The Antisymmetrical Interaction in Beta-Decay Theory

CHARLES L. CRITCHFIELD*
Research Laboratory of Physics, Harvard University, Cambridge, Massachusetts
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Forms of the beta-decay interaction are sought which are totally symmetric and totally antisymmetric to the interchange of any two of the particles involved in the emission. It is found that there is no symmetric interaction and one antisymmetric. The antisymmetric interaction allows transitions without change of parity and with a change of nuclear spin of either 0 or ± 1, including $0\rightarrow 0$. First forbidden transitions entail a change of parity and $\Delta i=0$, ± 1 or ± 2. Second forbidden transitions are characterized by $\Delta i=\pm 2$, ± 3 and no change in parity. The contributions of electron-neutrino states of definite angular momenta are presented explicitly as an aid in the study of the influence of the nucleus on the light particle waves. The influence of the Coulomb field is discussed.

I. INTRODUCTION

In the first paper describing a field theory of beta-activity Fermi[1] drew an analogy with electromagnetic theory and assumed the interaction to be the scalar product of a four-vector constructed from the wave functions of neutron and proton with a four-vector constructed from electron and neutrino waves. It is well known that besides the four-vector one can construct from products of two wave functions an axial four-vector, a skew-symmetric tensor, and two scalars, one of which changes sign under reflection and is commonly called a pseudoscalar. Invariant products of any of these five covariant quantities with the corresponding contravariant quantities formed from electron and neutrino wave functions present a fivefold infinite manifold of choices for an interaction between heavy and light particles. The experimental material that may make possible a particular choice of an interaction consists of (1) the selection rules obeyed in beta-emission and (2) the shape of the beta-ray spectrum, particularly the shape of the spectra in forbidden transitions.

An alternative choice of interaction has been proposed by Gamow and Teller[2] who noted that the selection rules imposed by the four-vector interaction were inconsistent with the beta-activity of ThCC′. The same violation of Fermi's selection rules appears to be made in the beta-activity of He^6 and kindred nuclei. The Gamow-Teller selection rules are less stringent than the Fermi rules in that they allow $\Delta i=0$, ± 1 (except $0\rightarrow 0$), whereas the Fermi rules allow only $\Delta i=0$ (including $0\rightarrow 0$). Interactions that give G-T rules are the tensor and the axial vector interactions.

Allowed transitions are defined as those transitions that are independent of the small components (i.e., the particle velocity) of the Dirac waves for the nuclear particles and in which the electron and neutrino waves have no node through the nucleus. The shape of the allowed beta-ray spectrum is the same for any one of the five possibilities of interaction listed above. It may be slightly different for linear combinations. Only in forbidden transitions will the spectra of the five possibilities be different.

Uhlenbeck and Konopinski[3] have compared the shapes to be expected from each of the five invariants with the experimental results on Na^{24}, P^{32}, and RaE. Their conclusion is that there is evidence against all these interactions taken singly except the tensor interaction. It is possible that actually none of these interactions is sufficient by itself to explain the experimental results but the tensor interaction cannot be excluded at present because of the great flexibility of this interaction which contains several unknown matrix elements. It is also possible that a linear combination of the five possibilities will account for experiment although no one taken singly is able to do so.

* Now at the Geophysical Laboratory, The Carnegie Institution of Washington, Washington, D. C.
[1] E. Fermi, Zeits. f. Physik **88**, 161 (1934).
[2] G. Gamow and E. Teller, Phys. Rev. **49**, 895 (1936).
[3] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

The supposition that a particular one of the five invariants discussed might be a complete description of fact is a kind of extension of Fermi's analogy to electromagnetic theory. It is what we would reasonably expect to obtain if beta-decay occurs in two steps, i.e., the proton emits a meson which then creates a positron and a neutrino. In this description one would naturally form covariant quantities with proton and neutron waves and contract them with the analogous light particle functions. The analogy with electromagnetism then leads to the hope that one of these covariants is enough. If, on the other hand, the beta-process is not analogous to radiation theory in this sense, linear combinations of the five invariants are as acceptable possibilities as the invariants taken singly.

It is the purpose of this paper to present the theoretical predictions of a particular linear combination of the five invariants. The basis on which this combination is chosen is as follows: The beta-process is considered to be an essentially four-particle process, and the requirement on the interaction is that it shall be either totally symmetric or totally antisymmetric to the interchange of role of any two of the participating particles.

II. THE INTERACTION

The symmetry properties of the interaction between fields of the four particles are most easily determined if the beta-process is described as the simultaneous creation of neutron, proton, electron, and neutrino. Since all these particles have spin $\frac{1}{2}\hbar$, this description can be permitted by assuming the Dirac wave equation and the hole theory extension of that equation for all four particles. A hole in the theoretical neutron sea of negative energy is then the experimentally observed neutron. In negative beta-emission such a neutron hole is filled at the same instant that a proton appears to take its place in the nucleus and electron and neutrino are created. Positive beta-emission is then a simultaneous disappearance of four particles: a proton of positive energy and "neutron," electron, and neutrino of negative energy. In this description it is evident that symmetrization of the interaction producing beta-activity is identical with symmetrization of an invariant product wave function of four particles of spin one-half.

