PHYSICAL REVIEW VOLUME 63, NUMBERS 11 AND 12 JUNE 1 AND 15, 1943

On the Theory of a Mixed Pseudoscalar and a Vector Meson Field

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(Received March 24, 1943)

Schwinger's modification of the Møller-Rosenfeld theory of nuclear forces based on a mixture of pseudoscalar and vector meson field is investigated in the strong coupling theory. The isobar separation turns out to be $\frac{1}{3}$ of its value in the pseudoscalar theory. The interaction energy between two nucleons is calculated, and in the most interesting case of the symmetrical theory, it is the same as the corresponding expression in the weak coupling theory with the spin and isotopic spin vectors replaced by the $\mathbf{e}^\alpha$ vectors of Pauli and Dancoff. A classification of the states of the two-nucleon system is made, and the lowest state for the deuteron is found to be the triplet state. An estimate is made for the values of the constants involved, and we find that a suitable choice of the coupling constant and the masses of the mesons can be made in such a way that the binding energy and the quadrupole moment of the deuteron agree with the observed values, and such that the conditions for small source (compared to the Compton wavelengths of the mesons) and for small effect of the higher spin states on the ground state are fulfilled. However the theory gives for the magnetic moment of the deuteron a value only a few percent of the observed value, and according to this theory highly charged nuclei would be unstable. We therefore conclude that the strong coupling theory, based on the assumption of an extended source, should be abandoned in favor of a weak coupling theory, based on a point source with the singularities of its field eliminated by means of a subtraction formalism.

I. INTRODUCTION

THE usual theories of nuclear forces derived from one type of meson field give rise, both in the weak and strong coupling approximations, to tensor forces with inadmissible r^{-3} singularity for small r, which consequently has to be cut off at a certain arbitrary radius. Møller and Rosenfeld[1] showed that the singularity can be removed by taking a mixture of pseudoscalar and vector meson fields with the same coupling constant. They, however, also took the masses of the two types of meson equal, and found that the tensor force vanished in the first approximation. Schwinger[2] then pointed out that the tensor force can be retained with only an admissible r^{-1} dependence at small distances if the masses are taken unequal, and that the right sign for the quadrupole moment is obtained if the vector meson has the larger mass. This is in agreement with the hypothesis that the vector meson is highly unstable and that it is responsible for the β-decay in the nucleus.

It may be argued that the introduction of the vector meson field is just another way of removing the inadmissible singularities by an arbitrary constant, and in a much more complicated way at that. The introduction of a new particle is admittedly undesirable, but we feel that further investigation in this direction is justified for the reason that this may be a first step in the direction of a more far-reaching theory in which the vector meson enters, not as a new independent particle, but as an excited state of the pseudoscalar meson.

We have investigated in this paper the mixed meson theory in the strong coupling approximation. The calculations of Section II for the isobar separation follows closely the development for the pseudoscalar meson given by Pauli and Dancoff.[3] The theory of nuclear forces in the strong coupling approximation has been developed by Serber and Dancoff[4] who treated the charged scalar and the neutral pseudoscalar meson fields, and the calculation in Section III follows their procedure. Hence the computations in these sections are not given in detail. However we wish to point out that in contrast to Serber and Dancoff, we do not consider the nucleons to be at rest, and consequently we do not discuss here the small oscillations around the minimum of the potential energy which occurs for sufficiently small separation of the nucleons. The

[1] C. Møller and L. Rosenfeld, K. Danske vidensk. Selsk. **17** (1940).
[2] J. Schwinger, Phys. Rev. **61**, 387A (1942).
[3] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942). This paper will henceforth be quoted as P-D.
[4] R. Serber and S. M. Dancoff, Phys. Rev. **63**, 143 (1943). This paper will henceforth be quoted as S-D. We wish to thank these authors for letting us see their calculations before publication.

results of these sections are applied in Sections IV and V to the deuteron and the heavy nuclei, respectively.

II. ISOBAR SEPARATION

We describe the pseudoscalar part by the real quantities $\varphi^\alpha(x)$ and the vector part by $\psi_i^\alpha(x)$, where the subscript i denotes the components in the ordinary space and takes the values 1, 2, 3. The superscript α in both cases denotes the components in the isotopic spin space. In the symmetrical theory it takes the three values 1, 2, 3; in the charged theory the values 1, 2; and in the neutral theory only the value 3. In the following we do not restrict the values which α may assume so that the results for these three types of theories may be obtained by making suitable restrictions on the values for α. However, unless otherwise stated, we restrict our attention to the most interesting case of the symmetrical theory. The Hamiltonian of the field and its interaction with a nucleon is given by

$$H=\tfrac{1}{2}\sum_\alpha \int \{(\pi^\alpha)^2+(\nabla\varphi^\alpha)^2+\kappa^2(\varphi^\alpha)^2\}dV$$
$$+\tfrac{1}{2}\sum_{\alpha,i}\int\{(\omega_i^\alpha)^2+(\nabla\psi_i^\alpha)^2+\mu^2(\psi_i^\alpha)^2\}dV$$
$$-(4\pi)^{\frac{1}{2}}\sum_{\alpha,i,j}\int\left\{f\frac{\partial\varphi^\alpha}{\partial x_i}\sigma_i+\frac{g}{2}\left(\frac{\partial\psi_j^\alpha}{\partial x_i}-\frac{\partial\psi_i^\alpha}{\partial x_j}\right)\sigma_{ij}\right\}\tau^\alpha U(x)dV, \quad (1)$$

where π^α and ω_i^α are the momenta conjugate to φ^α and ψ_i^α, respectively; κ and μ are the rest masses of the pseudoscalar and the vector meson, respectively (we are using natural units where $\hbar=c=1$); σ_i and τ^α are the spin and isotopic spin matrices, respectively, with $\sigma_{ij}=-\sigma_{ji}=\sigma_k$ if i, j, k are cyclic permutations of 1, 2, 3; and f and g are coupling constants for the pseudoscalar and vector field, respectively.[5] For the time being we do not make the assumption $f=g$. $U(x)$ is the source function of the nucleon which we assume to be spherically symmetrical and normalized according to

$$\int U(x)dV=1. \quad (2)$$

It determines a radius a of the nucleon

$$\frac{1}{a}=\int\int\frac{U(x)U(x')}{R}dVdV' \quad (3)$$

where $R=|\mathbf{x}-\mathbf{x}'|$.

As in P-D, we split the fields into two parts and write

$$\overset{0\alpha}{\varphi_i}=-(4\pi)^{\frac{1}{2}}\int\frac{\partial\varphi^\alpha}{\partial x_i}U(x)dV=(4\pi)^{\frac{1}{2}}\int\varphi^\alpha\frac{\partial U}{\partial x_i}dV,$$
$$\overset{0\alpha}{\psi_{ij}}=(2\pi)^{\frac{1}{2}}\int\left\{\frac{\partial\psi_j^\alpha}{\partial x_i}-\frac{\partial\psi_i^\alpha}{\partial x_j}\right\}U(x)dV$$
$$=(2\pi)^{\frac{1}{2}}\int\left\{\psi_i^\alpha\frac{\partial U}{\partial x_j}-\psi_j^\alpha\frac{\partial U}{\partial x_i}\right\}dV, \quad (4)$$

$$\int\varphi'^\alpha\frac{\partial U}{\partial x_i}dV=0,$$
$$\int\left\{\psi_i'^\alpha\frac{\partial U}{\partial x_j}-\psi_j'^\alpha\frac{\partial U}{\partial x_i}\right\}dV=0. \quad (5)$$

$\overset{0\alpha}{\psi_{ij}}$ is antisymmetric in i and j, and we shall sometimes write $\overset{0\alpha}{\psi_{ij}}\equiv\overset{0\alpha}{\psi_k}$ for i, j, k cyclic. The source function generates the potentials $X(x)$ and $Y(x)$ according to

$$(-\Delta+\kappa^2)X=4\pi U,\quad (-\Delta+\mu^2)Y=4\pi U; \quad (6)$$

so that

$$X(x)=\int U(x')(e^{-\kappa R}/R)dV',$$
$$Y(x)=\int U(x')(e^{-\mu R}/R)dV', \quad (7)$$

with $R=|\mathbf{x}-\mathbf{x}'|$. We define

$$I\delta_{ij}=\int\frac{\partial X}{\partial x_i}\frac{\partial U}{\partial x_j}dV, \quad (8)$$

and if we assume both κa and $\mu a\ll 1$, we have

$$\int\frac{\partial X}{\partial x_i}\frac{\partial U}{\partial x_i}dV\backsimeq\int\frac{dY}{\partial x_i}\frac{\partial U}{\partial x_i}dV. \quad (9)$$

[5] In this paper we find it more convenient to use these constants which have the dimension of κ^{-1}. Thus our f is equal to $g/\sqrt{2}\kappa$ where g is the coupling constant for the pseudoscalar field used in P-D.

This approximation gives some formal, although unessential, simplification and we shall make use of it. We therefore write

$$\xi(x)=X(x)/I, \quad \eta(x)=Y(x)/I; \tag{10}$$

then

$$\int \frac{\partial\xi}{\partial x_i}\frac{\partial U}{\partial x_j}dV=\int \frac{\partial\eta}{\partial x_i}\frac{\partial U}{\partial x_j}=\delta_{ij}. \tag{11}$$

We thus obtain from (4) and (5)

$$\begin{aligned}\varphi^\alpha(x)&=(4\pi)^{-\frac{1}{2}}\sum_i \varphi_i^{0\alpha}\frac{\partial\xi(x)}{\partial x_i}+\varphi'^\alpha(x),\\ \psi_i^\alpha(x)&=(8\pi)^{-\frac{1}{2}}\sum_j \overset{0\alpha}{\psi_{ij}}\frac{\partial\eta(x)}{\partial x_j}+\psi_i'^\alpha(x).\end{aligned} \tag{12}$$

The corresponding decompositions of the momenta are given by

$$\begin{aligned}\pi_i^{0\alpha}&=(4\pi)^{-\frac{1}{2}}\int \pi^\alpha\frac{\partial\xi}{\partial x_i}dV,\\ \overset{0\alpha}{\omega_{ij}}&=(8\pi)^{-\frac{1}{2}}\int\left\{\omega_i^\alpha\frac{\partial\eta}{\partial x_j}-\omega_j^\alpha\frac{\partial\eta}{\partial x_i}\right\}dV,\end{aligned} \tag{13}$$

with $\overset{0\alpha}{\omega_{ij}}\equiv-\overset{0\alpha}{\omega_{ji}}\equiv\overset{0\alpha}{\omega_k}$ (i, j, k cyclic);

$$\begin{aligned}&\int \pi'^\alpha\frac{\partial\xi}{\partial x_i}dV=0,\\ &\int\left\{\omega_i'^\alpha\frac{\partial\eta}{\partial x_j}-\omega_j'^\alpha\frac{\partial\eta}{\partial x_i}\right\}dV=0;\end{aligned} \tag{14}$$

$$N\delta_{ij}=4\pi\int \frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j}dV; \tag{15}$$

$$\begin{aligned}\pi^\alpha(x)&=(4\pi)^{\frac{1}{2}}\sum_i \pi_i^{0\alpha}\frac{\partial U(x)}{\partial x_i}+\pi'^\alpha(x),\\ \omega_i^\alpha(x)&=(2\pi)^{\frac{1}{2}}\sum_j \overset{0\alpha}{\omega_{ij}}\frac{\partial U(x)}{\partial x_j}+\omega_i'^\alpha(x).\end{aligned} \tag{16}$$

The commutation relations are

$$\begin{aligned}i[\pi_i^{0\alpha},\ \varphi_j^{0\beta}]&=\delta_{\alpha\beta}\delta_{ij},\\ i[\overset{0\alpha}{\omega_{ij}},\ \overset{0\beta}{\psi_{kl}}]&=\delta_{\alpha\beta}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}).\end{aligned} \tag{17}$$

