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# Cavitation in an Elastic Liquid

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An idealized theory of cavitation in the interior of a liquid is developed as an extension of the hydrodynamics of irrotational motion. It is assumed that cavitation occurs whenever the pressure sinks to a fixed breaking-pressure and that the pressure then rises at once to a fixed cavity pressure. The boundary of the cavitated region either advances as a breaking-front, moving with supersonic velocity, or remains stationary as a free surface, or recedes toward the cavitated region as a closing-front. The relevant formulas are obtained.

CAVITATION has been studied most extensively at the interface between a liquid and a solid, but apparently it may occur also within the liquid itself. The latter type of cavitation should be influenced by the elasticity of the liquid and should present certain hydrodynamical features of interest. In the present paper a contribution is offered to the theory of cavitation as a hydrodynamical phenomenon. The theory may also find application to the motion of liquids in pipes, even if the cavitation is actually initiated at the walls.[1]

## 1. FUNDAMENTAL ASSUMPTIONS

In order to develop a tractable analytical theory, it will be assumed that the liquid cavitates whenever its pressure sinks to a fixed *breaking-pressure* $p_b$, and that the pressure then rises at once to a definite cavity pressure $p_c$, the two pressures being such that

$$p_b \leqq p_c. \tag{1}$$

The pressure $p_b$ represents the minimum pressure that the liquid can stand; and, as a special case, it may equal $p_c$. In actual cases cavitation within a liquid takes the form of small bubbles, and these commonly contain a variable amount of air or other gases in addition to the vapor of the liquid; as the bubbles collapse, the foreign gas tends not to redissolve entirely, and for this reason the pressure in the bubbles is not constant. Such complications will be ignored here. It may be assumed that the bubbles are indefinitely small and contain only vapor of the liquid, so that $p_c$ is equal to the vapor pressure. The discussion will be limited to irrotational motion in a homogeneous liquid whose density remains almost but not quite constant. The one-dimensional case is easily treated completely and will be taken up first.

## 2. PLANE BREAKING-FRONTS

For one-dimensional motion the usual acoustic equations may be written

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\partial p}{\partial t} = -\rho c^2 \frac{\partial u}{\partial x}, \tag{2a, b}$$

where $p$ = excess of pressure above the hydrostatic pressure, $u$ = particle velocity, $t$ = time, $x$ = space coordinate, $\rho$ = undisturbed density, and $c$ = velocity of sound in the liquid. The derivatives occurring here will be assumed to be continuous functions. The equations are ordinarily assumed to hold only for small particle velocities, but they may also be applied generally to the relative motion of the liquid in the neighborhood of any point, that is, provided uniformly moving axes are employed relative to which the velocity of the liquid near the point is small.

Cavitation will obviously start where the pressure first sinks to $p_b$, hence at a point at which

$$p = p_b, \quad (\partial p/\partial t) < 0, \quad \text{or} \quad (\partial u/\partial x) > 0 \quad \text{(3a, b, c)}$$

by (2b). From this point, a *breaking-front* will advance toward both sides; that moving toward $+x$ or toward the right advances into regions where $\partial p/\partial x \geqq 0$, while the other, moving toward $-x$ or toward the left, advances into regions where $\partial p/\partial x \leqq 0$. Just ahead of each front, conditions (3a, b, c) must continue to hold.

To find the rate of advance of the breaking-

[1] Cf. Kennard, Phys. Rev. **35**, 428 (1930).

fronts, consider what happens as a front moving toward the right traverses, during a time $dt$, a layer of liquid of thickness $dx$ (Fig. 1). At the beginning of $dt$, the pressure in the unbroken liquid is $p_b$ over the left-hand face of the layer $F_1$ and is, therefore,

$$p_b+(\partial p/\partial x)dx$$

over the right-hand face $F_2$, whereas at the end of $dt$ it has sunk to $p_b$ at $F_2$ also. Thus the change in $p$ at $F_2$ during $dt$ is $(-\partial p/\partial x)dx$; or, if $U_b$ denotes the speed of the front relative to the liquid, so that $dx=U_b dt$, the change is $-U_b(\partial p/\partial x)dt$. If, for the moment, axes moving with the unbroken liquid are employed, this change can also be written $(\partial p/\partial t)dt$, the derivative being evaluated at $F_2$. Equating the last two expressions and using (2b), we obtain

$$U_b=\rho c^2(\partial u/\partial x)/(\partial p/\partial x). \qquad (4)$$

Here $U_b$ is the velocity of the front relative to the liquid, which may itself be in motion; but in evaluating the space derivatives axes moving in any manner may be employed.