Four particles of spin $\frac{1}{2}\hbar$ cannot form a totally symmetrical state that is invariant to Lorentz transformations. This may be seen at once with the aid of the calculation rules of Van der Waerden's spinor calculus.[4] If two of the four-wave functions have dotted indices and two undotted, the invariant forms are not symmetrical because of the skew-symmetry of the ϵ-operator. If all indices are undotted (or all dotted), the symmetric invariant vanishes because of the identity:

$$\epsilon^{\lambda\mu}\epsilon^{\kappa\nu}+\epsilon^{\kappa\lambda}\epsilon^{\mu\nu}+\epsilon^{\mu\kappa}\epsilon^{\lambda\nu}\equiv 0.$$

The symmetrization in this discussion is restricted, of course, to space and spin coordinates. If isotopic spin were introduced, symmetrization would have no significance.

There is one and only one totally antisymmetric invariant wave function constructed of the four Dirac ψ-functions, *viz.*,

$$\chi(\mathbf{x})\equiv\begin{vmatrix}\Psi_1 & \Phi_1 & \psi_1 & \phi_1\\ \Psi_2 & \Phi_2 & \psi_2 & \phi_2\\ \Psi_3 & \Phi_3 & \psi_3 & \phi_3\\ \Psi_4 & \Phi_4 & \psi_4 & \phi_4\end{vmatrix}_{\mathbf{x}}. \quad (1)$$

Different Greek letters are used to distinguish between particles and, in the following, Ψ shall stand for the neutron wave function, Φ for the proton, ψ, electron, and ϕ, neutrino. All are taken at the same point in space, $\mathbf{x}$. The subscripts in Eq. (1) refer to the four components of a Dirac wave equation, the particular form of which is immaterial as long as it is the same for all particles. That χ is an invariant is readily seen from the fact that the determinants of the matrices that represent Lorentz transformations are unity. Let S be the unitary matrix that represents a certain Lorentz transformation:

$$\psi'=S\psi. \quad (2)$$

This S applies to all four wave functions in the determinant, Eq. (1), so if we write

$$\chi\equiv|D|,$$

[4] B. Van der Waerden, *Göttingen Nachrichten* (1929), p. 100.

then the new antisymmetric wave function, χ', is

$$\chi' = |S\cdot D| = |S|\times|D| = |D| = \chi, \quad \text{Q.E.D.} \quad (3)$$

For the sake of definiteness the numerical subscripts in Eq. (1) will be assumed to refer to the four rows and columns in their usual order in the $\boldsymbol{\alpha}$ and β matrices in the familiar form of Dirac's equation:

$$\left[i\frac{\hbar}{c}\frac{\partial}{\partial t} - (\boldsymbol{\alpha}, \mathbf{p}) - \beta mc\right]\psi = 0. \quad (4)$$

If we consider the wave functions in Eq. (1) to be quantized, i.e., as operators, the form of the interaction between fields of the elementary particles becomes:

$$H' = g\int [\chi(\mathbf{x}) + \chi(\mathbf{x})^*]d\mathbf{x}, \quad (5)$$

where g is the "Fermi constant" having dimensions of energy times volume, and $\chi(\mathbf{x})$ gives rise to the disappearance of four particles at the point $\mathbf{x}$; $\chi(\mathbf{x})^*$ gives rise to their appearance. The transitions induced by H' are between states of equal energy and therefore the probability of beta-emission is given by the well-known formula

$$P(E)dE = 2\pi\hbar^{-1}|H'|^2\rho_n(w-E)\rho_e(E)dE. \quad (6)$$

In Eq. (6) $\rho_n(w-E)$ is the density of neutrino states of energy $w-E$ and $\rho_e(E)$ the density of electron states of energy E. The energy released by the nucleus is w.

An obvious difference between the form of this theory and that of preceding ones is that H' is an integral over products of Ψ and Φ or over products of Ψ^* and Φ^*. This is only a formal difference from the customary method of calculating matrix elements which contain Ψ^* and Φ together, etc., but it will be convenient all around to introduce Ψ^* in place of Ψ in Eq. (1). Then the physical interpretation of Eq. (5) will be the usual disappearance of a neutron when proton, electron, and neutrino are created or vice versa.

The replacement of Ψ by Ψ^* at the same time that positive and negative energy levels are exchanged has been studied by Furry[5] and Pauli.[6] These authors have shown that there is a unitary operator θ such that

$$\psi^* \rightarrow \psi\theta \quad (7)$$

when positive and negative energy levels are interchanged. In the spin coordinates pertinent to Eq. (4)

$$\theta = -i\beta\alpha_2. \quad (8)$$

Applying θ to the neutron waves in Eq. (1) and substituting in Eq. (5), we get the interaction for positron emission:

$$H_p' = -g\int \begin{vmatrix} \Psi_4(\mathbf{x})^* & \Phi_1(\mathbf{x}) & \psi_1(\mathbf{x}) & \phi_1(\mathbf{x}) \\ -\Psi_3(\mathbf{x})^* & \Phi_2(\mathbf{x}) & \psi_2(\mathbf{x}) & \phi_2(\mathbf{x}) \\ -\Psi_2(\mathbf{x})^* & \Phi_3(\mathbf{x}) & \psi_3(\mathbf{x}) & \phi_3(\mathbf{x}) \\ \Psi_1(\mathbf{x})^* & \Phi_4(\mathbf{x}) & \psi_4(\mathbf{x}) & \phi_4(\mathbf{x}) \end{vmatrix} d\mathbf{x}. \quad (9)$$

Electron emission is given by the complex conjugate of Eq. (9), but since in the expression for the probability of emission only $|H'|^2$ appears, it is sufficient to consider Eq. (9) alone.