The Hamiltonian (1) becomes

$$\begin{aligned}H=&\frac{N}{2}\sum_{\alpha,i}\{(\pi_i^{0\alpha})^2+(\omega_i^{0\alpha})^2\}\\ &+\frac{1}{2I}\sum_{\alpha,i}\{(\varphi_i^{0\alpha})^2+(\psi_i^{0\alpha})^2\}\\ &+\sum_{\alpha,i}\{f\varphi_i^{0\alpha}+\sqrt{2}g\psi_i^{0\alpha}\}\sigma_i\tau^\alpha\\ &+(4\pi)^{\frac{1}{2}}\sum_{\alpha,i,j}\left\{\pi_i^{0\alpha}\int\pi'^\alpha\frac{\partial U}{\partial x_i}dV\right.\\ &\left.+\frac{\overset{0\alpha}{\omega_{ij}}}{\sqrt{2}}\int\omega_i'^\alpha\frac{\partial U}{\partial x_j}dV\right\}\\ &+\tfrac{1}{2}\sum_{\alpha,i}\int\{(\pi'^\alpha)^2+(\omega_i'^\alpha)^2\}dV\\ &+\tfrac{1}{2}\sum_{\alpha,i}\int\{\varphi'^\alpha(-\Delta+\kappa^2)\varphi'^\alpha\\ &\qquad+\psi_i'^\alpha(-\Delta+\mu^2)\psi_i'^\alpha\}dV,\end{aligned} \tag{18}$$

and the potential energy in the zero state E^0 is given by

$$E^0=\sum_{\alpha,i}\left[\frac{1}{2I}\{(\varphi_i^{0\alpha})^2+(\psi_i^{0\alpha})^2\}+\{f\varphi_i^{0\alpha}+\sqrt{2}g\psi_i^{0\alpha}\}\sigma_i\tau^\alpha\right]. \tag{19}$$

The total angular momentum of the field and the nucleon is

$$\begin{aligned}L_{ij}=&-\sum_\alpha\int\left\{\pi^\alpha\left(x_i\frac{\partial\varphi^\alpha}{\partial x_j}-x_j\frac{\partial\varphi^\alpha}{\partial x_i}\right)\right\}dV\\ &-\sum_{\alpha,k}\int\left\{\omega_k^\alpha\left(x_i\frac{\partial\psi_k^\alpha}{\partial x_j}-x_j\frac{\partial\psi_k^\alpha}{\partial x_i}\right)\right\}dV\\ &+\sum_\alpha\int\{\psi_i^\alpha\omega_j^\alpha-\psi_j^\alpha\omega_i^\alpha\}dV+\tfrac{1}{2}\sigma_{ij};\end{aligned} \tag{20}$$

and the part L_{ij}^0 due to the zero state of the field is

$$L_{ij}^0=\sum_\alpha(\varphi_i^{0\alpha}\pi_j^{0\alpha}-\varphi_j^{0\alpha}\pi_i^{0\alpha})+\sum_{\alpha,k}(\overset{0\alpha}{\psi_{ik}}\overset{0\alpha}{\omega_{jk}}-\overset{0\alpha}{\psi_{jk}}\overset{0\alpha}{\omega_{ik}}),$$

or

$$L_{ij}^0=\sum_\alpha\{\varphi_i^{0\alpha}\pi_j^{0\alpha}-\varphi_j^{0\alpha}\pi_i^{0\alpha}+\psi_i^{0\alpha}\omega_j^{0\alpha}-\psi_j^{0\alpha}\omega_i^{0\alpha}\}. \tag{21}$$

Now E^0 takes its minimum value when

$$\varphi_i^{0\alpha}=fIe_i^\alpha, \quad \psi_i^{0\alpha}=\sqrt{2}gIe_i^\alpha; \tag{22}$$

so that it is convenient to introduce new variables defined by the orthogonal transformation

$$\begin{aligned}\Phi_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{f\varphi_i^{0\alpha}+\sqrt{2}g\psi_i^{0\alpha}\},\\ \Psi_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{\sqrt{2}g\varphi_i^{0\alpha}-f\psi_i^{0\alpha}\};\end{aligned}\tag{23}$$

with the inverse transformation

$$\begin{aligned}\varphi_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{f\Phi_i^{0\alpha}+\sqrt{2}g\Psi_i^{0\alpha}\},\\ \psi_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{\sqrt{2}g\Phi_i^{0\alpha}-f\Psi_i^{0\alpha}\}.\end{aligned}\tag{24}$$

The corresponding transformation equations for the momenta are

$$\begin{aligned}\Pi_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{f\pi_i^{0\alpha}+\sqrt{2}g\omega_i^{0\alpha}\},\\ \Omega_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{\sqrt{2}g\pi_i^{0\alpha}-f\omega_i^{0\alpha}\};\end{aligned}\tag{25}$$

with the inverse

$$\begin{aligned}\pi_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{f\Pi_i^{0\alpha}+\sqrt{2}g\Omega_i^{0\alpha}\},\\ \omega_i^{0\alpha}&=(f^2+2g^2)^{-\frac{1}{2}}\{\sqrt{2}g\Pi_i^{0\alpha}-f\Omega_i^{0\alpha}\}.\end{aligned}\tag{26}$$

Then we have

$$\overset{0}{L}_{ij}=\overset{00}{L}_{ij}+\overset{01}{L}_{ij},\tag{27}$$

where

$$\overset{00}{L}_{ij}=\textstyle\sum_\alpha\{\Phi_i^{0\alpha}\Pi_j^{0\alpha}-\Phi_j^{0\alpha}\Pi_i^{0\alpha}\},\tag{28}$$

$$\overset{01}{L}_{ij}=\textstyle\sum_\alpha\{\Psi_i^{0\alpha}\Omega_j^{0\alpha}-\Psi_j^{0\alpha}\Omega_i^{0\alpha}\}.\tag{29}$$

As in P-D, we write

$$\Phi_i^{0\alpha}=De_i^\alpha+\textstyle\sum_\beta q^{\alpha\beta}e_i^\beta,\tag{30}$$

where $q^{\alpha\beta}\equiv q^{\beta\alpha}$, e_i^α are the components of an orthogonal matrix introduced in Eqs. (51), (51a) of P-D and can be considered as components of an orthogonal system of three unit vectors, and according to (22)

$$D=(f^2+2g^2)^{\frac{1}{2}}I;\tag{31}$$

and

$$\Pi_i^{0\alpha}=\textstyle\sum_\beta\{p^{\alpha\beta}+(1/2D)L^{00\alpha\beta}\}e_i^\beta,\tag{32}$$

where $p^{\alpha\beta}\equiv p^{\beta\alpha}$ and

$$L^{00\alpha\beta}=\sum_{i,j}e_i^\alpha e_j^\beta\overset{00}{L}_{ij}=-D\sum_i\{e_i^\alpha\Pi_i^{0\beta}-e_i^\beta\Pi_i^{0\alpha}\}.\tag{33}$$

In (32) and (33) terms of higher order in $q^{\alpha\beta}/D$ have been neglected [cf. P-D, Eq. (72)]. $p^{\alpha\beta}$ and $q^{\alpha\beta}$ satisfy the commutation relation

$$i[p^{\alpha\beta},q^{\gamma\delta}]=\tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}),\tag{34}$$

[cf. P-D, Eq. (57)]. If we insert these new variables in (18) and perform the S-transformation, we obtain for the Hamiltonian

$$\begin{aligned}H=\frac{N}{4D^2}\mathbf{s}^2-\tfrac{3}{2}(f^2+2g^2)I&+\frac{\pi^{\frac{1}{2}}f}{D(f^2+2g^2)^{\frac{1}{2}}}\sum_{\alpha,\beta,i}L^{00\alpha\beta}e_i^\beta\int\frac{\partial U}{\partial x_i}\pi'^\alpha dV\\&+\frac{\pi^{\frac{1}{2}}g}{D(f^2+2g^2)^{\frac{1}{2}}}\sum_{\alpha,\beta,i}L^{00\alpha\beta}\int[\nabla U\times\mathbf{e}^\beta]_i\omega_i'^\alpha dV\\&+\tfrac{1}{2}\sum_{\alpha,i}\int\{(\pi_i'^\alpha)^2+(\omega_i'^\alpha)^2\}dV+\frac{1}{2I^2}\sum_{\alpha,\beta}(q^{\alpha\beta})^2,\end{aligned}\tag{35}$$

where

$$\mathbf{s}^2=\tfrac{1}{2}\sum_{\alpha,\beta}(L^{00\alpha\beta})^2.$$[6]

The terms which do not influence the isobar separation have been omitted; in particular the terms due to $\Psi_i^{0\alpha}$ and $\Omega_i^{0\alpha}$ which describe free mesons. By shifting the origin of π'^α and $\omega_i'^\alpha$ according to

$$\pi'^\alpha=\pi''^\alpha-\frac{\pi^{\frac{1}{2}}f}{D(f^2+2g^2)^{\frac{1}{2}}}\sum_{\beta,i}L^{00\alpha\beta}e_i^\beta\times\left\{\frac{\partial U}{\partial x_i}-\frac{3\partial\xi/\partial x_i}{\int(\nabla\xi)^2dV}\right\},\tag{36}$$

$$\omega_i'^\alpha=\omega_i''^\alpha+\frac{\pi^{\frac{1}{2}}g}{D(f^2+2g^2)^{\frac{1}{2}}}\sum_\beta L^{00\alpha\beta}\times\left[\mathbf{e}^\beta\times\left(\nabla U-\frac{3\nabla\xi}{\int(\nabla\xi)^2dV}\right)\right]_i,\tag{36}$$

we can get rid of the terms linear in π'^α and $\omega_i'^\alpha$ and of the order $1/D$, and still retain the orthogonality relations (14) [cf. P-D, Eq. (75)].

The final result for the isobar energy is

$$\frac{3}{4}\frac{\mathbf{s}^2}{(f^2+2g^2)}\frac{4\pi}{\int(\nabla X)^2dV},$$

and the isobar separation ΔE is given by

$$\Delta E=\frac{3}{4}\frac{\{s(s+1)-\frac{3}{4}\}}{(f^2+2g^2)}\frac{4\pi}{\int(\nabla X)^2dV}.\tag{37}$$

Now for a small source (κa and μa both $\ll1$)

$$\frac{4\pi}{\int(\nabla X)^2dV}\cong\frac{1}{a}$$

[6] We have **s** instead of **L** as in P-D since we want to use the usual spectroscopic notation in the discussion of the two-nucleon system in Section IV.

[cf. P-D, Eq. (80a)], hence

$$\Delta E=\frac{3a}{4(f^2+2g^2)}\{s(s+1)-\tfrac{3}{4}\}. \qquad (38)$$

We obtain the result (80a) in P-D for pseudoscalar meson alone if we put $g=0$ and $f=g/\sqrt{2}\kappa$. For the vector theory alone, we put $f=0$ and find that in this case ΔE is 1/2 of the value in the pseudoscalar theory (for the same coupling constant). For the Møller-Rosenfeld mixture, we put $f=g$ and find ΔE in this case is 1/3 of the value in the pseudoscalar theory.

III. NUCLEAR FORCES

A. Pseudoscalar Theory

We make the simplifying assumptions that the source size a is small compared to both κ^{-1} and the distance $r=|\mathbf{x}_{\text{I}}-\mathbf{x}_{\text{II}}|$ between the two nucleons. Let $U_{\text{I}}(x)$ and $U_{\text{II}}(x)$ be the source functions for particles I and II, respectively, and normalize them according to

$$\int U_A(x)dV=1 \text{ for } A=\text{I, II}. \qquad (39)$$

The potentials $X_{\text{I}}(x)$ and $X_{\text{II}}(x)$ which they generate according to

$$(-\Delta+\kappa^2)X_A=4\pi U_A \text{ for } A=\text{I, II}, \qquad (40)$$

are given by

$$X_A(x)=\int U_A(x')\frac{e^{-\kappa R}}{R}dV', \qquad (41)$$

where $R=|\mathbf{x}-\mathbf{x}'|$. Also let

$$I\delta_{ij}=\int\frac{\partial X_A}{\partial x_i}\frac{\partial U_A}{\partial x_j}dV \text{ for } A=\text{I, II}; \qquad (42)$$

$$J_{ij}=\int\frac{\partial X_A}{\partial x_i}\frac{\partial U_B}{\partial x_j}dV \text{ for } A,B=\text{I, II and } A\neq B$$
$$=-\int\int U_{\text{I}}(x)U_{\text{II}}(x')\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{e^{-\kappa R}}{R}\right)dVdV'. \qquad (43)$$

It is evident that $J_{ij}=J_{ji}$. Now I is of the order a^{-3}, and J_{ij} is for small r of the order r^{-3} so that in the following we can neglect all quantities of the second and higher orders in J_{ij}/I.[7] Moreover we can substitute for J_{ij} its value for point sources;[8] i.e.,

$$J_{ij}=-\frac{\partial^2}{\partial x_{\text{I}i}\partial x_{\text{I}j}}\left(\frac{e^{-\kappa r}}{r}\right) \qquad (44)$$

where $r=|\mathbf{x}_{\text{I}}-\mathbf{x}_{\text{II}}|$.