At the front there is a discontinuity in the pressure. To find its impulsive effect, consider the motion of the layer of liquid of thickness $dx$ (Fig. 1). On the right-hand face $F_2$ the pressure is nearly equal to the breaking-pressure $p_b$. It sinks to this value as the front comes up to the face, whereas on the left-hand face there is acting the steady cavity pressure $p_c$. The momentum in the layer changes, therefore, during $dt$ by $(p_c-p_b)dt$ per unit area. Let $u_b$ denote the particle velocity just ahead of the front, $u_c$ that just behind it. Then the velocity changes impulsively from $u_b$ to $u_c$ as the front passes, and since $dx=U_b dt$,

$$(p_c-p_b)dt=(u_c-u_b)\rho dx=(u_c-u_b)\rho U_b dt,$$

$$u_c=u_b+\frac{p_c-p_b}{\rho U_b}=u_b+\frac{p_c-p_b}{\rho^2 c^2}\frac{\partial p/\partial x}{\partial u/\partial x}, \qquad (5)$$

from (4).

The same equations are obtained for a breaking-front advancing toward the left. In view of (3c), Eq. (4) makes $U_b$ positive where $\partial p/\partial x>0$ and negative where $\partial p/\partial x<0$, the two signs referring to fronts traveling in opposite directions. The sign of $u_c-u_b$ is likewise opposite in the two cases, the liquid being accelerated in the direction of advance of the front. At the initial point, where $p$ first sinks to $p_b$, $\partial p/\partial x=0$ and $U_b=\pm\infty$.

It remains to be shown, however, that the motion of the liquid represented by $u_c$ as defined by Eq. (5) is a hydrodynamically possible one. In doing this, an additional condition for the propagation of a breaking-front is obtained. The argument takes very different courses according as $p_c=p_b$ or $p_c>p_b$.

If $p_c=p_b$, by (5) $u_c=u_b$, the liquid being left at each point with that particle velocity which it happens to possess at the instant of arrival of the front. It is necessary that the liquid be not left with a motion of contraction; hence the values of $u_c$ left behind the front can only vary in such a way that

$$(\partial u_c/\partial x)\geqq 0. \qquad (6)$$

Now, if $u$ has the value $u_{b1}$ at the left-hand face $F_1$ of a layer of thickness $dx$ (Fig. 1), at the instant when a front moving toward the right passes this face, then, at the same instant, at the right-hand face $F_2$,

$$u=u_{b1}+(\partial u/\partial x)dx.$$

Let uniformly moving axes be employed such that momentarily $u_{b1}$ is zero or small. Then, by the time that the front has reached $F_2$, $u$ will have changed at $F_2$ to

$$u_{b2}=u_{b1}+(\partial u/\partial x)dx+(\partial u/\partial t)dt.$$

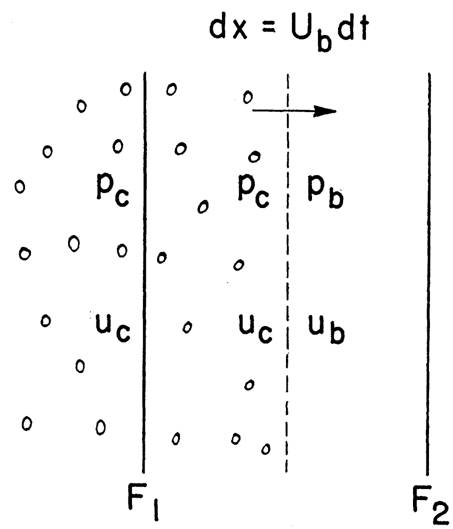


FIG. 1. Motion of a breaking-front.