It is convenient to expand Eq. (9) for H_p' as a linear combination of the familiar invariants. For this purpose we shall use the Dirac operators, ρ and $\boldsymbol{\sigma}$, *acting on the heavy particle waves only*

$$\begin{array}{lll} \rho_1 = i\alpha_x\alpha_y\alpha_z, & \rho_2 = \beta\alpha_x\alpha_y\alpha_z, & \rho_3 = \beta, \\ \sigma_x = i\alpha_y\alpha_z, & \sigma_y = i\alpha_z\alpha_x, & \sigma_z = i\alpha_x\alpha_y. \end{array} \quad (10)$$

In this formulation Eq. (9) becomes

$$\begin{aligned} H_p' = g\cdot\tfrac{1}{2}\int \Psi^*\Bigg\{ & (\rho_1 + i\rho_2)\begin{vmatrix}\psi_1 & \phi_1 \\ \psi_2 & \phi_2\end{vmatrix} \\ & - (\rho_1 - i\rho_2)\begin{vmatrix}\psi_3 & \phi_3 \\ \psi_4 & \phi_4\end{vmatrix} \\ & + \rho_3\left[\begin{vmatrix}\psi_2 & \phi_2 \\ \psi_3 & \phi_3\end{vmatrix} - \begin{vmatrix}\psi_1 & \phi_1 \\ \psi_4 & \phi_4\end{vmatrix}\right] \\ & - (\sigma_x + i\sigma_y)\begin{vmatrix}\psi_1 & \phi_1 \\ \psi_3 & \phi_3\end{vmatrix} \\ & + (\sigma_x - i\sigma_y)\begin{vmatrix}\psi_2 & \phi_2 \\ \psi_4 & \phi_4\end{vmatrix} \\ & + \sigma_z\left[\begin{vmatrix}\psi_2 & \phi_2 \\ \psi_3 & \phi_3\end{vmatrix} + \begin{vmatrix}\psi_1 & \phi_1 \\ \psi_4 & \phi_4\end{vmatrix}\right]\Bigg\}\Phi d\mathbf{x}. \quad (11) \end{aligned}$$

From this expansion selection rules can be

[5] W. H. Furry, Phys. Rev. **51**, 125 (1936); **54**, 56 (1938).

[6] W. Pauli, Inst. H. Poincaré Ann. **6**, 130 (1936).

deduced at once. The terms proportional to the σ's and to ρ_1 constitute an axial vector interaction and give $\Delta i=0, \pm 1$ ($0\rightarrow 0$ excluded) with no change in parity in allowed transitions. The ρ_3 term is the scalar interaction and gives $\Delta i=0$ ($0\rightarrow 0$ included) no change in parity. The ρ_2 term is the pseudoscalar and gives no allowed transitions.

A curious form of the rule for determining the change in parity is evident from Eq. (11). It will be noted that the light particle functions that are associated with ρ_3 and $\boldsymbol{\sigma}$ (which mix large terms of proton waves with large terms of neutron waves) contain products of one small and one large component of the light particle waves. Thus, if we follow the customary method of giving the electron (and neutrino) the orbital quantum number L that applies to the spin eigenfunction, i.e., the same as the large component, we obtain the rule: There is no change in the parity of the nucleus if the sum of orbital quantum numbers of electron and neutrino is odd; the parity changes if the sum is even. This is just the reverse of the rule that applies to particles of integral spin and also of the rule that applies to the absorption and subsequent emission of a single particle of any spin. The root of the paradox lies in assigning a definite parity to wave functions of spin $\frac{1}{2}h$. The form of the matrix that represents inversion of the coordinate system[5] shows that the phase of these waves is changed by $\frac{1}{2}\pi$ rather than 0 or π, the phase of the large components increasing if that of the small decreases and vice versa. Thus two successive inversions lead to the original wave function but with opposite sign, the same result as that obtained by rotating the coordinates by 360° about some axis. It is well known that there is no paradox in the case of the rotation because the space of all rotations, although continuous, is two-sided.[7]

A short discussion of the above results has been presented in a letter to the editor of *The Physical Review.*[8] It is pointed out in the letter that addition of scalar and axial vector operators increases the lifetimes of the $4n\pm 2$ nuclei relative to the $4n\pm 1$ nuclei in allowed transitions.[9] This improves the agreement of theory with experiment for C^{10} and F^{18} but makes it worse for He^6. Furthermore, the ratio of the transitions to excited and to ground states of the nuclei considered by Grönblom[9] is predicted more closely by the antisymmetric theory than by the axial vector (or tensor) alone. The reason for this is again the relatively stronger transitions without change of spin which the antisymmetric theory predicts.