To obtain the interaction energy between two nucleons, it is sufficient to consider just the potential energy of the system E which is

$$E=\sum_\alpha\int\varphi^\alpha(-\Delta+\kappa^2)\varphi^\alpha dV-(4\pi)^{\frac{1}{2}}f\sum_{A,\alpha,i}\int\frac{\partial\varphi^\alpha}{\partial x_i}\sigma_{Ai}\tau_A{}^\alpha U_A(x)dV, \qquad (45)$$

where the subscript A is summed over I and II. As in S-D, Eq. (71), we split the field as follows:

$$\varphi^\alpha(x)=(4\pi)^{-\frac{1}{2}}\sum_{A,i}\overset{0\alpha}{\varphi}_{Ai}\xi_{Ai}(x)+\varphi'^\alpha(x), \qquad (46)$$

where φ'^α satisfies the condition

$$\int\varphi'^\alpha\frac{\partial U_A}{\partial x_i}dV=0 \text{ for } A=\text{I, II}, \qquad (47)$$

and where the functions ξ_{Ai} span the linear subspace of the functions $\partial X_A/\partial x_i$ in such a way that

$$\int\xi_{Ai}\frac{\partial U_B}{\partial x_j}dV=\delta_{AB}\delta_{ij}. \qquad (48)$$

From these relations we obtain

$$\overset{0\alpha}{\varphi}_{Ai}=(4\pi)^{\frac{1}{2}}\int\varphi^\alpha\frac{\partial U_A}{\partial x_i}dV. \qquad (49)$$

The above conditions are fulfilled up to terms of the order J_{ij}/I if we put

$$\xi_{Ai}=\frac{1}{I}\frac{\partial X_A}{\partial x_i}-\frac{1}{I^2}\sum_j J_{ij}\frac{\partial X_B}{\partial x_j} \quad \text{for } A,B=\text{I, II and } A\neq B. \qquad (50)$$

In terms of these variables, the potential energy (45) becomes

$$E=\sum_{A,\alpha,i,j}\left\{\frac{1}{2I}(\overset{0\alpha}{\varphi}_{Ai})^2-\frac{J_{ij}}{I^2}\overset{0\alpha}{\varphi}_{\text{I}i}\overset{0\alpha}{\varphi}_{\text{II}j}\right\}+f\sum_{A,\alpha,i}\overset{0\alpha}{\varphi}_{Ai}\sigma_{Ai}\tau_A{}^\alpha+\sum_\alpha\int\varphi'^\alpha(-\Delta+\kappa^2)\varphi'^\alpha dV. \qquad (51)$$

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[7] A similar situation holds for the integral $\int(\partial X_{\text{I}}/\partial x_i)\times(\partial X_{\text{II}}/\partial x_j)dV$ in comparison to $\int(\partial X_{\text{I}}/\partial x_i)(\partial X_{\text{I}}/\partial x_i)dV$ which are of the order of magnitude r^{-1} and a^{-1}, respectively. It is shown in S-D that, due to this circumstance, the neglect of the dependence of the isobar levels on r is justified as long as the latter integral is large compared to the former.

[8] The substitution of the point source is in reality not so simple and holds only for distances r considerably larger than a. This was pointed out by Oppenheimer and Serber.

The justification for the assumed decomposition of φ^α lies in the circumstance that no cross terms between $\overset{0\alpha}{\varphi}_{Ai}$ and φ'^α occur in (51). φ'^α describes free mesons and can be omitted for our purpose.

For the case of infinite separation of the two nucleons ($J_{ij}=0$), the lowest eigenvalue of the interaction energy is obtained analogously to the treatment of the one-source problem in P-D [cf. Eqs. (82), (64), (65), (67) there]. We put

$$\overset{0\alpha}{\varphi}_{Ai}=fI\overset{\alpha}{e}_{Ai} \tag{52}$$

where the $\overset{\alpha}{e}_{Ai}$ satisfy the conditions

$$\textstyle\sum_\alpha \overset{\alpha}{e}_{Ai}\overset{\alpha}{e}_{Aj}=\delta_{ij},\quad \sum_i \overset{\alpha}{e}_{Ai}\overset{\beta}{e}_{Ai}=\delta_{\alpha\beta},\qquad \text{for } A=\text{I, II}; \tag{53}$$

and by selecting the state where

$$\sum_{\alpha,i}\sigma_{Ai}\tau_A{}^\alpha\overset{\alpha}{e}_{Ai}=-3 \text{ for } A=\text{I, II}; \tag{54}$$

we obtain for $A=\text{I, II}$

$$\sigma_A{}'^\alpha\tau_A{}^\alpha=-1 \text{ for each value of } \alpha \tag{55}$$

with

$$\sigma_A{}'^\alpha=\textstyle\sum_i \overset{\alpha}{e}_{Ai}\sigma_{Ai} \tag{56}$$

and

$$\sigma_A{}'^\alpha\tau_A{}^\beta+\sigma_A{}'^\beta\tau_A{}^\alpha=0 \text{ for } \alpha\neq\beta. \tag{57}$$

To obtain the minimum value E^0 of E to the first order in J_{ij}/I in the absence of free mesons, we simply insert (52), (53), (54) in (51). The result is[9]

$$E^0=-3f^2I-f^2\sum_{\alpha,i,j}J_{ij}\overset{\alpha}{e}_{\text{I}i}\overset{\alpha}{e}_{\text{II}j}, \tag{58}$$

or omitting the first term which gives the self-energy, we have for the interaction energy

$$\overset{0}{E}_{\text{I II}}=-f^2\sum_{\alpha,i,j}J_{ij}\overset{\alpha}{e}_{\text{I}i}\overset{\alpha}{e}_{\text{II}j}. \tag{59}$$

Inserting for J_{ij} the expression (44), we obtain

$$\overset{0}{E}_{\text{I II}}=f^2\textstyle\sum_\alpha(\mathbf{e}_{\text{I}}{}^\alpha\cdot\nabla)(\mathbf{e}_{\text{II}}{}^\alpha\cdot\nabla)(e^{-\kappa r}/r), \tag{60}$$

and if the differentiation is carried out,

$$\overset{0}{E}_{\text{I II}}=\frac{f^2}{3}\left\{\Gamma\frac{\kappa^2}{r}e^{-\kappa r}+\Lambda\frac{1}{r^3}(3+3\kappa r+\kappa^2r^2)e^{-\kappa r}\right\} \tag{61}$$

[9] If we had taken a more general expression

$$\overset{0\alpha}{\varphi}_{Ai}=\textstyle\sum_\beta(fI\delta_{\alpha\beta}+q_A{}^{\alpha\beta})\overset{\alpha}{e}_{Ai}$$

instead of (52), we would have obtained in the expression for E^0 terms in $q_A{}^{\alpha\beta}$ which are of the relative order J_{ij}/I, and hence give corrections of the order $(J_{ij}/I)^2$ to E^0.

where

$$\Gamma=\textstyle\sum_\alpha(\mathbf{e}_{\text{I}}{}^\alpha\cdot\mathbf{e}_{\text{II}}{}^\alpha),$$

$$\Lambda=\sum_\alpha\left\{\frac{3}{r^2}(\mathbf{e}_{\text{I}}{}^\alpha\cdot\mathbf{x})(\mathbf{e}_{\text{II}}{}^\alpha\cdot\mathbf{x})-(\mathbf{e}_{\text{I}}{}^\alpha\cdot\mathbf{e}_{\text{II}}{}^\alpha)\right\} \tag{62}$$

with $\mathbf{x}=\mathbf{x}_{\text{I}}-\mathbf{x}_{\text{II}}$.

It follows from the method of derivation that our theory differs from the usual perturbation treatment in the weak coupling theory only in the fact that $\boldsymbol{\sigma}_A\tau_A{}^\alpha$ is replaced by $\mathbf{e}_A{}^\alpha$ ($A=\text{I, II}$).

As stated at the beginning of Section II, the symmetrical theory is obtained by letting α take the values 1, 2, 3; the charged theory for α taking the values 1, 2; and the neutral theory for α restricted to the single value 3.

B. Mixed Theory

As in Part A we consider two source functions normalized according to (39). They generate the potentials according to

$$\begin{rcases}(-\Delta+\kappa^2)X_A=4\pi U_A\\(-\Delta+\mu^2)Y_A=4\pi U_A\end{rcases}\text{ for } A=\text{I, II}; \tag{63}$$

and hence

$$X_A(x)=\int U_A(x')\frac{e^{-\kappa R}}{R}dV',\qquad Y_A(x)=\int U_A(x')\frac{e^{-\mu R}}{R}dV'\qquad\text{for } A=\text{I, II}. \tag{64}$$

We have

$$I\delta_{ij}=\int\frac{\partial X_A}{\partial x_i}\frac{\partial U_A}{\partial x_j}dV\simeq\int\frac{\partial Y_A}{\partial x_i}\frac{\partial U_A}{\partial x_j}dV\qquad\text{for } A=\text{I, II}; \tag{65}$$

$$J_{ij}=J_{ji}=\int\frac{\partial X_A}{\partial x_i}\frac{\partial U_B}{\partial x_j}dV\simeq-\frac{\partial^2}{\partial x_{\text{I}i}\partial x_{\text{I}j}}\left(\frac{e^{-\kappa r}}{r}\right)\qquad\text{for } A, B=\text{I, II and } A\neq B. \tag{66}$$

$$K_{ij}=K_{ji}=\int\frac{\partial Y_A}{\partial x_i}\frac{\partial U_B}{\partial x_j}dV\simeq-\frac{\partial^2}{\partial x_{\text{I}i}\partial x_{\text{I}j}}\left(\frac{e^{-\mu r}}{r}\right)$$

The potential energy of the system is given by

$$E=\sum_{\alpha,i}\int\{\varphi^\alpha(-\Delta+\kappa^2)\varphi^\alpha+\psi_i{}^\alpha(-\Delta+\mu^2)\psi_i{}^\alpha\}dV-(4\pi)^{\frac{1}{2}}f\sum_{A,\alpha,i,j}\int\left\{\frac{\partial\varphi^\alpha}{\partial x_i}\sigma_{Ai}+\frac{1}{2}\left(\frac{\partial\psi_j{}^\alpha}{\partial x_i}-\frac{\partial\psi_i{}^\alpha}{\partial x_j}\right)\sigma_{Aij}\right\}\tau_A{}^\alpha U_A dV, \tag{67}$$

where we have assumed the equality of the coupling constants.