After the passage of the front, $u$ does not change with time because of the absence of a pressure gradient in the cavitated region. Hence the difference, $u_{b2}-u_{b1}$, is also the difference between the values of $u_c$ at $F_1$ and at $F_2$, or $du_c$, so that

$$du_c = u_{b2} - u_{b1} = (\partial u/\partial x)dx + (\partial u/\partial t)dt,$$

or, by (2a) and the relation $dx = U_b dt$,

$$du_c = [U_b(\partial u/\partial x) - (1/\rho)(\partial p/\partial x)]dt.$$

The condition expressed by (6) now requires that

$$U_b \geqq (1/\rho)(\partial p/\partial x)/(\partial u/\partial x), \quad (7)$$

since $\partial u/\partial x > 0$. Multiplication of this inequality by Eq. (4) gives

$$U_b^2 \geqq c^2; \quad (8)$$

and division of this latter inequality by Eq. (4) squared gives

$$1 \geqq (1/\rho^2 c^2)[(\partial p/\partial x)/(\partial u/\partial x)]^2, \quad (9)$$

whence

$$|\partial p/\partial x| \leqq \rho c(\partial u/\partial x). \quad (10)$$

The same results are obtained for fronts moving toward the left.

If, on the other hand, $p_c > p_b$, the argument just given does not apply, because by Eq. (5) $u_c \neq u_b$ at the front. The situation appears to be essentially different. For a front running to the right ($\partial u/\partial x > 0$, $\partial p/\partial x > 0$), $u_c > u_b$, so that the liquid behind the front tends to overrun that just ahead of it, which is physically impossible. The condition that prevents this happening can be found as follows.

While the front is traversing the layer of thickness $dx$ (Fig. 1), during a time $dt$, the face $F_2$ of the layer advances a distance $u_b dt$ (to the first order), whereas the other face $F_1$ advances a distance $u_c dt$. The volume occupied by the layer, per unit area, thus decreases by

$$(u_c - u_b)dt = [(p_c - p_b)/(\rho U_b)]dt \quad (11)$$

by (5). During the same time, however, the liquid in the layer contracts, as its pressure rises suddenly from $p_b$ to $p_c$. The elasticity being $\rho c^2$, the contraction decreases the volume actually occupied by liquid by the amount

$$(1/\rho c^2)(p_c - p_b)dx = (1/\rho c^2)(p_c - p_b)U_b dt. \quad (12)$$

Hence the liquid can continue to fit into the space allowed for it, with or without the occurrence of cavitation, provided

$$(1/\rho c^2)(p_c - p_b)U_b \geqq [(p_c - p_b)/\rho U_b], \quad U_b^2 \geqq c^2.$$

Thus expressions (8) and (10) are obtained again.

In case $p_c > p_b$, however, it is not possible to show that necessarily $\partial u_c/\partial x \geqq 0$. Apparently it can happen that behind a front $\partial u_c/\partial x < 0$; in this case, any cavitation bubbles that may have been formed at the instant of passage of the front will proceed to decrease in size and ultimately to disappear again.

According to Eq. (8) as written it can happen that $U_b = \pm c$. It is easily seen, however, that if $U_b = \pm c$ for a finite length of time, cavitation does not actually occur. Thus *a true breaking-front travels always at a speed exceeding that of sound in the liquid*. The conditions found for its propagation, including (10), may be summarized as follows:

$$p = p_b, \quad (\partial u/\partial x) > 0, \quad |\partial p/\partial x| < \rho c(\partial u/\partial x); \quad (13a, b, c)$$

in addition, it is necessary that $\partial p/\partial x \geqq 0$ for propagation toward $+x$ or $\partial p/\partial x \leqq 0$ for propagation toward $-x$. In these expressions the derivatives are to be evaluated in the unbroken liquid just ahead of the breaking-front. These values are not affected in any way by the approach of the front; for no hydrodynamic influence of the first order can be propagated through a fluid at a speed exceeding that of sound.

Once started, a breaking-front will travel until it arrives at a point beyond which the necessary conditions are not satisfied. At a point where $\partial p/\partial x = 0$, the front may meet another one coming from the opposite direction, whereupon both fronts will disappear. At a point where $|\partial p/\partial x|$ becomes equal to $\rho c \partial u/\partial x$, with larger values beyond, the front, approaching at speed $c$ relative to the liquid (provided $\partial u/\partial x > 0$), must suddenly stop advancing. What happens next at such a point will be considered presently.