III. MATRIX ELEMENTS

The shapes of the forbidden spectra under the theory presented in this paper can be derived from the results of reference 3 on the scalar, axial vector, and pseudoscalar interactions. A slight extension is necessary to include some cross product terms in the first forbidden spectra. I have repeated the work of Konopinski and Uhlenbeck insofar as it applies to the antisymmetrical theory of beta-decay by the method which uses the eigenfunctions of both electron and neutrino in polar coordinates. Except for the addition of cross product terms between different kinds of interaction, the results are, of course, the same as in reference 3. Nevertheless an outline of the method is presented, and the results are given in detail for several reasons. The chief reason is to have the contributions of various electron and neutrino angular momenta presented separately because the possible influence of the nucleus on these contributions is very likely different. Beta-spectra may furnish a useful experimental means of determining the influence of nuclei on electron and neutrino waves. It appears certain from the work of Rabi *et al.*[10] that there is no spin dependent force between electron and nucleus except that due to magnetic moments; but it may well be that there are spin-independent forces in addition to the Coulomb field. Another reason for presenting the calculations in polar coordinates is to throw light on the detailed effect of the Coulomb field on the shape of the beta-spectrum and also on the life-

[7] Cf. E. Wigner, *Gruppentheorie und ihre anwendung auf die quantenmechanik der atomspektren* (Braunschweig, F. Vieweg & sohn, 1931), p. 99.

[8] C. Critchfield and E. Wigner, Phys. Rev. **60**, 412 (1941).

[9] E. Wigner, Phys. Rev. **56**, 526 (1939), and in the same issue, B. Grönblom, Phys. Rev. **56**, 508 (1939).

[10] Rabi, Millman, Kusch, and Zacharias, Phys. Rev. **55**, 526 (1939); Kusch, Millman, and Rabi, Phys. Rev. **57**, 765 (1940).

times of the beta-activities. The latter questions are taken up in Section IV.

In this section it will be assumed that there is no interaction between the electron and the electromagnetic field of the nucleus. We have, therefore, to use the eigenfunctions of kinetic energy for both electron and neutrino in Eq. (11) and substitute H_p' in turn in Eq. (6).

The "matrix element" of beta-decay theory, M, is related to our H' by

$$M \equiv -i(2\pi h^2 R/g p_e p_n)H', \tag{12}$$

where R is the radius of the sphere in which quantization is effected, p_e is the momentum of the electron, and p_n is the momentum of the neutrino. Using M in place of H_p' we find the probability of emission of a positron and neutrino (absorption from states ψ and ϕ, respectively) with total angular momentum jh and z-component mh to be

$$P(\psi,\phi)_{j,m} = (g^2/2\pi^3c^5h^7)Ep_e(w-E)^2 \times |M(\psi,\phi)_{j,m}|^2. \tag{13}$$

An electron and neutrino emitted into a state specified by j and m may cause a maximum change in the nuclear spin of jh. The nuclear spin may also change by integral multiples of h less than jh provided that the total angular momentum is conserved. It is necessary to calculate the matrix element only for one particular m, and it is convenient to choose the highest value, $m=j$. The total probability of emission into $(\psi,\phi)_j$ is the sum over all m, and this sum will be considered automatically accomplished in the square of matrix elements.

Consider the particular case in which an electron disappears from an $S_{\frac{1}{2}}$ state of energy $-E$, and the neutrino from a $P_{\frac{1}{2}}$ state of energy $E-w$. If the total angular momentum of these two particles is zero, the state will be denoted by the symbol

$$(S_{\frac{1}{2}}P_{\frac{1}{2}})_0.$$

The electron wave will always be given first. A more expanded form of the wave function is

$$2^{-\frac{1}{2}}[S_{\frac{1}{2}}^{(\frac{1}{2})}P_{\frac{1}{2}}^{(-\frac{1}{2})} - S_{\frac{1}{2}}^{(-\frac{1}{2})}P_{\frac{1}{2}}^{(\frac{1}{2})}],$$

where superscripts denote the values of m for individual waves. Now we may calculate $M(S_{\frac{1}{2}}, P_{\frac{1}{2}})_0$ by substituting this form into Eqs. (11) and (12) and evaluating the expressions at the nuclear radius. Only the leading terms in the series expansions in powers of r/λ_e or r/λ_n will be retained. Under these conditions

$$M(S_{\frac{1}{2}}P_{\frac{1}{2}})_0 = \tfrac{1}{2}a\int \rho_3. \tag{14}$$

Here $\int \rho_3$ stands for $\int \Psi^* \rho_3 \Phi d\mathbf{x}$. In general

$$\int A = \int \Psi^* A \Phi d\mathbf{x}.$$

The factor a and the related factor b which appears later are defined by

$$a \equiv [(E+mc^2)/2E]^{\frac{1}{2}}, \quad b \equiv [(E-mc^2)/2E]^{\frac{1}{2}}. \tag{15}$$

In some expressions a', a'', b', and b'' will be used for those a's and b's that are changed by the effect of the Coulomb field. Their values in the absence of a field are the same as the unprimed letters.