As before, we split the field as follows:

$$\begin{aligned}\varphi^\alpha(x)&=(4\pi)^{-\frac{1}{2}}\sum_{A,i}\overset{0\alpha}{\varphi}_{Ai}\xi_{Ai}(x)+\varphi'^\alpha(x),\\ \psi_i^\alpha(x)&=(8\pi)^{-\frac{1}{2}}\sum_{A,j}\overset{0\alpha}{\psi}_{Aij}\eta_{Aj}(x)+\psi_i'^\alpha(x);\end{aligned}\tag{68}$$

with the orthogonality conditions

$$\left.\begin{aligned}&\int\varphi'^\alpha\frac{\partial U_A}{\partial x_i}dV=0\\ &\int\psi_i'^\alpha\frac{\partial U_A}{\partial x_j}dV=0\end{aligned}\right\}\quad\text{for } A=\mathrm{I},\ \mathrm{II};\tag{69}$$

and

$$\begin{aligned}&\int\xi_{Ai}\frac{\partial U_B}{\partial x_j}dV=\delta_{AB}\delta_{ij},\\ &\int\eta_{Ai}\frac{\partial U_B}{\partial x_j}dV=\delta_{AB}\delta_{ij}.\end{aligned}\tag{70}$$

Then

$$\left.\begin{aligned}\overset{0\alpha}{\varphi}_{Ai}&=(4\pi)^{\frac{1}{2}}\int\varphi^\alpha\frac{\partial U_A}{\partial x_i}dV\\ \overset{0\alpha}{\psi}_{Aij}&=(8\pi)^{\frac{1}{2}}\int\psi_i{}^\alpha\frac{\partial U_A}{\partial x_j}dV\end{aligned}\right\}\quad\text{for } A=\mathrm{I},\ \mathrm{II},\tag{71}$$

where we do not assume yet the antisymmetry of $\overset{0\alpha}{\psi}_{Aij}$ in i and j. The above conditions are satisfied up to term linear in J_{ij}/I and K_{ij}/I by

$$\xi_{Ai}=\frac{1}{I}\frac{\partial X_A}{\partial x_i}-\frac{1}{I^2}\sum_j J_{ij}\frac{\partial X_B}{\partial x_j}$$

$$\text{for } A,\ B=\mathrm{I},\ \mathrm{II} \text{ and } A\neq B.\tag{72}$$

$$\eta_{Ai}=\frac{1}{I}\frac{\partial Y_A}{\partial x_i}-\frac{1}{I^2}\sum_j K_{ij}\frac{\partial Y_A}{\partial x_j}$$

The potential energy now becomes

$$\begin{aligned}E=&\frac{1}{2I}\sum_{A,\alpha,i,j}\{(\overset{0\alpha}{\varphi}_{Ai})^2+\tfrac{1}{2}(\overset{0\alpha}{\psi}_{Aij})^2\}\\ &-\frac{1}{I^2}\sum_{\alpha,i,j,k}\{J_{ij}\overset{0\alpha}{\varphi}_{\mathrm{I}i}\overset{0\alpha}{\varphi}_{\mathrm{II}j}+\tfrac{1}{2}K_{ij}\overset{0\alpha}{\psi}_{\mathrm{I}ik}\overset{0\alpha}{\psi}_{\mathrm{II}jk}\}\\ &-f\sum_{A,\alpha,i,j}\left\{\sigma_{Ai}\tau_A{}^\alpha\overset{0\alpha}{\varphi}_{Ai}+\frac{1}{\sqrt{2}}\sigma_{Aij}\tau_A{}^\alpha\overset{0\alpha}{\psi}_{Aij}\right\}\\ &+\sum_{\alpha,i}\int\{\varphi'^\alpha(-\Delta+\kappa^2)\varphi'^\alpha\\ &\qquad+\psi_i'^\alpha(-\Delta+\mu^2)\psi_i'^\alpha\}.\end{aligned}\tag{73}$$

To obtain the minimum value of E to the first order in J_{ij}/I and K_{ij}/I, it is sufficient to put

$$\overset{0\alpha}{\varphi}_{Ai}=fI\overset{\alpha}{e}_{Ai}$$

$$\text{for } A=\mathrm{I},\ \mathrm{II} \text{ and } i,\ j,\ k \text{ cyclic},\tag{74}$$

$$\overset{0\alpha}{\psi}_{Aij}=-\overset{0\alpha}{\psi}_{Aji}\equiv\overset{0\alpha}{\psi}_{Ak}=\sqrt{2}fI\overset{\alpha}{e}_{Ak}$$

$$\tfrac{1}{2}\textstyle\sum_i\{\sigma_{Ai}\overset{\alpha}{\tau}_A\overset{\beta}{e}_{Ai}+\sigma_{Ai}\overset{\beta}{\tau}_A\overset{\alpha}{e}_{Ai}\}=\delta_{\alpha\beta}$$

$$\text{for } A=\mathrm{I},\ \mathrm{II}.\tag{75}$$

We find the interaction energy to be

$$\overset{0}{E}_{\mathrm{I\,II}}=-f^2\sum_{\alpha,i,j}\{(J_{ij}-K_{ij})\overset{\alpha}{e}_{\mathrm{I}i}\overset{\alpha}{e}_{\mathrm{II}j}-K_{ii}\overset{\alpha}{e}_{\mathrm{I}j}\overset{\alpha}{e}_{\mathrm{II}j}\},\tag{76}$$

and inserting the expressions (66) for J_{ij}, K_{ij} for point source, we obtain

$$\begin{aligned}\overset{0}{E}_{\mathrm{I\,II}}=f^2\sum_\alpha\bigg\{&(\mathbf{e}_{\mathrm{I}}{}^\alpha\cdot\nabla)(\mathbf{e}_{\mathrm{II}}{}^\alpha\cdot\nabla)\left(\frac{e^{-\kappa r}-e^{-\mu r}}{r}\right)\\ &+(\mathbf{e}_{\mathrm{I}}{}^\alpha\cdot\mathbf{e}_{\mathrm{II}}{}^\alpha)\Delta\left(\frac{e^{-\mu r}}{r}\right)\bigg\}.\end{aligned}\tag{77}$$

The evaluation of the differential operators yields

$$\overset{0}{E}_{\mathrm{I\,II}}=\tfrac{1}{3}\{\Gamma J(r)+\Lambda K(r)\}\tag{78}$$

with Γ and Λ as defined in (62) and with the radial functions

$$\begin{aligned}J(r)&=\frac{f^2}{r}(\kappa^2e^{-\kappa r}+2\mu^2e^{-\mu r}),\\ K(r)&=\frac{f^2}{r^3}\{(3+3\kappa r+\kappa^2r^2)e^{-\kappa r}\\ &\qquad-(3+3\mu r+\mu^2r^2)e^{-\mu r}\}.\end{aligned}\tag{79}$$

It is to be noted that for small r, both J and K behave as r^{-1}, and $K(r)\equiv 0$ for $\kappa=\mu$ in agreement with the results of Møller and Rosenfeld for the weak coupling theory.

The remarks made at the end of Part A also hold in this case; i.e., by letting α take the values 1, 2, 3; 1, 2; or 3, we obtain the symmetrical, the charged, and the neutral theory, respectively.

IV. THE DEUTERON

A. Classification of the States

The stationary states of the one-nuclear system in the symmetrical theory can be described completely by the numbers s, m, and n which are the eigenvalues of the following operators:[10]

$$\mathbf{s}^2=\mathbf{t}^2=s(s+1), \quad s_3=m, \quad t^3=n. \tag{80}$$

s is a positive half-odd integer, and m and n are half-odd integers such that $-s\leqq m,n\leqq s$. It is to be noted that there is complete symmetry between spin and isotopic spin.

For a system containing two nucleons, the Hamiltonian is

$$H=\frac{1}{M}\left\{p_r^2+\frac{\mathbf{L}^2}{r^2}\right\}+\frac{a}{4f^2}\{\mathbf{s}_{\mathrm{I}}^2+\mathbf{s}_{\mathrm{II}}^2-\tfrac{3}{2}\} +\tfrac{1}{3}\{\Gamma J(r)+\Lambda K(r)\}. \tag{81}$$

The first part gives the kinetic energy of relative motion,[11] the second the isobar energy, and the third the interaction energy. The motion of the center of gravity of the system is not considered, and of course this is in the non-relativistic approximation.

Let us first consider states with no orbital angular momentum ($\mathbf{L}=0$); then there is still complete symmetry between spin and isotopic spin as in the one-nucleon system. Their operators $\mathbf{S}=\mathbf{s}_{\mathrm{I}}+\mathbf{s}_{\mathrm{II}}$ and $\mathbf{T}=\mathbf{t}_{\mathrm{I}}+\mathbf{t}_{\mathrm{II}}$ are still constant, but now their magnitude need not be the same. Hence the stationary states can be described by the eigenvalues S, T, M, and N of the following operators:

$$\begin{aligned}\mathbf{S}^2&=S(S+1),\\ \mathbf{T}^2&=T(T+1),\\ S_3&=M,\\ T^3&=N,\end{aligned} \tag{82}$$

where S and T are positive integers or zero such that $|s_{\mathrm{I}}-s_{\mathrm{II}}|\leqq S$, $T\leqq(s_{\mathrm{I}}+s_{\mathrm{II}})$, and M and N are integers or zero such that $-S\leqq M\leqq S$ and $-T\leqq N\leqq T$. If there is an orbital angular momentum $\mathbf{L}$, then the spin angular momentum $\mathbf{S}$ is no longer constant, and it has to be replaced by the total angular momentum $\mathbf{J}$ given by

$$\mathbf{J}=\mathbf{L}+\mathbf{S}, \tag{83}$$

and the eigenvalues S and M by J and J_M, respectively, where

$$\mathbf{J}^2=J(J+1), \quad J_M=L_M+M. \tag{84}$$

In addition there is the quantum number provided by the parity, the eigenvalue of the spatial reflection operator. The Hamiltonian (81) is also invariant under any rotation in the isotopic spin space. Such a rotation leaves the s_{Ai} unchanged and transforms the $t_A{}^\alpha$ and $\overset{\alpha}{e}_{Ai}$ (for each i) in the same way. In particular, a rotation which changes the sign of the 3-component, as for instance the rotation about the 1-axis through 180° which results in the transformation

$$\begin{aligned} &t_A{}^1\rightarrow t_A{}^1, \quad t_A{}^2\rightarrow -t_A{}^2, \quad t_A{}^3\rightarrow -t_A{}^3,\\ &\mathbf{e}_A{}^1\rightarrow\mathbf{e}_A{}^1, \quad \mathbf{e}_A{}^2\rightarrow-\mathbf{e}_A{}^2, \quad \mathbf{e}_A{}^3\rightarrow-\mathbf{e}_A{}^3,\end{aligned} \tag{85}$$

will turn out to be very useful. [Note that this transformation conserves the commutation relations satisfied by the $e_i{}^\alpha$ given in (135) of the Appendix.] The transformation (85) multiplies the spin function by a phase factor $e^{iN\times\mathrm{const}}$, and changes the sign of n_{I}, n_{II}, and N. Thus the operator shows the degeneracy in the energy levels of the excited spin states which differ only by the sign of N. In particular for the case $N=0$, that is for the deuteron, the phase factor is unity, and the transformation does not change the state to which the spin function belongs. Consequently this operator provides another quantum number which is the signature of the spin function under the substitution $n_{\mathrm{I}}\rightarrow -n_{\mathrm{I}}$, $n_{\mathrm{II}}\rightarrow -n_{\mathrm{II}}$.

With the aid of these quantum numbers the complete wave function of a stationary state can be written in the form

$$\Psi(r,\vartheta,\varphi;J,T,J_M,N)=\sum_{\substack{L,S\\ L_M+M=J_M}} c^J_{L_M M}\frac{u_{L,S}(r)}{r} \times Y_L^{L_M}(\vartheta,\varphi)\Xi(S,T,M,N), \tag{86}$$

where $c^J_{L_M M}$ is the normalized coefficient of the Clebsch-Gordon series; $u_{L,S}(r)$ is the radial function which we normalize, for bound states where this is possible, according to

$$\sum_{L,S}\int_0^\infty \bar{u}_{L,S}(r)u_{L,S}(r)dr=1; \tag{87}$$

$Y_L{}^{L_M}(\vartheta,\varphi)$ is the normalized spherical harmonic;

[10] Cf. P-D. We use s instead of j there to denote the eigenvalue of the spin. Also we denote the spin operator by **s** instead of **L** as mentioned already in footnote 6, and the isotopic spin operator by **t** instead of **T**.

[11] We identify M with the empirical mass of the nucleon from which the part of the mass due to the meson field has been subtracted. It was pointed out to one of us by Dr. Schwinger that in the neutral theories the latter is a tensor with respect to the direction of $\mathbf{e}^3$. In the symmetrical theories, however, the field mass is also a scalar.

and $\Xi(S, T, M, N)$ is the spin-isotopic spin function (we shall refer to this hereafter simply as the spin function when no possibility of ambiguity exists). Of course the numbers L and S do not determine the radial functions uniquely since there is still the radial quantum number, but in the present work we consider only the radial functions belonging to the lowest energy level. In accordance with the exclusion principle, we assume Ψ to be antisymmetric in the interchange of particle I and II. Then since the interchange of the space coordinates gives rise to a factor $(-1)^L$, the spin function Ξ must be symmetric or antisymmetric in the interchange of particles I and II depending on whether L is odd or even, respectively. Further since the parity quantum number is $(-1)^L$, any state is restricted to spherical harmonics of even or of odd orders.