A simpler rule than that just stated might have been supposed to be preferable, namely, that the front travels as far as pressures so low as $p_b$ occur. This statement, however, although true, is less convenient than the conditions just stated, because the occurrence of cavitation results indirectly in a contraction of the region within which the pressure sinks to $p_b$. This results from the fact that the pressure is higher in the cavitated region than it would have been if cavitation had not occurred, and, after the breaking-front has ceased advancing, propagation of this higher pressure into adjoining regions prevents the occurrence of pressures so low as $p_b$ at some points where such pressures would otherwise have occurred. It is impossible, in fact, for the breaking-front to touch the surface which, in the absence of cavitation would be the boundary of the region of pressures below $p_b$; for, at this surface, when $p=p_b$, $\partial p/\partial t=0$, else $p$ would sink to $p_b$ at points outside the surface as well (provided $\partial p/\partial t$ is continuous), and then by (2b) $\partial u/\partial x=0$ and condition (13c) is not satisfied.

### 3. THE CAVITATED REGION

In the cavitated region the pressure is assumed to be uniformly equal to $p_c$, so that there is no pressure gradient; hence the particle velocity $u_c$ at each point remains as it was left by the passage of the breaking-front. Let $\eta$ denote the fraction of the space which is occupied by bubbles. If $p_c=p_b$, $\eta$ is zero immediately back of the breaking-front. If $p_c>p_b$, however, $\eta$ starts from an initial value $\eta_0$ representing the difference between the space freed by compression from pressure $p_b$ to $p_c$ and the space filled by incoming liquid. When a front travels toward $+x$, the intial free space in a layer $dx$ thick just back of the front equals the difference between expressions (12) and (11), so that, since $dx=U_b dt$,

$$\eta_0 dx=(1/\rho c^2)(p_c-p_b)U_b dt-[(p_c-p_b)/\rho U_b]dt,$$

$$\eta_0=[(p_c-p_b)/\rho c^2][1-(c^2/U_b^2)]. \quad (14)$$

The same result is obtained for fronts traveling toward $-x$. Thereafter $\eta$ changes at the rate $\partial u_c/\partial x$. Hence, at any time $t$ subsequent to the passage of the front at time $t_b$,

$$\eta=\int_{t_b}^{t}(\partial u_c/\partial x)dt + [(p_c-p_b)/\rho c^2][1-(c^2/U_b^2)]. \quad (15)$$

The value of $\eta$ may vary from point to point in the cavitated region. The initial value $\eta_0$ will be small, in accordance with the assumption that the pressure remains within the linear range; but subsequently $\eta$ may increase to any magnitude.

### 4. MOTION OF PLANE BREAKING-FRONTS IN TERMS OF WAVE TRAINS

Since no influence can be propagated with supersonic velocity past a breaking-front, the pressure and particle velocity ahead of the front will be determined by initial or boundary conditions elsewhere; hence, in the above formulas the values of $p$ and $u$ may be regarded as given. When, however, the boundary of the cavitated region moves more slowly than the speed of sound, the presence of cavitation is able to influence the values of $p$ and $u$ in the nearby unbroken liquid. In such cases analysis in terms of fundamental wave trains is convenient. It will be worth while to restate, also, the conditions for the propagation of a breaking-front in terms of such trains.

Any one-dimensional disturbance in unbroken liquid can be resolved into a train of waves traveling at speed $c$ toward $+x$ and another traveling toward $-x$; if $p_1$, $p_2$ denote the corresponding pressures and $u_1$ and $u_2$, particle velocities, then

$$p_1=\rho c u_1, \qquad p_2=-\rho c u_2, \quad (16a, b)$$

$$p=p_1+p_2, \qquad u=u_1+u_2, \quad (17a, b)$$

$$p_1=\tfrac{1}{2}(p+\rho c u), \quad p_2=\tfrac{1}{2}(p-\rho c u). \quad (18a, b)$$

Here the significance of $p_1$ and $p_2$ is really only mathematical, hence $u_1$, $u_2$, and $u$ need not be restricted to be small. Substitution from these equations for $u$ in (13b) gives

$$(\partial p_1/\partial x)-(\partial p_2/\partial x)>0,$$

and substitution in (13c) squared gives

$$[(\partial p_1/\partial x)+(\partial p_2/\partial x)]^2<[(\partial p_1/\partial x)-(\partial p_2/\partial x)]^2,$$

$$(\partial p_1/\partial x)(\partial p_2/\partial x)<0.$$

The first and last of these inequalities can be satisfied simultaneously only if $\partial p_1/\partial x$ is positive and $\partial p_2/\partial x$ negative. Hence the conditions for the propagation of a breaking front and Eqs. (4)