The calculation for the state $(P_{\frac{1}{2}}S_{\frac{1}{2}})_0$ yields

$$M(P_{\frac{1}{2}}S_{\frac{1}{2}})_0 = -\tfrac{1}{2}b\int \rho_3. \tag{16}$$

In order to simplify the presentation of the formulas, we shall combine related matrix elements into one equation when feasible. Thus Eqs. (14) and (16) may be condensed into

$$M(S_{\frac{1}{2}}P_{\frac{1}{2}})_0/a = -M(P_{\frac{1}{2}}S_{\frac{1}{2}})_0/b = \tfrac{1}{2}\int \rho_3. \tag{17}$$

The allowed transitions may also cause emissions into states of unit angular momentum for which:

$$\mathbf{M}(S_{\frac{1}{2}}P_{\frac{1}{2}})_1/a = -\mathbf{M}(P_{\frac{1}{2}}S_{\frac{1}{2}})_1/b = \tfrac{1}{2}\int \boldsymbol{\sigma}. \tag{18}$$

Matrix elements (17) and (18) comprise all the allowed transitions, and it is evident from Eq. (15) that the sum of squares of these matrix elements will be independent of light particle wave-lengths. We now present the matrix elements that lead to first forbidden transitions. The latter are of two types: *dipole* matrix elements which are proportional to ρ_3 and $\boldsymbol{\sigma}$ but also contain the first power of the momentum of one of the light particles, and *pseudomonopole* matrix elements, proportional to velocity terms of the heavy particle waves, i.e., to ρ_1 and ρ_2. In both cases there is a change in parity, so it is necessary to consider all possible changes in angular momentum:

$$
\begin{aligned}
M(S_{\frac{1}{2}}S_{\frac{1}{2}})_0 &= \tfrac{1}{2}ia\int(\rho_1-i\rho_2) \\
&\quad +(ap_n+b'p_e)/6\hbar\int(\boldsymbol{\sigma},\mathbf{r}), \\
M(P_{\frac{1}{2}}P_{\frac{1}{2}})_0 &= \tfrac{1}{2}ib\int(\rho_1+i\rho_2) \\
&\quad -(bp_n+a'p_e)/6\hbar\int(\boldsymbol{\sigma},\mathbf{r}),
\end{aligned}
\tag{19}
$$

$$
\mathbf{M}(S_{\frac{1}{2}}S_{\frac{1}{2}})_1=(ap_n-b'p_e)/6\hbar\int[\rho_3\mathbf{r}-i(\boldsymbol{\sigma}\times\mathbf{r})],
$$

$$
\mathbf{M}(P_{\frac{1}{2}}P_{\frac{1}{2}})_1=-(bp_n-a'p_e)/6\hbar\int[\rho_3\mathbf{r}+i(\boldsymbol{\sigma}\times\mathbf{r})],
$$

$$
\begin{aligned}
\mathbf{M}(S_{\frac{1}{2}}D_{1\frac{1}{2}})_1/ap_n &= -\mathbf{M}(D_{1\frac{1}{2}}S_{\frac{1}{2}})_1/bp_e \\
&= \mathbf{M}(P_{\frac{1}{2}}P_{1\frac{1}{2}})_1{}^*/bp_n \\
&= -\mathbf{M}(P_{1\frac{1}{2}}P_{\frac{1}{2}})_1{}^*/ap_e \\
&= (\sqrt{2}/12\hbar)\int[2\rho_3\mathbf{r}+i(\boldsymbol{\sigma}\times\mathbf{r})],
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\mathbf{M}(S_{\frac{1}{2}}D_{1\frac{1}{2}})_2/ap_n &= -\mathbf{M}(D_{1\frac{1}{2}}S_{\frac{1}{2}})_2/bp_e \\
&= -\mathbf{M}(P_{\frac{1}{2}}P_{1\frac{1}{2}})_2/bp_n \\
&= \mathbf{M}(P_{1\frac{1}{2}}P_{\frac{1}{2}})_2/ap_e \\
&= (3^{\frac{1}{2}}/6\hbar)\int[\boldsymbol{\sigma}\mathbf{r}],
\end{aligned}
\tag{21}
$$

where

$$
[\boldsymbol{\sigma}\mathbf{r}]_{ik}\equiv\tfrac{1}{2}(\sigma_i r_k+\sigma_k r_i)-\tfrac{1}{3}(\sigma,r)\delta_{ik}.
$$

Equations (19), (20), and (21) give all the matrix elements that lead to first forbidden transitions. Following are the results for second forbidden transitions. In these $\rho_3[\mathbf{rr}]$ and $[\mathbf{r}(\boldsymbol{\sigma}\times\mathbf{r})]$ are defined analogously to $[\boldsymbol{\sigma}\mathbf{r}]$ above, and the symbols $[\boldsymbol{\sigma}\mathbf{rr}]$ and $\mathbf{Q}$ will be used:

$$
[\boldsymbol{\sigma}\mathbf{rr}]_{ijk}\equiv\tfrac{1}{3}(\sigma_i r_j r_k+\sigma_j r_i r_k+\sigma_k r_i r_j)
-\frac{1}{15}(\boldsymbol{\sigma}r^2)-\frac{2}{15}\mathbf{r}(\sigma,r),
$$