In order to evaluate the eigenvalues of the operators Γ and Λ, it is necessary to evaluate the matrix elements of e_i^α in the (s, m, n) representation and express the spin function $\Xi(S, T, M, N)$ in terms of the one-nucleon spin functions $\xi_{\mathrm{I}}(s_{\mathrm{I}}, m_{\mathrm{I}}, n_{\mathrm{I}})$ and $\xi_{\mathrm{II}}(s_{\mathrm{II}}, m_{\mathrm{II}}, n_{\mathrm{II}})$. The calculation of the matrix elements of e_i^α is given in the Appendix. The expression for $\Xi(S, T, M, N)$ in terms of $\xi_{\mathrm{I}}(s_{\mathrm{I}}, m_{\mathrm{I}}, n_{\mathrm{I}})$ and $\xi_{\mathrm{II}}(s_{\mathrm{II}}, m_{\mathrm{II}}, n_{\mathrm{II}})$ is obtained by means of the Clebsch-Gordon series. In fact they can be expressed as products of the spin and isotopic spin functions

$$\Xi(S, T, M, N) = \mathrm{X}(S, M)\mathrm{Z}(T, N), \tag{88}$$

$$\xi_A(s_A, m_A, n_A) = \chi_A(s_A, m_A)\zeta_A(s_A, n_A) \quad \text{for } A = \mathrm{I}, \mathrm{II}; \tag{89}$$

and

$$\mathrm{X}(S, M) = \sum_{\substack{s_{\mathrm{I}}, s_{\mathrm{II}} \\ m_{\mathrm{I}}+m_{\mathrm{II}}=M}} \overset{S}{C}_{m_{\mathrm{I}} m_{\mathrm{II}}} \{\chi_{\mathrm{I}}(s_{\mathrm{I}}, m_{\mathrm{I}})\chi_{\mathrm{II}}(s_{\mathrm{II}}, m_{\mathrm{II}}) \pm \chi_{\mathrm{II}}(s_{\mathrm{I}}, m_{\mathrm{I}})\chi_{\mathrm{I}}(s_{\mathrm{II}}, m_{\mathrm{II}})\},$$

$$\mathrm{Z}(T, N) = \sum_{\substack{s_{\mathrm{I}}, s_{\mathrm{II}} \\ n_{\mathrm{I}}+n_{\mathrm{II}}=N}} \overset{T}{C}_{n_{\mathrm{I}} n_{\mathrm{II}}} \{\zeta_{\mathrm{I}}(s_{\mathrm{I}}, n_{\mathrm{I}})\zeta_{\mathrm{II}}(s_{\mathrm{II}}, n_{\mathrm{II}}) \pm \zeta_{\mathrm{II}}(s_{\mathrm{I}}, n_{\mathrm{I}})\zeta_{\mathrm{I}}(s_{\mathrm{II}}, n_{\mathrm{II}})\}. \tag{90}$$

The symmetrization of these expressions in particles I and II is necessary for the reason given above. However since Γ and Λ have matrix elements between states with values of s_{I} and s_{II} which differ by 1, s_{I} and s_{II} are not constants of the motion, and hence the series (90) are actually infinite series, and the calculation of the eigenvalues of Γ and Λ are very complicated.

We therefore make the following approximation: we replace Γ and Λ by Γ_0 and Λ_0 which are the parts of Γ and Λ diagonal in s_{I} and s_{II}. This treatment is of course rigorous in the limit of infinitely large isobar energy. We shall show for the ground state of the deuteron, by treating the off-diagonal parts Γ_1 and Λ_1 of Γ and Λ as perturbations, that even when the isobar energy is not too large, the effect of the perturbing terms is small. This justifies our method of approximation at least of the ground state of the deuteron in which we are interested.

In the unperturbed system, s_{I} and s_{II} are also quantum numbers of the system, and hence, combining (88), (89), and (90), we obtain for the spin functions of the unperturbed system

$$\Xi(s_{\mathrm{I}}, s_{\mathrm{II}}; S, T, M, N) = \sum_{\substack{m_{\mathrm{I}}+m_{\mathrm{II}}=M \\ n_{\mathrm{I}}+n_{\mathrm{II}}=N}} \overset{S}{C}_{m_{\mathrm{I}} m_{\mathrm{II}}} \overset{T}{C}_{n_{\mathrm{I}} n_{\mathrm{II}}} \times \{\xi_{\mathrm{I}}(s_{\mathrm{I}}, m_{\mathrm{I}}, n_{\mathrm{I}})\xi_{\mathrm{II}}(s_{\mathrm{II}}, m_{\mathrm{II}}, n_{\mathrm{II}}) \pm \xi_{\mathrm{II}}(s_{\mathrm{I}}, m_{\mathrm{I}}, n_{\mathrm{I}})\xi_{\mathrm{I}}(s_{\mathrm{II}}, m_{\mathrm{II}}, n_{\mathrm{II}})\}. \tag{91}$$

It is readily seen that these spin functions are eigenfunctions of the operator given in (85) with the eigenvalues $(-1)^{s_{\mathrm{I}}+s_{\mathrm{II}}-T}$. It is only necessary to use the following property of the Clebsch-Gordon coefficients:

$$\overset{T}{C}_{n_{\mathrm{I}} n_{\mathrm{II}}} = (-1)^{s_{\mathrm{I}}+s_{\mathrm{II}}-T} \overset{T}{C}_{-n_{\mathrm{I}} -n_{\mathrm{II}}}, \tag{92}$$

and to remember that the operation (85) on the spin functions (91) for the deuteron simply replaces n_{I} and n_{II} by $-n_{\mathrm{I}}$ and $-n_{\mathrm{II}}$, respectively.

The spin functions (91) are also eigenfunctions of the operator Γ_0, and its eigenvalues are found by simple calculation with the matrix elements of the e_i^α. They are given by the general formula

$$\Gamma_0 = \frac{\{S(S+1) - s_{\mathrm{I}}(s_{\mathrm{I}}+1) - s_{\mathrm{II}}(s_{\mathrm{II}}+1)\} \times \{T(T+1) - s_{\mathrm{I}}(s_{\mathrm{I}}+1) - s_{\mathrm{II}}(s_{\mathrm{II}}+1)\}}{4s_{\mathrm{I}}(s_{\mathrm{I}}+1)s_{\mathrm{II}}(s_{\mathrm{II}}+1)}. \tag{93}$$

Thus if s is the smaller and t the larger of the two numbers s_{I} and s_{II}, we have

$$-\frac{s}{s+1} \leq \Gamma_0 \leq \frac{s(t+1)}{(s+1)t}; \tag{94}$$

TABLE I. Classification of the deuteron states.

s_I	s_{II}	S	T	(I, II)	$(n\to -n)$	Γ_0
1/2	1/2	0	0	+	−	+1
1/2	1/2	0	1	−	+	−1/3
1/2	1/2	1	0	−	−	−1/3
1/2	1/2	1	1	+	+	+1/9
3/2	1/2	1	1	+, −	−	+5/9
3/2	1/2	1	2	+, −	+	−1/3
3/2	1/2	2	1	+, −	−	−1/3
3/2	1/2	2	2	+, −	+	+1/5
3/2	3/2	0	0	+	−	+1
3/2	3/2	0	1	−	+	+11/15
3/2	3/2	0	2	+	−	+2/5
3/2	3/2	0	3	−	+	−3/5
3/2	3/2	1	0	−	−	+11/15
3/2	3/2	1	1	+	+	+121/225
3/2	3/2	1	2	−	−	+11/75
3/2	3/2	1	3	+	+	−11/25
3/2	3/2	2	0	+	−	+2/5
3/2	3/2	2	1	−	+	+11/75
3/2	3/2	2	2	+	−	+1/25
3/2	3/2	2	3	−	+	−3/25
3/2	3/2	3	0	−	−	−3/5
3/2	3/2	3	1	+	+	−11/25
3/2	3/2	3	2	−	−	−3/25
3/2	3/2	3	3	+	+	+9/25

so that for large s_I and s_{II}, Γ_0 lies between -1 and 1. Furthermore, Γ_0 is negative when either S or T is near the maximum value s_I+s_{II} and the other near the minimum value $|s_I-s_{II}|$, while Γ_0 is positive when both S and T are either near their maximum or near their minimum value. This result is very useful since $\frac{1}{3}\Gamma_0 J(r)$ is the dominant part of the interaction energy, and, in general, the sign of Γ_0 determines whether the interaction is attractive or repulsive. Since the operator Λ_0 contains angular coordinates besides the operators $\overset{\alpha}{e}_{Ai}$, the calculation of its eigenvalues is too complicated in the general case, and they are computed only for the special states considered in Part B.

The first few states of the deuteron system are classified according to their quantum numbers in Table I. The fifth column under (I, II) indicates the symmetry of the spin function under the interchange of particles I and II. For $s_I\neq s_{II}$, both symmetric and antisymmetric functions are possible for each value of S and T; but for $s_I=s_{II}$, only a symmetric or an antisymmetric function is possible for a given value of S and T, depending on whether $(S+T)$ is even or odd, respectively. The sixth column under $(n\to -n)$ indicates the signature of the spin functions under the operation (85). As mentioned above, it is positive or negative depending on whether $(s_I+s_{II}-T)$ is even or odd. The last column under Γ_0 gives the eigenvalues of this operator.

B. The Ground State

The lowest state of the unperturbed system is for $s_I=s_{II}=\frac{1}{2}$, and for the spatial part of the wave function to consist mostly of the S-wave $(L=0)$, we must consider antisymmetric spin functions (in the interchange of particles). We see from Table I that both the singlet $(S=0)$ and the triplet $(S=1)$ states have $\Gamma_0=-\frac{1}{3}$. Thus these states will evidently be bound states. Now the tensor interaction given by Λ_0 vanishes for the singlet state, and we shall find below that it contributes additional attractive potential to the triplet state. Thus in agreement with experiment the ground state is the triplet state.

To obtain the matrix elements of Λ_0, we use the matrix elements of the e_i^α and the expansion (91) for Ξ in terms of ξ_I and ξ_{II}. The result is[12]

$$\Lambda_0\Xi_0(1,0,M)=-\frac{4(2\pi)^{\frac{1}{2}}}{3}\times\sum_{M'+M''=M}\overset{1}{c}_{M'M''}Y_2^{M'}\Xi_0(1,0,M''). \quad (95)$$

Now if we use the spectroscopic notation to designate the states; i.e. write ${}^3S_M{}^1$ for the triplet S state with $J=1$ and $J_M=M$, ${}^3D_M{}^1$ for the triplet D state etc., and if we let $\Sigma_0({}^3S_M{}^1)$, $\Sigma_0({}^3D_M{}^1)$, etc. denote the spin and angular parts of the wave function, we have

$$\Sigma_0({}^3S_M{}^1)=(4\pi)^{-\frac{1}{2}}\Xi_0(1,0,M),$$
$$\Sigma_0({}^3D_M{}^1)=\sum_{M'+M''=M}\overset{1}{c}_{M'M''}Y_2^{M'}\Xi_0(1,0,M''). \quad (96)$$

Thus combining (95) and (96) we obtain

$$\Lambda_0\Sigma_0({}^3S_M{}^1)=-\frac{2\sqrt{2}}{3}\Sigma_0({}^3D_M{}^1),$$
$$\Lambda_0\Sigma_0({}^3D_M{}^1)=\tfrac{2}{3}\Sigma\,({}^3D_M{}^1)-\frac{2\sqrt{3}}{3}\Sigma_0({}^3S_M{}^1). \quad (97)$$

No higher values of the angular momentum

[12] Instead of writing the spin values s_I and s_{II} as in (91), we now use the subscripts 0, 1, and 2 to denote quantities belonging to the states $s_I=s_{II}=\frac{1}{2}$; $s_I=\frac{3}{2}$, $s_{II}=\frac{1}{2}$; and $s_I=s_{II}=\frac{3}{2}$, respectively. Also we drop the designation 0 for N since we do not consider states where N has values different from 0.

occur in this state, and we have

$$\Psi_0(r,\vartheta,\varphi;1,0,M)=\frac{u_0(r)}{r}\sum{}_0({}^3S_M{}^1)+\frac{v_0(r)}{r}\sum{}_0({}^3D_M{}^1). \quad (98)$$

Inserting this expression into the Schrödinger equation

$$H\Psi=E\Psi, \quad (99)$$

and using the expression (81) for H, we obtain with the aid of (97) the simultaneous set of differential equations for $u_0(r)$ and $v_0(r)$

$$\begin{aligned}\frac{d^2u_0}{dr^2}+M\left\{E_0+\frac{1}{9}J(r)\right\}u_0&=-\frac{2\sqrt{2}}{9}MK(r)v_0,\\ \frac{d^2v_0}{dr^2}-\frac{6v_0}{r^2}+M\left\{E_0+\frac{1}{9}J(r)-\frac{2}{9}K(r)\right\}v_0&=-\frac{2\sqrt{2}}{9}MK(r)u_0.\end{aligned} \quad (100)$$

Thus we see that the tensor force contributes additional attractive potential. The above equations are to be solved with the normalization condition

$$\int_0^\infty (u_0{}^2+v_0{}^2)dr=1, \quad (101)$$

and the boundary conditions u_0, $v_0=0$ at both $r=0$ and $r=\infty$. The solution of these equations will be studied in Part C.