and (5) for $U_b$ and $u_c$ can be written, since $u_b=u$,

$$p_1+p_2=p_b,\quad (\partial p_1/\partial x)>0,\quad (\partial p_2/\partial x)<0; \tag{19a, b, c}$$

and, for travel toward $+x$,

$$(\partial p_1/\partial x)\geqq -(\partial p_2/\partial x), \tag{20a}$$

or, for travel toward $-x$,

$$(\partial p_1/\partial x)\leqq -(\partial p_2/\partial x); \tag{20b}$$

$$U_b=c\frac{(\partial p_1/\partial x)-(\partial p_2/\partial x)}{(\partial p_1/\partial x)+(\partial p_2/\partial x)}, \tag{21}$$

$$u_c=\frac{1}{\rho c}\left[p_1-p_2+(p_c-p_b)\times\frac{(\partial p_1/\partial x)+(\partial p_2/\partial x)}{(\partial p_1/\partial x)-(\partial p_2/\partial x)}\right]. \tag{22}$$

Equations (19a, b, c) lead to the useful conclusion that a wave of tension can give rise to cavitation only where it overruns a wave of *decreasing* pressure traveling in the opposite direction.

## 5. PLANE RESTING BOUNDARIES AND CLOSING-FRONTS

The next question to consider is the behavior of the boundary of a cavitated region when it cannot advance as a breaking front. It may stand still, relatively to the liquid, or it may advance into the cavitated region, leaving the liquid unbroken again behind it; in the latter case, the boundary may be called a *closing-front.* Which alternative occurs will depend, in general, both upon conditions in the cavitated region and upon the magnitude of the wave that is incident from the unbroken side.

Let $p'$ denote pressure in the component wave train that approaches the boundary in the unbroken liquid and $p''$ that in the receding train; let $u$ denote the particle velocity on the unbroken side and $u_c$ that on the cavitated side, but let the positive direction for velocity be taken now always toward the cavitated region. As before, let $p_c$ denote the uniform pressure in the cavitated region and $\eta$ the fraction of the space that is occupied by bubbles.

Whether the boundary remains at rest or advances is found to depend upon the magnitudes of $p'$, $u_c$, $\partial u_c/\partial x$, and $\eta$. These quantities may be regarded as given; $p''$ is then determined by conditions at the boundary, and $p$ and $u$ are given by the relations, obtained from (17a, b) and (16a, b),

$$p=p'+p'',\quad u=(1/\rho c)(p'-p''), \tag{23a, b}$$

whence

$$p+\rho cu=2p'. \tag{23c}$$

Four cases may be distinguished.

### (1) Intrinsic Closing-Fronts

$$\eta=0,\quad \partial u_c/\partial x<0,\quad -\partial u_c/\partial x\geqq c\partial\eta/\partial x>0. \tag{24a, b, c}$$

Here $x$ denotes distance measured from the boundary into the cavitated region, and the relations refer to conditions at the boundary.

Because of (24a, b), the cavitation will proceed to disappear in the layers next to the boundary; hence the boundary will advance into the cavitated region as a closing-front, leaving unbroken liquid behind it. A front of this type, being propelled by conditions inside the cavitated region, may be called an *intrinsic* closing-front.

To find its speed of advance, consider a layer of the cavitated liquid $dx$ thick (Fig. 2). If the particle velocity is $u_c$ over the face toward $-x$ or $F_1$, that over the other face $F_2$ is

$$u_c+(\partial u_c/\partial x)dx.$$

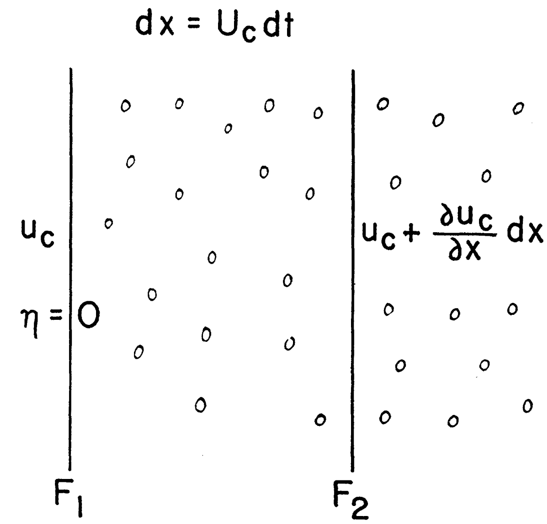


FIG. 2. Layer of cavitated liquid $dx$ thick.