$$
\begin{aligned}
\mathbf{Q} &\equiv\int\{\rho_3[\mathbf{rr}]-i[\mathbf{r}(\boldsymbol{\sigma}\times\mathbf{r})]\}/(12)^{\frac{1}{2}}\hbar^2, \\
\mathbf{M}(S_{\frac{1}{2}}P_{1\frac{1}{2}})_2 &= -(b'p_ep_n/3-ap_n{}^2/5)\mathbf{Q}, \\
\mathbf{M}(P_{1\frac{1}{2}}S_{\frac{1}{2}})_2 &= (ap_ep_n/3-b''p_e{}^2/5)\mathbf{Q}, \\
\mathbf{M}(P_{\frac{1}{2}}D_{1\frac{1}{2}})_2 &= (a'p_ep_n/3-bp_n{}^2/5)\mathbf{Q}^*, \\
\mathbf{M}(D_{1\frac{1}{2}}P_{\frac{1}{2}})_2 &= -(bp_ep_n/3-a''p_e{}^2/5)\mathbf{Q}^*,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\mathbf{M}(S_{\frac{1}{2}}F_{2\frac{1}{2}})_2/ap_n{}^2 &= -\mathbf{M}(F_{2\frac{1}{2}}S_{\frac{1}{2}})_2/bp_e{}^2 \\
&= -\mathbf{M}(P_{\frac{1}{2}}D_{2\frac{1}{2}})_2{}^*/bp_n{}^2 \\
&= \mathbf{M}(D_{2\frac{1}{2}}P_{\frac{1}{2}})_2{}^*/ap_e{}^2 \\
&= \int\{3\rho_3[\mathbf{rr}] \\
&\quad +2i[\mathbf{r}(\boldsymbol{\sigma}\times\mathbf{r})]\}/30\sqrt{2}\hbar^2,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\mathbf{M}(P_{1\frac{1}{2}}D_{1\frac{1}{2}})_2/a &= -\mathbf{M}(D_{1\frac{1}{2}}P_{1\frac{1}{2}})_2/b \\
&= (p_ep_n/6\cdot 3^{\frac{1}{2}}\hbar^2)\int\rho_3[\mathbf{rr}],
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\mathbf{M}(S_{\frac{1}{2}}F_{2\frac{1}{2}})_3/ap_n{}^2 &= -\mathbf{M}(F_{2\frac{1}{2}}S_{\frac{1}{2}})_3/bp_e{}^2 \\
&= -\mathbf{M}(P_{\frac{1}{2}}D_{2\frac{1}{2}})_3/bp_n{}^2 \\
&= \mathbf{M}(D_{2\frac{1}{2}}P_{\frac{1}{2}})_3/ap_e{}^2 \\
&= \hbar^{-2}(120)^{-\frac{1}{2}}\int[\boldsymbol{\sigma}\mathbf{rr}],
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\mathbf{M}(P_{1\frac{1}{2}}D_{1\frac{1}{2}})_3 &= -\mathbf{M}(D_{1\frac{1}{2}}P_{1\frac{1}{2}})_3 \\
&= (p_np_e/6\hbar^2)\int[\boldsymbol{\sigma}\mathbf{rr}].
\end{aligned}
\tag{26}
$$

IV. EFFECT OF THE COULOMB FIELD

The Coulomb field of the final nucleus has an important effect on the beta-spectrum of most forbidden transitions. In fact, it is only for the very light nuclei and (or) rather large electron energies that the effect of the nuclear charge can be neglected. This has already been pointed out by Konopinski and Uhlenbeck.[3] The effect may be described as follows: In the absence of a field, electron waves may be classified as S

waves, P waves, etc., according to the transformation properties of the "large components," the radial dependence of these waves being *one*, r, etc., (r^L in general) near $r=0$. In the presence of the field, the angular dependence remains unchanged, but the radial dependence acquires two new properties. The first is that some of the solutions (total angular momentum $\frac{1}{2}\hbar$) are not bounded at $r=0$. The integral of the square of the wave function is bounded, however, and in calculating transition probabilities for beta-activities the value of the electron wave at the surface of the nucleus is used. This singularity is not important to the shapes of the beta-spectra.

The second, and more important, effect of the electric field is to lower by unity the power of r with which a wave approaches $r=0$. Neglecting the small effect mentioned above, a $P_{\frac{1}{2}}$ wave approaches $r=0$ for the most part as r itself, but there is a small admixture of a wave (with the same angular dependence) which approaches $r=0$ as r^0, i.e., as a finite constant. The coefficient of this S part of the $P_{\frac{1}{2}}$ wave is proportional to $Z/137$ and is usually very small, but the more favorable radial dependence of this part near the origin makes it of importance compared with the P part itself. The physical interpretation of the electron state is that there is a finite probability of finding an electron extremely close to the center of attraction in spite of the fact that the electron has one unit of angular momentum. In the relativistic theory of the electron the increase in kinetic energy that arises upon concentrating the electron within a radius r of the center is proportional to $c\hbar/r$. On the other hand, the decrease in potential energy in the Coulomb field is also proportional to $1/r$ so that it is possible to have the electron very close to the nucleus although it has angular momentum.

So far as we carry the analysis of forbidden spectra in this paper it is sufficient to consider the amount of S part in the P waves and of the P part in D waves. There are P waves in both $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ eigenfunctions, however, and the coefficient of the S part will be different in the two. In order to determine the coefficients with which these anomalous parts of the wave appear the work of Dirac[11] on the radial functions is used. Neglecting quantities of the order $(Z/137)^2$ compared with unity we find the results are adequately expressed as modifications of the factors a and b in Eqs. (19) and (22) as already indicated by the use of primes. All other matrix elements are unaffected. The values of the primed letters are:

$$\begin{aligned} a'/a &= 1-3Ze^2/2(E+mc^2)\rho, \\ b'/b &= 1-3Ze^2/2(E-mc^2)\rho, \\ a''/a &= 1-5Ze^2/4(E+mc^2)\rho, \\ b''/b &= 1-5Ze^2/4(E-mc^2)\rho, \end{aligned} \tag{27}$$

where ρ is the radius of the nucleus at which the wave functions are evaluated.