First we want to see whether (98) is a good approximation to the actual ground state wave function; i.e., to see whether the approximation we have made in taking only the parts of Γ and Λ diagonal in $s_{\rm I}$ and $s_{\rm II}$ is justified. We do this by calculating the matrix elements of the perturbations Γ_1 and Λ_1 between the state described by (98) and the other states. The first excited spin state occurs when one of the particles has spin 3/2 while the other has the unexcited value 1/2. We see from Table I, however, that none of the states for this case has the same eigenvalues as any of the states with $s_{\rm I}=s_{\rm II}=\frac{1}{2}$, and since neither Γ_1 nor Λ_1 change these quantum numbers (except $s_{\rm I}$ and $s_{\rm II}$), both Γ_1 and Λ_1 have no matrix element between these states.

For the second excited spin state, $s_{\rm I}=s_{\rm II}=\frac{3}{2}$, the states which combine with the ground state ($S=1$, $T=0$) are only the triplet and the septet terms with $T=0$ (cf. Table I). The spin-angular functions which occur in this state are

$$\begin{aligned}\Sigma_2({}^3S_M{}^1)&=(4\pi)^{-\frac{1}{2}}\Xi_2(1,0,M),\\ \Sigma_2({}^3D_M{}^1)&=\sum_{M'+M''=M} \overset{1}{c}_{M'M''}Y_2^{M'}\Xi_2(1,0,M''),\\ \Sigma_2({}^7D_M{}^1)&=\sum_{M'+M''=M} \overset{1}{c}_{M'M''}Y_2^{M'}\Xi_2(3,0,M''),\\ \Sigma_2({}^7G_M{}^1)&=\sum_{M'+M''=M} \overset{1}{c}_{M'M''}Y_4^{M'}\Xi_2(3,0,M'').\end{aligned} \quad (102)$$

The operator Λ_0 applied to these functions gives the following results:

$$\begin{aligned}\Lambda_0\Sigma_2({}^3S_M{}^1)&=\frac{4\sqrt{7}}{25}\Sigma_2({}^7D_M{}^1)-\frac{34\sqrt{2}}{75}\Sigma_2({}^3D_M{}^1),\\ \Lambda_0\Sigma_2({}^3D_M{}^1)&=\frac{12\sqrt{6}}{25\sqrt{7}}\Sigma_2({}^7G_M{}^1)-\frac{4\sqrt{2}}{25\sqrt{7}}\Sigma_2({}^7D_M{}^1)+\frac{34}{75}\Sigma_2({}^3D_M{}^1)-\frac{34\sqrt{2}}{75}\Sigma_2({}^3S_M{}^1),\\ \Lambda_0\Sigma_2({}^7D_M{}^1)&=-\frac{12\sqrt{3}}{175}\Sigma_2({}^7G_M{}^1)+\frac{144}{175}\Sigma_2({}^7D_M{}^1)-\frac{4\sqrt{2}}{25\sqrt{7}}\Sigma_2({}^3D_M{}^1)+\frac{4\sqrt{7}}{25}\Sigma_2({}^3S_M{}^1),\\ \Lambda_0\Sigma_2({}^7G_M{}^1)&=\frac{6}{7}\Sigma_2({}^7G_M{}^1)-\frac{12\sqrt{3}}{175}\Sigma_2({}^7D_M{}^1)+\frac{12\sqrt{6}}{25\sqrt{7}}\Sigma_2({}^3D_M{}^1).\end{aligned} \quad (103)$$

Writing the wave function in the form

$$\Psi_2(r,\vartheta,\varphi;1,0,M)=\frac{u_2(r)}{r}\Sigma_2({}^3S_M{}^1)+\frac{v_2(r)}{r}\Sigma_2({}^3D_M{}^1)+\frac{w_2(r)}{r}\Sigma_2({}^7D_M{}^1)+\frac{y_2(r)}{r}\Sigma_2({}^7G_M{}^1), \quad (104)$$

we find the following set of equations for the radial functions:

$$\frac{d^2u_2}{dr^2}+M\left\{E_2-\frac{3a}{2f^2}-\frac{11}{15}J(r)\right\}u_2$$
$$=-\frac{2}{225}MK(r)\{17\sqrt{2}\,v_2-6\sqrt{7}\,w_2\},$$
$$\frac{d^2v_2}{dr^2}-\frac{6v_2}{r^2}+M\left\{E_2-\frac{3a}{2f^2}-\frac{11}{15}J(r)-\frac{34}{225}K(r)\right\}v_2$$
$$=-\frac{2}{225}MK(r)\left\{17\sqrt{2}\,u_2+\frac{6\sqrt{2}}{7}w_2-\frac{18\sqrt{6}}{\sqrt{7}}y_2\right\},$$
$$\frac{d^2w_2}{dr^2}-\frac{6w_2}{r^2}+M\left\{E_2-\frac{3a}{2f^2}+\frac{1}{5}J(r)-\frac{48}{175}K(r)\right\}w_2 \qquad (105)$$
$$=-\frac{4}{75}MK(r)\left\{-\sqrt{2}\,u_2+\frac{\sqrt{2}}{\sqrt{7}}v_2+\frac{3\sqrt{3}}{\sqrt{7}}y_2\right\},$$
$$\frac{d^2y_2}{dr^2}-\frac{20y_2}{r^2}+M\left\{E_2-\frac{3a}{2f^2}+\frac{1}{5}J(r)-\frac{2}{7}K(r)\right\}y_2$$
$$=-\frac{4}{25}MK(r)\left\{-\frac{\sqrt{6}}{\sqrt{7}}v_2+\frac{\sqrt{3}}{\sqrt{7}}w_2\right\}.$$

No higher spin states need be considered since neither Γ_1 nor Λ_1 has matrix elements between states whose spin value differ by more than 1. Hence the excited state considered above is the only state which combines with the ground state (98).

To calculate the matrix elements between them, we have

$$\Gamma_1\Sigma_0(^3S_M{}^1)=\frac{\sqrt{5}}{3}\Sigma_2(^3S_M{}^1),$$
$$\Gamma_1\Sigma_0(^3D_M{}^1)=\frac{\sqrt{5}}{3}\Sigma_2(^3D_M{}^1),$$
$$\Lambda_1\Sigma_0(^3S_M{}^1)=-\frac{\sqrt{7}}{\sqrt{5}}\Sigma_2(^7D_M{}^1)+\frac{\sqrt{2}}{3\sqrt{5}}\Sigma_2(^3D_M{}^1), \qquad (106)$$
$$\Lambda_1\Sigma_0(^3D_M{}^1)=-\frac{3\sqrt{6}}{\sqrt{35}}\Sigma_2(^7G_M{}^1)+\frac{\sqrt{2}}{\sqrt{35}}\Sigma_2(^7D_M{}^1)-\frac{1}{3\sqrt{5}}\Sigma_2(^3D_M{}^1)+\frac{\sqrt{2}}{3\sqrt{5}}\Sigma_2(^3S_M{}^1).$$

Hence if we write $(2|H|0)$ for the matrix element of the perturbation

$$H_1=\tfrac{1}{3}\{\Gamma_1J(r)+\Lambda_1K(r)\} \qquad (107)$$

between these states, we have

$$(2|H|0)=\frac{1}{3}\int_0^\infty\Bigg[\frac{\sqrt{5}}{3}\{\bar{u}_2u_0+\bar{v}_2v_0\}J(r)+\Bigg\{\frac{\sqrt{2}}{3\sqrt{5}}\bar{u}_2v_0+\frac{1}{3\sqrt{5}}\bar{v}_2(\sqrt{2}\,u_0-v_0)-\frac{1}{\sqrt{35}}\bar{w}_2(7u_0-\sqrt{2}\,v_0)-\frac{3\sqrt{6}}{\sqrt{35}}\bar{y}_2v_0\Bigg\}K(r)\Bigg]dr. \qquad (108)$$

From (105) we see that for the dominant S part u_2 of the wave function the direct interaction gives rise to a repulsive potential $11/15J(r)$. The tensor force gives rise to an attractive potential when coupled with the triplet D part v_2, and a repulsive potential when coupled with the septet D part w_2, but these contributions will be small compared to the direct interaction. Hence it seems that the state (104) will not be a bound state. Assuming this to be the case, we can estimate the correction to the wave function (98) due to the perturbation (107) by approximating the wave function (104) with a plane wave. If we denote this correction term by Ψ_{01}, then we find

$$\int|\Psi_{01}|^2dV\sim 0.2, \qquad (109)$$

by using numerical values for the constants a, f, κ and μ which will be found in Part C and for isobar separation of 20 Mev. Thus the correction is quite small, and it will not alter appreciably the results obtained for the unperturbed ground state.

On the other hand, if the state (105) is a bound state, the effect of the perturbation is proportional to the ratio

$$(2|H|0)/(E_2-E_0)\backsimeq(2|H|0)/E_{\text{isob}} \qquad (110)$$

where $E_{\text{isob}}=3a/2f^2$. We would certainly obtain an upper bound for $(2|H|0)$ if we insert in (108) u_0, v_0 for u_2, v_2 and neglect the other small quantities. This gives for $(2|H|0)$ a value a few times the binding energy of the deuteron, $E_{\text{bind}}=|E_0|$. Hence in this case the approximation is justified since the upper limit of the ratio (110) is about 0.3, and actually it will be much smaller.

In either case, the validity of the approximation depends on the smallness of the quantity

$$E_{\text{bind}}/E_{\text{isob}}. \tag{111}$$

C. Determination of the Constants

In the present theory the following constants occur: the size of the nucleon a, the strength of the coupling f, and the masses of the pseudoscalar and the vector mesons, κ and μ. Of these, f must be chosen so that the binding energy of the deuteron is equal to the observed value of 2.17 Mev, and κ equal to the observed value of the mass of the meson of ~ 200 electron masses (we take for convenience $\kappa = 177$ m). The tensor force depends on the ratio of the masses μ/κ and vanishes for $\mu = \kappa$. Thus we shall choose μ so that the quadrupole moment of the deuteron comes out equal to the observed value of 2.73×10^{-27} cm². However we have the following conditions to satisfy for the validity of our calculation:

(1) small source, i.e., $a\kappa \ll 1$;
(2) strong coupling, i.e., $(f\kappa)^2 \gg (a\kappa)^2$;
(3) small effect of the higher spin states on the ground state, i.e., $E_{\text{bind}}/E_{\text{isob}} \ll 1$.

It is not possible to say *a priori* whether these conditions can be satisfied with the above determination of the constants, but as we shall see below, this is possible.