Hence during a time $dt$ the volume of the layer changes by $(\partial u_c/\partial x)dxdt$, for each unit of area of its faces and, since the pressure is constant, the change must occur in the cavitated space and not in the liquid itself. Hence,

$$(\partial u_c/\partial x)dxdt = (\partial/\partial t)(\eta dx)dt,$$
$$\partial\eta/\partial t = \partial u_c/\partial x, \tag{25}$$

in which the time derivative is to be evaluated at a point moving with the liquid.

On the other hand, if $U_c$ is the velocity of advance of the boundary relatively to the liquid, $dx = U_c dt$. At the beginning of $dt$, $\eta = 0$ at $F_1$, hence at $F_2$ $\eta = (\partial\eta/\partial x)dx$; whereas at the end of $dt$, $\eta = 0$ at $F_2$. Hence, during $dt$, $\eta$ changes at $F_2$ by $d\eta = -(\partial\eta/\partial x)dx$. This can also be written $d\eta = (\partial\eta/\partial t)\partial t$. Hence, using $dx = U_c dt$, we obtain

$$(\partial\eta/\partial t) = -U_c(\partial\eta/\partial x), \tag{26}$$

and

$$U_c = -(\partial\eta/\partial t)/(\partial\eta/\partial x)$$
$$= -(\partial u_c/\partial x)/(\partial\eta/\partial x) \tag{27}$$

by (25). Here the positive directions for $x$ and $u_c$ are taken toward the cavitated region and the signs of the derivatives are such as to make $U_c$ positive.

So far as can be seen, the value of $U_c$ as given by (27) might be either greater or less than $c$. If it is greater than $c$, everything is satisfactory. The advancing front leaves behind it a region in which $p = p_c$ and $u = u_c$, $u_c$ being the local value of the particle velocity as encountered by the advancing front. The two component wave trains in the unbroken liquid are thereby progressively built out in accord with the equations

$$p' = \tfrac{1}{2}(p + \rho cu) = \tfrac{1}{2}(p_c + \rho cu_c), \tag{28a}$$

$$p'' = \tfrac{1}{2}(p - \rho cu) = \tfrac{1}{2}(p_c - \rho cu_c). \tag{28b}$$

The part of the $p'$ wave train that is already established and is moving toward the front is unable to overtake it. Even if $U_c = c$, there is no difficulty, although in this case a fixed discontinuity in $p$ and $u$ may occur at the front.

If, however, the value of $U_c$ as defined by Eq. (27) is less than $c$, a front moving at speed $U_c$ would be overtaken continually by values of $p'$ coming out of the unbroken region. In such cases conditions at the boundary must be determined in part by the arriving values of $p'$, and they will usually not be such as to satisfy Eqs. (28a, b). A different type of action must then occur at the boundary. Thus *intrinsic closing-fronts, when they occur, move through the liquid at velocities not less than that of sound.*

### (2) Resting boundary

$$2p' \leqq p_c + \rho cu_c. \tag{29}$$

Let it be assumed that the pressure is continuous at the boundary and equal, therefore, to $p_c$. Then a wave train must be reflected from the boundary of such magnitude that, by (23a, c)

$$p'' = p_c - p', \quad u = (1/\rho c)(2p' - p_c). \tag{30a, b}$$

These equations represent a possible motion provided $u \leqq u_c$, so that the unbroken liquid does not collide with the cavitated liquid; and this condition, in turn, is met provided $p'$ is such that (29) holds. It may be concluded that, whenever (29) holds but the boundary cannot advance as an intrinsic closing-front moving at supersonic velocity, the boundary will remain stationary and the incident waves will be reflected from it in accord with Eq. (30a).

If $2p'$ is actually less than $p_c + \rho cu_c$, so that by (30b) $u < u_c$, separation of the two portions of the liquid must occur, with formation of a crevasse or a special layer of expanding bubbles. If $2p' = p_c + \rho cu_c$, the unbroken liquid remains in peaceful contact with the cavitated liquid.

If, however, (29) does not hold, no solution of the type just described is possible and a different process must occur.