V. SHAPES OF SPECTRA

The contributions of the individual matrix elements found above may be changed in different ways by further, non-Coulombic interaction between electron and nucleus. In this section the electric field only will be taken into account. It will be further assumed that the nuclear wave functions are eigenfunctions of the time reversal operator[12] so that the operators in Eq. (11) that change sign under time reversal, $\boldsymbol{\sigma}$ and ρ_2, lead to different states of the final nucleus than the operators that do not change sign, ρ_1 and ρ_3. This assumption eliminates most of the cross products of different operators in the squares of matrix elements.

The probability of allowed transitions is calculated by adding the absolute squares of all matrix elements leading to such transitions. The positions of the nuclear particles do not matter to those matrix elements, Eqs. (17) and (18), and no change in parity takes place in the transition. Such transitions may thus be called *monopole* radiations. From the values of a and b as given in Eq. (15), the sum of squares of matrix elements, Eq. (17), that apply if initial and final nuclei have spin zero is

$$\sum|M|^2=\tfrac{1}{4}|\textstyle\int\rho_3|^2 \quad 0\rightarrow 0. \tag{28}$$

Similarly, if the nuclear spin changes by one unit the appropriate sum of squares is

$$\sum|\mathbf{M}|^2=\tfrac{1}{4}|\textstyle\int\boldsymbol{\sigma}|^2 \quad \Delta i=\pm 1. \tag{29}$$

[11] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford, 1935), p. 265.

[12] E. Wigner, *Göttingen Nachrichten* (1932), p. 546.

If the transition involves no change in nuclear spin and the spin is not zero, the sum of (28) and (29) is to be used:

$$\sum|\mathbf{M}|^2=\tfrac{1}{4}|\int\rho_3|^2+\tfrac{1}{4}|\int\boldsymbol{\sigma}|^2 \quad \Delta i=0 \ (i\neq 0). \quad (30)$$

Since Eq. (30) represents a larger factor in the probability of decay than Eq. (29), transitions without change of spin will be more prominent than under the axial vector or tensor theory alone and this is desirable in Grönblom's analysis mentioned above.[7] Equation (13) represents the normal Fermi spectrum of electrons if M^2 is constant so that allowed transitions in the antisymmetric theory give normal spectra.

First forbidden transitions are conveniently divided into three classes: monopole, dipole, and mixed. The monopole radiations differ from the allowed emissions in that a change of parity is entailed, and for this reason the first forbidden monopole transitions are called *pseudomonopole.* The classification of beta-radiations as various multipole radiations is somewhat more convenient than the broader: allowed, first, and second forbidden classification. This is especially true for light nuclei because the matrix elements characteristic of first forbidden transitions, say, may vary in magnitude over a much wider range than the difference between allowed and the strongest first forbidden.[13]

Pseudomonopole matrix elements are proportional to ρ_1 and ρ_2, i.e., to the small components of the nuclear waves. The two operators ρ_1 and ρ_2 appear in Eq. (19) and, according to the assumption stated above, do not mix. The sum of squares of M applicable to pseudomonopole emissions is then

$$\sum|M|^2=\tfrac{1}{4}|\int\rho_1|^2+\tfrac{1}{4}|\int\rho_2|^2 \quad \Delta i=0. \quad (31)$$

The spectrum is normal.

There are four distinct matrix elements that lead to dipole transitions. The sum of squares for each of these will be presented along with the allowed changes in nuclear spin. An indicated change of spin that is primed, as the 0′ in $\Delta i=0', 1$, means that the sum of spins of initial and final nucleus must be as large as the unprimed number in the same Δi.

$$\sum|M(\sigma, r)|^2=(1/36h^2)\times[(p_n+cp_e^2/E-3Ze^2/2c\rho)^2+m^2c^4p_e^2/E^2]|\int(\sigma, r)|^2, \quad \Delta i=0 \quad (32)$$

$$\sum|\mathbf{M}(\rho_3\mathbf{r})|^2=(1/36h^2)\times[(p_n-cp_e^2/E+3Ze^2/2c\rho)^2+2(p_n^2+p_e^2)+m^2c^4p_e^2/E^2]\times|\int\rho_3\mathbf{r}|^2, \quad \Delta i=0', 1 \quad (33)$$

$$\sum|\mathbf{M}(\boldsymbol{\sigma}\times\mathbf{r})|^2=(1/36h^2)\times[(p_n-cp_e^2/E+3Ze^2/2c\rho)^2+\tfrac{1}{2}(p_n^2+p_e^2)+m^2c^4p_e^2/E^2]\times|\int(\sigma\times r)|^2, \quad \Delta i=0', 1 \quad (34)$$

$$\sum|\mathbf{M}(\boldsymbol{\sigma}\mathbf{r})|^2=(1/12h^2)(p_n^2+p_e^2)|\int[\boldsymbol{\sigma}\mathbf{r}]|^2 \quad \Delta i=0', 1', 2. \quad (35)$$

Ordinarily the term containing Z^2 will far outweigh other terms in a given expression and in every one except Eq. (35) the spectrum will be quite the same as the normal one. Deviations from the normal spectrum should be apparent in the high energy transitions in light elements if $\Delta i<2$ or in any case if $\Delta i=2$.