In order to determine f and μ, we must consider the differential equations (100). f is to be determined so that the lowest eigenvalue E_0 of these equations is equal to $-E_{\text{bind}}$, and μ so that the quadrupole moment[13]

$$Q = \frac{\sqrt{2}}{10} \int_0^\infty r^2 \left(uv - \frac{1}{2\sqrt{2}} v^2 \right) dr \tag{112}$$

agrees with the experimental value. Since the Eqs. (100) cannot be solved in terms of any known function and since the conditions determining f and μ are interdependent, we have to assume some reasonable value for μ, solve (100) approximately, and see whether it gives a good value for Q. Let us first simplify the equations by introducing the dimensionless variable

$$x = 2(M|E_0|)^{\frac{1}{2}} r \tag{113}$$

which transforms (101) to

$$\begin{aligned} d^2u/dx^2 - \{\tfrac{1}{4} - j(x)\} u &= -\sqrt{2} k(x) v, \\ \frac{d^2v}{dx^2} - \frac{6v}{x^2} - \{\tfrac{1}{4} - j(x) + k(x)\} v &= -\sqrt{2} k(x) u, \end{aligned} \tag{114}$$

with

$$\begin{aligned} j(x) &= \frac{\gamma}{x} \{\alpha^2 e^{-\alpha x} + 2\beta^2 e^{-\beta x}\}, \\ k(x) &= \frac{2\gamma}{x^3} \{(\alpha^2 x^2 + 3\alpha x + 3) e^{-\alpha x} \\ &\qquad - (\beta^2 x^2 + 3\beta x + 3) e^{-\beta x}\}, \end{aligned} \tag{115}$$

where

$$\begin{aligned} \alpha &= \frac{\kappa}{2(M|E_0|)^{\frac{1}{2}}}, \quad \beta = \frac{\mu}{2(M|E_0|)^{\frac{1}{2}}}, \\ \gamma &= \frac{2}{9} f^2 M (M|E_0|)^{\frac{1}{2}}. \end{aligned} \tag{116}$$

The normalization condition (101) becomes

$$\int_0^\infty (u^2 + v^2) dx = 1, \tag{117}$$

and the quadrupole moment (112) becomes

$$Q = \frac{\sqrt{2}}{10} \int_0^\infty x^2 \left(uv - \frac{1}{2\sqrt{2}} v^2 \right) dx, \tag{118}$$

and it is now measured in units of $(4M|E_0|)^{-1}$.

We have made an estimate of the constants by carrying out a variational calculation to obtain the minimum value for γ, and approximate expressions for $u(x)$ and $v(x)$ in order to calculate Q. Hulthén[14] has shown that for the simple Schrödinger equation with the potential $e^{-\alpha x}/x$, a good approximate solution is

$$\{a_1(1-e^{-x}) + a_2(1-e^{-x})^2\} e^{-x/2}.$$

Hence we have taken this expression for $u(x)$, and an analogous expression for $v(x)$. That is, we take

$$\begin{aligned} u(x) &= \{a_1(1-e^{-x}) + a_2(1-e^{-x})^2\} e^{-x/2}, \\ v(x) &= \{b_1(1-e^{-x})^4 + b_2(1-e^{-x})^5\} \\ &\qquad \times \left(1 + \frac{6}{x} + \frac{12}{x^2}\right) e^{-x/2}. \end{aligned} \tag{119}$$

[13] Since from now on we consider only the ground state wave function (98), we drop the subscript 0.

[14] L. Hulthén, Arkiv för mat. astr. och fysik A28, No. 5 (1942).

They have the required behavior at both small and large values of x. We insert these expressions in the energy which is

$$\frac{1}{4}=\int_0^{\infty}\left[u\frac{d^2u}{dx^2}+v\frac{d^2v}{dx^2}-\frac{6v^2}{x^2}+u^2j(x)\right.$$
$$\left.+2\sqrt{2}uvk(x)+v^2\{j(x)-k(x)\}\right]dx, \quad (120)$$

and determine the constants a_1, a_2, b_1, and b_2 so that γ is a minimum and the normalization condition (117) is satisfied. The calculation is tedious but straightforward, and we give only the results. Taking $|E_0|=2.17$ Mev, $\kappa=177$ m, and $\mu/\kappa=2$, we obtain

$$\gamma=0.433, \quad (121)$$

$$\begin{aligned} a_1&=3.18, & a_2&=-2.01,\\ b_1&=0.095, & b_2&=-0.072. \end{aligned} \quad (122)$$

(121) gives $(f\kappa)^2=0.375$, and the evaluation of the quadrupole moment with (119) and (122) gives $Q=1.6\times10^{-27}$ cm^2. The latter seems to indicate that the ratio μ/κ should be larger to give the right value for Q. However, it is well known that the variational method may give quite good results for the quantity which is minimized (in this case f), but that the use of the approximate wave functions to calculate other properties may give very bad results. In our case this is quite possible since Q is proportional to $v(x)$ which contributes very little to the energy, and moreover the main contribution to Q comes from the parts of the wave functions at large distances from the origin which do not affect the energy very much. In fact the same calculation made with $\mu/\kappa=3$ actually gives a smaller value for Q. Thus it seems that it is not possible to obtain anything but a rough estimate $\mu/\kappa\sim2$ without going into a much more careful computation.

We can, however, show that the conditions given in the beginning of this part can be satisfied with our estimated values for the constants. Now since $E_{\text{isob}}=3a/2f^2$,

$$\frac{E_{\text{bind}}}{E_{\text{isob}}}=\frac{2(f\kappa)^2}{3(a\kappa)}\left(\frac{E_{\text{bind}}}{\kappa}\right)\simeq\frac{0.0056}{(a\kappa)}. \quad (123)$$

Thus there is a region of values for a (and consequently for E_{isob}) for which all the inequalities in the conditions given above hold by a factor of about 10.

D. The Magnetic Moment

The calculation of the magnetic moment of the deuteron is exactly analogous to that given in P-D for a single nucleon. The current vector is[15]

$$\begin{aligned}\mathbf{j}=\varphi^2\nabla\varphi^1-\varphi^1\nabla\varphi^2 &+(4\pi)^{\frac{1}{2}}f\textstyle\sum_A\boldsymbol{\sigma}_A(\varphi^1\tau_A{}^2-\varphi^2\tau_A{}^1)U_A\\ &+\boldsymbol{\psi}^2\times(\nabla\times\boldsymbol{\psi}^1)-\boldsymbol{\psi}^1\times(\nabla\times\boldsymbol{\psi}^2)\\ &+(4\pi)^{\frac{1}{2}}f\textstyle\sum_A\{(\boldsymbol{\sigma}_A\times\boldsymbol{\psi}^1)\tau_A{}^2\\ &\qquad-(\boldsymbol{\sigma}_A\times\boldsymbol{\psi}^2)\tau_A{}^1\}U_A. \end{aligned} \quad (124)$$

Let us first consider just the part due to the pseudoscalar meson. Inserting in the expansion (46) the value corresponding to the absence of free mesons ($\varphi'^{\alpha}=0$) and the lowest eigenvalue of the interaction energy (52), we obtain with the aid of (50)

$$\varphi^{\alpha}=\frac{f}{(4\pi)^{\frac{1}{2}}}\sum_{A,i}\overset{\alpha}{e}_{Ai}\frac{\partial X_A}{\partial x_i}. \quad (125)$$

We have neglected terms in J_{ij}/I in (50) since they only give rise to higher order terms. Putting (125) in the pseudoscalar part of (124), we obtain

$$\begin{aligned} j_k=\frac{f^2}{4\pi}\sum_{A,B,i,j}\Bigg[&\left(\overset{2}{e}_{Ai}\frac{\partial X_A}{\partial x_i}\right)\frac{\partial}{\partial x_k}\left(\overset{1}{e}_{Bj}\frac{\partial X_B}{\partial x_j}\right)\\ &-\left(\overset{1}{e}_{Ai}\frac{\partial X_A}{\partial x_i}\right)\frac{\partial}{\partial x_k}\left(\overset{2}{e}_{Bj}\frac{\partial X_B}{\partial x_j}\right)\\ &-4\pi\left\{\left(\overset{1}{e}_{Ai}\frac{\partial X_A}{\partial x_i}\right)\overset{2}{e}_{Bk}-\left(\overset{2}{e}_{Ai}\frac{\partial X_A}{\partial x_i}\right)\overset{1}{e}_{Bk}\right\}U_B\Bigg],\end{aligned}$$

or

$$j_k=\frac{f^2}{4\pi}\sum_{A,B,i,j}\{\overset{2}{e}_{Ai}\overset{1}{e}_{Bj}-\overset{1}{e}_{Ai}\overset{2}{e}_{Bj}\}$$
$$\times\left\{\frac{\partial X_A}{\partial x_i}\frac{\partial^2X_B}{\partial x_k\partial x_j}+4\pi\delta_{jk}\frac{\partial X_A}{\partial x_i}U_B\right\}. \quad (126)$$

The magnetic moment is given by

$$M_{kl}=\frac{e}{2}\int(x_kj_l-x_lj_k)dV, \quad (127)$$

[15] We have not written down here the terms which would occur if the vector meson had an anomalous magnetic moment, but this will not change our results in any way.

and inserting (126) for j_k, we obtain

$$M_{kl}=\frac{ef^2}{8\pi}\sum_{A,B,i,j}\{\overset{2}{e}_{Ai}\overset{1}{e}_{Bj}-\overset{1}{e}_{Ai}\overset{2}{e}_{Bj}\}(A_{kl}^{ij}-A_{lk}^{ij}), \quad (128)$$

where

$$A_{kl}^{ij}=\int\left[x_k\left\{\frac{\partial X_A}{\partial x_i}\frac{\partial^2 X_B}{\partial x_l\partial x_j}+4\pi\delta_{jl}\frac{\partial X_A}{\partial x_i}U_B\right\}\right]dV. \quad (129)$$

In the double sum over A, B in (128), the terms with $A=B$ give the moments of the individual particles, and they are just the expressions calculated in P-D. Since it was found there that the moments of the proton and the neutron are equal in magnitude and opposite in sign, these terms will cancel out, and we need only take the terms with $A\neq B$. Since A_{kl}^{ij} is not altered by the interchange of particles I and II, (128) can be written

$$M_{kl}=\frac{ef^2}{8\pi}\sum_{i,j}P_{ij}(A_{kl}^{ij}-A_{lk}^{ij}) \quad (130)$$

with

$$P_{ij}=\{\overset{2}{e}_{\mathrm{I}i}\overset{1}{e}_{\mathrm{II}j}-\overset{1}{e}_{\mathrm{I}i}\overset{2}{e}_{\mathrm{II}j}+\overset{2}{e}_{\mathrm{II}i}\overset{1}{e}_{\mathrm{I}j}-\overset{1}{e}_{\mathrm{II}i}\overset{2}{e}_{\mathrm{I}j}\}. \quad (131)$$

We see that P_{ij} is antisymmetric in i, j so that it has only 3 independent components. In order to compute the component of $\mathbf{M}$ in a given direction, we need the diagonal matrix elements of the P_{ij} in the (S, T, M, N) representation. However the P_{ij} change sign under the operation (85) so that they cannot have any matrix elements between states with the same signature. Thus this theory gives zero magnetic moment for the deuteron.

There is no need to continue the calculation for the vector meson since the reason that the magnetic moment vanishes in the above case is so general that it is clear the same result will be obtained in all cases of the strong coupling theory.

Of course the existence of the tensor force gives a magnetic moment due to the orbital motion of the proton, but this contribution is very small (of the order of a few percent of the observed value).

V. HEAVY NUCLEI

It was shown in S-D that the charged scalar and the neutral pseudoscalar meson theories give rise to nuclear forces which have no saturation property. These results are due to the fact that in these theories, any number of nucleons can be arranged in a symmetrical configuration in such a way that each particle interacts with an attractive potential with all the others: in the charged scalar theory by having the isotopic spin vectors of all the particles in the same direction, and in the neutral pseudoscalar theory by having all the particles lie in a plane with all their spin vectors parallel and perpendicular to this plane. In the neutral mixed theory such configurations do not lead to attractive interaction between the particles.[16] In the symmetrical pseudoscalar and symmetrical mixed theories, we have a system of 3 orthonormal vectors $\mathbf{e}^\alpha$ instead of a single spin or isotopic spin vector for each particle. The only symmetrical configuration in these cases is obtained by having all the vectors parallel, and the interaction for this arrangement is repulsive. Thus for these theories the potential energy will be proportional to the number of particles and not to its square as in the cases treated in S-D.

However, for the forces to have the saturation property, it is also necessary that the kinetic energy of the particles increase faster than the attractive potential energy as the radius of the nucleus decreases in order that there exist a minimum for the total energy of the system. This condition is fulfilled in the usual weak coupling theory[17] since, if we denote by r_0 the radius corresponding to the volume occupied by each particle, the kinetic energy increases as r_0^{-2}, and the dependence of the potential energy, of course, is the same as the behavior of the radial potential function, but this cannot increase faster than r^{-1}. In the strong coupling theory however, the existence of isobar states increases the number of allowed states in a given volume of space with the same upper limit for the energy, and this will change the dependence on r_0 of both the kinetic and potential energies.