### (3) Forced Closing-Fronts

$$2p' > p_c + \rho cu_c, \quad \eta > 0. \tag{31a, b}$$

When condition (31a) obtains, the incident waves are so strong that the boundary advances into the cavitated region, closing up the bubbles. The action is one of successive impacts, a finite discontinuity of pressure occurring at the moving boundary if $\eta > 0$. A boundary moving in this manner may be called a *forced closing-front.*

Let $U_c$ be the speed of advance of the front relative to the cavitated liquid ahead of it.

Then, during a time $dt$, a layer of cavitated liquid of thickness $dx$ becomes consolidated. The argument is now similar to that in Section 2.

The mass $(1-\eta)\rho dx$ of liquid per unit area is accelerated from $u_c$ to $u$; the added momentum is supplied by the action of a pressure $p$ on one face and $p_c$ on the other face. Hence,

$$(1-\eta)\rho(u-u_c)dx=(p-p_c)dt,$$

and since $dx=U_c dt$,

$$p-p_c=(1-\eta)\rho U_c(u-u_c), \tag{32}$$

where $p$ and $u$ denote pressure and particle velocity in the unbroken liquid just behind the front.

Furthermore, since one face of the layer moves at speed $u$ and the other at $u_c$, its thickness changes during $dt$ by $(u_c-u)dt$. Part of this change is taken up by compression of actual liquid from $p_c$ to $p$, which changes the total thickness of liquid, initially equal to $(1-\eta)dx$, by the amount

$$-[(p-p_c)/\rho c^2](1-\eta)dx,$$

$\rho c^2$ representing the elasticity. In addition, cavitated spaces of total thickness $\eta dx$ are closed up. Hence,

$$\begin{aligned}(u_c-u)dt&=-[(p-p_c)/\rho c^2](1-\eta)dx-\eta dx,\\ \rho c^2(u-u_c)&=[\rho c^2\eta+(1-\eta)(p-p_c)]U_c.\end{aligned} \tag{33}$$

In addition, there is Eq. (23c) or

$$p+\rho cu=2p'. \tag{34}$$

Equations (32), (33), and (34) suffice to fix $U_c$, $p$, and $u$. The solution can be written in a convenient explicit form by introducing the auxiliary quantity

$$v=(1/\rho c)(2p'-p_c)-u_c.$$

According to (31a) $v$ is positive; it may be regarded as known. Then

$$U_c=cv/[v+(c-v)\eta], \tag{35}$$

$$p-p_c=\frac{(1-\eta)\rho cv^2}{2v+(c-2v)\eta},\qquad u-u_c=v\frac{v+(c-v)\eta}{2v+(c-2v)\eta}. \tag{36a, b}$$

In these equations, as in all equations not referring to breaking-fronts, $u$ and $u_c$ are taken positive toward the cavitated region.

Since $v>0$ and $1\geqq\eta>0$, these formulas make $p>p_c$ and $u>u_c$, as is obviously necessary for physical reasons. Furthermore, from (35) it is clear that $U_c<c$. Or, if we consider the velocity of the front relative to the unbroken liquid behind it or $U_c'=U_c-(u-u_c)$, we find that $U_c'<c$. Thus the front cannot run away but must remain subject to influences propagated up to it from behind at speed $c$. To sum up, *all forced closing-fronts move at a speed less than that of sound relative either to the unbroken liquid behind them or to the cavitated liquid in front.*

The requirement, $\eta>0$, has been included in the definition of forced closing-fronts because boundaries at which $\eta=0$ appear to constitute a singular class. One type of such boundaries has already been discussed.

### (4) Boundaries at which $\eta=0$

If $2p'>p_c+\rho cu_c$, a little consideration shows that the boundary will advance as a closing-front of the intrinsic type if it can do so, otherwise as one of the forced type; in the latter case, its velocity of advance is initially equal to $c$, so that any tendency to form a front closing by intrinsic action but advancing at a speed below $c$ is cut short. The front will usually advance at once into regions where $\eta>0$.

If, however, $2p'<p_c+\rho cu_c$, a resting boundary as described by Eqs. (30a, b) should certainly occur, accompanied by a finite discontinuity in the particle velocity and by the formation of a gap or a special layer of bubbles lying between the unbroken and the cavitated parts of the liquid. It appears, however, that the opposite side of this gap may also advance into the cavitated region as a closing-front, so that the original boundary splits into three, two resting boundaries separated by a gap and a closing-front.

