu a^*_{r-\nu}\xi_0^r.$$

For $in_2\gg in_1$, $in_2\gg 1$, whence $\xi_0=(-4n_1/n_2)$, we

[4] A. Sommerfeld and A. W. Maue, Ann. d. Physik **23**, 589 (1935).

find at length

$$L^2 = (-1/2n_1^2) \sum_{r=0}^{\infty} r \sum_{\nu=0}^{r} f_{2\nu} f^*_{2,r-\nu},$$

where $f_{2\nu}$ is the quantity defined in the preceding section.

LIMITATIONS OF THE RESULTS

It is well to keep in mind the limits to the applicability of the results of this paper; these consist, briefly, of the following. In addition to the neglect of relativity and retardation of potential, there is inherent the neglect of screening by the atomic electrons of the target material —i.e., the assumption of a pure Coulomb field as seen by the incident electron during the emission process. "This is only strictly justified for an electron whose de Broglie wave-length is small compared with the radius of the K shell of the scattering atom. For $Z=28$ and $V=15$ kilovolts, for example, these lengths are of the same order of magnitude, so that at best the assumption of a pure Coulomb field is only approximate—the higher the voltage and the smaller Z, the better the approximation."[5] Finally, the x-radiation upon which measurement is made for comparison with the theory must originate from a *thin* target; the theory takes no account of multiple scattering within the target.

[5] Quoted from reference 1.

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The Relative Abundance of the Isotopes of Potassium in Pacific Kelps and in Rocks of Different Geologic Age

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The Dempster double focusing mass spectrograph has been used to determine the K^{39}/K^{41} isotope ratio in certain Pacific kelps, fossils, and rocks of different geologic age. Within the 1 percent accuracy obtainable, the potassium isotope ratio for the kelps and two Upper Cambrian fossils examined is the same as that of rocks. Rocks whose ages vary from Early pre-Cambrian to Tertiary show no measurable difference in the potassium isotope ratio, within 1 percent error. The average value of the potassium isotope ratio was found to be 14.12 ± 0.28. Fluctuations in the ratio are attributed to isotope effects of the hot-filament ion source.

INTRODUCTION

DURING the past seven years there have been reported variations in the relative abundance of the stable isotopes of three elements. Variations in the relative abundance of the stable isotopes of oxygen were found by Dole[1] in his careful measurements of the density of water in different localities. Variations of 5 percent in the isotope ratio of the stable isotopes of carbon (C^{12}/C^{13}) have been observed by Nier[2] and his collaborators.[3] Variations of as much as 15 percent in the ratio of the relative abundance of the stable isotopes of potassium (K^{39}/K^{41}) have been reported by Brewer.[4] As emphasized by Brewer, variations of this order of magnitude are too small to be detected by any of the present chemical techniques of measuring atomic weights of the above elements. Therefore, more precise determinations, such as those obtainable with the modern mass spectrograph or by careful density measurements, must ordinarily be employed.

It seemed desirable to have the above observations substantiated and supplemented by further studies. Goodman[5] has already pointed out the

[1] M. Dole, J. Chem. Phys. **4**, 268, 778 (1936).
[2] A. O. Nier and E. A. Gulbransen, J. Am. Chem. Soc. **61**, 697 (1939).
[3] B. F. Murphy and A. O. Nier, Phys. Rev. **59**, 771 (1941).
[4] A. K. Brewer, J. Am. Chem. Soc. **58**, 365 (1936).
[5] C. Goodman, J. App. Phys. **13**, 276 (1942).