In order to see how this feature changes our results, we have made a calculation similar to that carried out in Bethe and Bacher. We make

[16] The second configuration gives attraction only when $K(r)>J(r)$. For $\mu/\kappa\sim 2$, this condition holds only when r is greater than about $2/\kappa$ and for such distances, both $J(r)$ and $K(r)$ are very small.

[17] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936), in particular §25.

a Hartree approximation for the wave function of the system with free particle solutions for the individual wave functions, both for the spatial and for the spin parts. The latter are hypergeometric functions and have been investigated by Rademacher and Reiche[18] in connection with the quantum theory of a symmetrical top. The integrals which we need are all tabulated there. We shall not give the details of the calculation but just make a few remarks on the points which differ from the calculation given in Bethe and Bacher.

The number of particles N is now the product of N_k and N_s where N_k is the number of states with kinetic energy less than $k_0^2/2M$, and N_s is the number of states with isobar energy less than $(a/4f^2)\{s_0(s_0+1)-\frac{3}{4}\}$. The total energy of the system is

$$E=E_k+E_s+E_p \tag{132}$$

where E_k, E_s, and E_p denote the kinetic, isobar, and the potential energy, respectively. If we consider s_0 to be large compared to 1 and keep only the highest powers of it, then N_k, N_s and E_k, E_s depend on k_0, s_0 in the same way. Thus for (132) to be a minimum,

$$k_0^2/2M=(a/4f^2)s_0^2, \tag{133}$$

and

$$E_k=E_s. \tag{134}$$

Moreover they are proportional to N/r_0. E_p on the other hand turns out to be proportional to $N/(r_0)^{\frac{1}{2}}$, and there does exist a value of r_0 at which E is a minimum.

In the above consideration we have neglected the Coulomb energy, and though this may not affect the saturation property, it does bring in a serious difficulty in connection with the stability of heavy nuclei with high charge. As pointed out by Fierz,[19] in a nucleus like that of uranium, the transition of a proton or a neutron to a particle with negative charge is energetically favorable unless the isobar separation is about 50 Mev. However such a value for the isobar separation is inadmissible since we would then have to take the size of the source of the same magnitude as the Compton wave-lengths of the mesons. That is, the size of the source would have to be taken as the same order of magnitude as the range of the nuclear forces, and hence the shape of the source would completely determine the nature of the nuclear forces. Furthermore, such a value also removes the main reason for the introduction of the strong coupling theory; namely, to give a small value for the meson scattering cross section.

VI. CONCLUSIONS

Our results show that the theory considered here suffers from two grave difficulties; it gives a magnetic moment for the deuteron a value only a few percent of the observed value, and it predicts instability of highly charged nuclei. These results seem to be fundamental properties of all strong coupling theories, and there does not seem to be any way of overcoming them.

These difficulties are not present in the weak coupling theory, and it thus seems advisable to go back and reconsider the arguments which led us to take up the strong coupling theory in favor of the weak coupling theory. The main difficulties in the weak coupling theories are the divergences due to the treatment of the heavy particles as a point source, and the large scattering cross section of the meson. As already pointed out by one of us,[20] the first difficulty can be overcome by using the λ-process developed by Wentzel and Dirac, and the second by using the theory of radiation damping developed by Heitler and Wilson. In addition, the weak coupling theory developed in this way has the advantage of relativistic invariance which the strong coupling theory does not have on account of the finite size of the source. Thus there is no reason now to consider the strong coupling theory, and we should go back to the weak coupling theory.

As stated at the end of Part A of Section III, the transition from the strong coupling to the weak coupling theory can be made by a simple replacement of the spin and isotopic spin vectors in place of the vectors $\mathbf{e}^\alpha$. Thus the radial wave functions for the ground state of the deuteron in the weak coupling theory satisfy exactly the

[18] H. Rademacher and F. Reiche, Zeits. f. Physik **39**, 444 (1926), and **41**, 453 (1927).

[19] M. Fierz, Helv. Phys. Acta **14**, 105 (1941).

[20] W. Pauli, Bull. Am. Phys. Soc. New York Meeting, Jan. 22–23, 1943, Abstract No. 25: Phys. Rev. **63**, 221 (1943).

same differential equations (114) except that as shown in the Appendix, γ is redefined as

$$\gamma = 2f'^2 M(M|E_0|)^{\frac{1}{2}},$$

where f' is the coupling constant in this theory, and there is no longer any approximation due to the higher isobar states since they do not exist. Hence the computation in Part C of Section IV applies just as well for this case, and we obtain $(f'\kappa)^2 = \frac{1}{9}(f\kappa)^2 \cong 0.042$ for the strength of the coupling. The ratio of the masses is unchanged.

APPENDIX

Calculation of the Matrix Element of e_i^α

The e_i^α satisfy the following commutation relations with s_i and t^α [cf. P-D, Eq. (55)]:

$$[s_i, e_j^\alpha] = ie_k^\alpha, \quad i, j, k, \text{ cyclic}; \qquad [t^\alpha, e_i^\beta] = ie_i^\gamma, \quad \alpha, \beta, \gamma, \text{ cyclic}. \tag{135}$$

It is more convenient to introduce the imaginary components

$$s_+ = s_1 + is_2, \quad s_- = s_1 - is_2, \quad s_3; \qquad t^+ = t^1 + it^2, \quad t^- = t^1 - it^2, \quad t^3; \tag{136}$$

and

$$\begin{aligned} &e_+^{\ +} = e_1^1 - e_2^2 + i(e_2^1 + e_1^2), && e_+^{\ 3} = e_1^3 + ie_2^3,\\ &e_+^{\ -} = e_1^1 + e_2^2 + i(e_2^1 - e_1^2), && e_-^{\ 3} = e_1^3 - ie_2^3,\\ &e_-^{\ +} = e_1^1 + e_2^2 - i(e_2^1 - e_1^2), && e_3^{\ +} = e_3^1 + ie_3^2,\\ &e_-^{\ -} = e_1^1 - e_2^2 - i(e_2^1 + e_1^2), && e_3^{\ -} = e_3^1 - ie_3^2,\\ &e_3^3. \end{aligned} \tag{137}$$

For the matrix elements of the e_i^α between the states (s, m, n) and (s', m', n'), we have the following selection rules:

(1) for any e_i^α, $s' = s-1$, s, or $s+1$;
(2) for e_+^α, $m' = m-1$;
(3) for e_3^α, $m' = m$;
(4) for e_-^α, $m' = m+1$;
(5) for e_i^+, $n' = n-1$;
(6) for e_i^3, $n' = n$;
(7) for e_i^-, $n' = n+1$.

Matrix elements which do not satisfy these conditions all vanish.

The non-vanishing matrix elements of s_i and t^α are:

$$\begin{aligned} (m|s_+|m-1) &= [(s+m)(s-m+1)]^{\frac{1}{2}},\\ (m|s_3|m) &= m,\\ (m|s_-|m+1) &= [(s+m+1)(s-m)]^{\frac{1}{2}}, \end{aligned} \tag{138}$$

with exactly the same expressions for t^+, t^3, t^- with m replaced by n. Hence in order to satisfy the commutation relations (135), we must have the m dependence of the e_i^α as follows:

$$\begin{aligned} &(s, m, n|e_+^\alpha|s-1, m-1, n') = -(s, n|c^\alpha|s-1, n')[(s+m)(s+m-1)]^{\frac{1}{2}},\\ &(s, m, n|e_+^\alpha|s, m-1, n') = (s, n|c^\alpha|s, n')[(s+m)(s-m+1)]^{\frac{1}{2}},\\ &(s, m, n|e_+^\alpha|s+1, m-1, n') = (s, n|c^\alpha|s+1, n')[(s-m+1)(s-m+2)]^{\frac{1}{2}},\\ &(s, m, n|e_3^\alpha|s-1, m, n') = (s, n|c^\alpha|s-1, n')(s^2-m^2)^{\frac{1}{2}},\\ &(s, m, n|e_3^\alpha|s, m, n') = (s, n|c^\alpha|s, n')m,\\ &(s, m, n|e_3^\alpha|s+1, m, n') = (s, n|c^\alpha|s+1, n')[(s+1)^2-m^2]^{\frac{1}{2}},\\ &(s, m, n|e_-^\alpha|s-1, m+1, n') = (s, n|c^\alpha|s-1, n')[(s-m)(s-m-1)]^{\frac{1}{2}},\\ &(s, m, n|e_-^\alpha|s, m+1, n') = (s, n|c^\alpha|s, n')[(s-m)(s+m+1)]^{\frac{1}{2}},\\ &(s, m, n|e_-^\alpha|s+1, m+1, n') = -(s, n|c^\alpha|s+1, n')[(s+m+1)(s+m+2)]^{\frac{1}{2}}. \end{aligned}$$

Similarly to satisfy the commutation relations, the n-dependence must be as follows:

$$\begin{aligned} &(s, n|c^+|s-1, n-1) = -(s|b|s-1)[(s+n)(s+n-1)]^{\frac{1}{2}},\\ &(s, n|c^+|s, n-1) = (s|b|s)[(s+n)(s-n+1)]^{\frac{1}{2}},\\ &(s, n|c^+|s+1, n-1) = (s|b|s+1)[(s-n+1)(s-n+2)]^{\frac{1}{2}},\\ &(s, n|c^3|s-1, n) = (s|b|s-1)[s^2-n^2]^{\frac{1}{2}},\\ &(s, n|c^3|s, n) = (s|b|s)n,\\ &(s, n|c^3|s+1, n) = (s|b|s+1)[(s+1)^2-n^2]^{\frac{1}{2}},\\ &(s, n|c^-|s-1, n+1) = (s|b|s-1)[(s-n)(s-n-1)]^{\frac{1}{2}},\\ &(s, n|c^-|s, n+1) = (s|b|s)[(s-n)(s+n+1)]^{\frac{1}{2}},\\ &(s, n|c^-|s+1, n+1) = -(s|b|s+1)[(s+n+1)(s+n+2)]^{\frac{1}{2}}. \end{aligned}$$

To obtain the s-dependence, we use the normalization condition which the e_i^α satisfy [cf. P-D, Eq. (51a)]. Using (137) we find

$$\begin{aligned} e_+^{\ 3}e_-^{\ 3} + (e_3^{\ 3})^2 &= 1,\\ e_+^{\ +}e_-^{\ -} + e_-^{\ +}e_+^{\ -} + 2e_3^{\ +}e_3^{\ -} &= 4, \end{aligned} \tag{139}$$

and from these conditions, we obtain the equations

$$\begin{aligned} &s^3(2s-1)(s|b|s-1)^2 + (s+1)^3(2s+3)(s|b|s+1)^2 = 1,\\ &s^2(s-1)(2s-1)(s|b|s-1)^2 + s^2(s+1)^2(s|b|s)^2\\ &\qquad + (s+1)^2(s+2)(2s+3)(s|b|s+1)^2 = 2. \end{aligned} \tag{140}$$

Then from the relation

$$(\mathbf{s}\cdot\mathbf{e}^3) = -t^3 \tag{141}$$

we have

$$(s|b|s) = -1/s(s+1). \tag{142}$$

Thus (140) can now be solved, and we finally obtain

$$\begin{aligned} (s|b|s-1) &= 1/s[(2s-1)(2s+1)]^{\frac{1}{2}},\\ (s|b|s+1) &= 1/(s+1)[(2s+1)(2s+3)]^{\frac{1}{2}}. \end{aligned} \tag{143}$$

In particular for $s = s' = \frac{1}{2}$, we have the relation

$$(\tfrac{1}{2}, m, n|\mathbf{e}^\alpha|\tfrac{1}{2}, m', n') = \tfrac{1}{3}(m, n|\boldsymbol{\sigma}\tau^\alpha|m', n') \tag{144}$$

which gives the connection between the strong and the weak coupling theories; namely, the only difference for the ground state is the occurrence of a factor $\frac{1}{9}$ in the interaction energy which merely changes the values of the coupling constants.