The one mixed transition, between ρ_2 and (σ, r) leads to

$$\sum|M(\rho_2; (\sigma, r))|^2=(1/6h)(p_n+cp_e^2/E-3Ze^2/2c\rho)|\int\rho_2|\times|\int(\sigma, r)| \quad \Delta i=0. \quad (36)$$

Again the Z term should predominate and the spectrum be normal. All first forbidden transitions cause a change in the parity of the nucleus.

So-called second forbidden transitions can be of two types, pseudodipole and quadrupole. There are no pseudodipole transitions in the antisymmetric theory, however, because the only possible matrix elements of this type, $\rho_1\mathbf{r}$ and $\rho_2\mathbf{r}$, do not permit $\Delta i=2$; and $\Delta i<2$, without change of parity, is allowed in monopole transitions. Thus all transitions are quadrupole, $\Delta i=2$ or 3, in the second forbidden category. There are three distinct matrix elements and no cross

[13] Discussions of and formulas for the half-lives of beta-activities have been presented in three independent papers appearing in *The Physical Review* **61** (1942): C. L. Critchfield, p. 249 (light nuclei only); R. E. Marshak, p. 431, and E. Greuling, p. 568.

products under the above assumptions:

$$\sum|\mathbf{M}(\rho_3\mathbf{rr})|^2=(1/360\hbar^4)[3p_e^4-4cp_np_e^4/E+10p_e^2p_n^2-4cp_n^3p_e^2/E+3p_n^4+(Ze^2/c\rho)(6p_n^2-10cp_n^2p_e^2/E+5p_e^2p_n-3p_e^4/E)+(15/8)(Ze^2/c\rho)^2\times(4p_n^2+p_e^2)]|\textstyle\int\rho_3\mathbf{rr}|^2\quad\Delta i=2,\qquad(37)$$

$$\sum|\mathbf{M}(\boldsymbol{\sigma}\mathbf{rr})|^2=(1/360\hbar^4)(3p_e^4+10p_e^2p_n^2+3p_n^4)|\textstyle\int\boldsymbol{\sigma}\mathbf{rr}|^2\quad\Delta i=2',\,3,\qquad(38)$$

$$\sum|\mathbf{M}(\mathbf{r}\boldsymbol{\sigma}\times\mathbf{r})|^2=\{\sum|M(\rho_3\mathbf{rr})|^2/|\textstyle\int\rho_3\mathbf{rr}|^2-(1/3)\sum|M(\boldsymbol{\sigma}\mathbf{rr})|^2/|\int\sigma\mathbf{rr}|^2\}\times|\int(\mathbf{r}\boldsymbol{\sigma}\times\mathbf{r})|^2\quad\Delta i=2.\qquad(39)$$

If Z is fairly large or the energy particularly small, the normalization factor for waves in a Coulomb field should be included in the electron waves. An approximate form of the squares of the normalization factor is

$$f(\zeta)=\zeta/(e^\zeta-1),\quad\zeta\equiv2\pi Ze^2E/\hbar cp_e,\qquad(40)$$

in which ζ is positive for positron emission and negative for electron emission. The approximation involved in Eq. (40) as well as throughout the calculations on the Coulomb effect is that in which quantities of the order $(Z/137)^2$ are neglected in comparison with unity. Our results thus apply only to light nuclei. The final form of $P(E)$ may be written

$$P(E)=(g^2/2\pi^3c^5\hbar^7)f(\zeta)Ep_e(w-E)^2\sum|\mathbf{M}|^2,\qquad(41)$$

in which the sum is taken over all elements that contribute to the transition. The integral of $P(E)dE$ over all electron energies then gives the decay constant. These integrals have been presented in the papers quoted in reference 13.

The foregoing calculations have been carried through for positron emission. In order to apply them to electron emission, it is simply necessary to change the sign of Z throughout.

VI. CONCLUSION

Full application of the results of this paper would require many experimentally determined beta-spectra of forbidden transitions. Furthermore, the spectra must be free of distortion such as introduced by thick sources, etc. The spectra of P^{32} and Na^{24} as determined by Lawson[14] (cf. reference 3) are probably in the desired category. Both transitions are slower than allowed transitions, Na^{24} by a factor 40 and P^{32} by a factor 2×10^4. Assignment of a definite multipole to these transitions is complicated by the fact that we do not know the magnitudes of the matrix elements and also by the intermultiplet character of the transitions (cf. Wigner, reference 9). According to the estimates made of matrix elements in the first reference 13, Na^{24} could be a pseudo-monopole transition that depends upon ρ_2. The Kurie plot predicted for such a transition would be a straight line, and in fact, the high energy end of the experimental curve does appear to be quite straight. The low energy end is probably complicated by another transition.

The high energy end of the Kurie plot for P^{32} is also straight but its lifetime is too long to ascribe the decay to the matrix element $M(\rho_2)$. In addition, the spectrum is probably simple so that the deviation at the low energy end cannot be explained as the influence of a second spectrum. The lifetime of P^{32} suggests the matrix element $M(\rho_1)$ according to the estimates in the first reference 13, but in order to obtain the observed spectrum, it would be necessary for the matrix element to become appreciably dependent upon the energy of the electron when this is low. This or any other elaborate explanation is to be avoided at this stage of the theory of forbidden spectra.

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[14] J. L. Lawson, Phys. Rev. **56**, 131 (1939).