The principal type of boundary at which $\eta=0$ is the boundary that has just ceased moving as a breaking-front. For there, usually, the inequality expressed by (13c) has just failed to exist because its members have momentarily become equal; hence by (4) at this instant, $U_b^2=c^2$ and by (14) $\eta=\eta_0=0$. At this instant, furthermore, Eq. (5)

becomes, after reversal of the signs of $u_c$ and $u_b$ because of the difference in the direction chosen here for positive velocities,

$$u_c = u_b - (p_c - p_b)/\rho c,$$

and insertion of the value of $u_b$ given by this equation for $u$ and of $p = p_b$ in (23c) gives

$$2p' = p_c + \rho c u_c.$$

The slightest increase in $p'$ will thus cause the boundary to start back as a closing-front. Otherwise it remains as a resting boundary.

### 6. THREE-DIMENSIONAL CAVITATION

Insofar as the three-dimensional theory can be formulated in compact form, it is easily written out as a generalization of the one-dimensional theory.

Equations (2a, b) are replaced by

$$\partial \mathbf{u}/\partial t = -\nabla p, \quad \partial p/\partial t = -\rho c^2 \operatorname{div} \mathbf{u}. \quad (37a, b)$$

It is clear that wherever $\partial u/\partial x$ occurs in the one-dimensional formulas as the equivalent of $\partial p/\partial t$, it is to be replaced by div $\mathbf{u}$. The theory of boundaries as given above then becomes applicable to three-dimensional cavitation, provided $x$ in the one-dimensional equations is interpreted as distance along the normal to the boundary, and provided $u$ is replaced by the component of the particle velocity taken in the direction of this normal. In addition, it is to be assumed that, as a boundary moves, no change occurs in any component of the particle velocity tangential to it, the impulsive changes being confined to the normal component.

Cavitation will begin at a point where a local minimum of pressure sinks to the breaking pressure $p_b$, and from this point a breaking-front will spread out in the form of a closed surface surrounding the point of initiation and enclosing a cavitated region. In the unbroken liquid this surface, at which $p = p_b$, constitutes a surface of uniform pressure, hence the normal to the surface lies everywhere in the direction of $\nabla p$. The conditions to be satisfied in the unbroken liquid in order that such a front may advance can at once be written down as a generalization of Eqs. (13a, b, c):

$$p = p_b, \quad \operatorname{div} \mathbf{u} > 0,$$
$$0 \leqq (\partial p/\partial n) < \rho c \operatorname{div} \mathbf{u}, \quad (38a, b, c)$$

where $n$ stands for distance measured into the unbroken liquid along the normal to the boundary. As before, the front cannot travel to the boundary of the region in which, in the absence of cavitation, the pressure would sink to $p_b$, at least provided all derivatives of $p$ are continuous functions of the space variables. For $U_b$, the velocity of propagation of the front along its normal relative to the unbroken liquid ahead, Eqs. (4) and (8) are replaced by

$$U_b = \rho c^2 \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \bigg/ \left( \frac{\partial p}{\partial n} \right) \geqq c. \quad (39)$$

In the same way the theory of the cavitated region, as developed in Section 3, can at once be generalized. Those components of the particle velocity which are tangential to the breaking-front are unaltered by the passage of the front, whereas the component perpendicular to the front, taken positive toward the unbroken liquid, is changed from $u_{bn}$ to $u_{cn}$ where

$$u_{cn} = u_{bn} + (p_c - p_b)/\rho U_b. \quad (40)$$

The fractional cavitation $\eta$, or the fraction of the space that is occupied by bubbles, is left by the passage of the front at the value stated in Eq. (14) or

$$\eta_0 = [(p_c - p_b)/\rho c^2][1 - (c^2/U_b^2)]; \quad (41)$$

after the lapse of a further time $t - t_b$ it becomes

$$\eta = \int_{t_b}^{t} \operatorname{div} \mathbf{u}_c dt + [(p_c - p_b)/\rho c^2] \times [1 - (c^2/U_b^2)]. \quad (42)$$

Here $\mathbf{u}_c$ is the particle velocity in the cavitated region, which remains constant in time as long as the cavitation lasts.

From the physical standpoint, the behavior of resting boundaries and of closing-fronts, also, is essentially the same as in the one-dimensional case. For intrinsic closing-fronts the mathematical theory likewise is similar and can at once be written down. Equation (27) for the propagation velocity of such a front, relative to the cavitated liquid, is replaced by

$$U_c = -(\text{div } \mathbf{u}_c)/(\partial\eta/\partial n). \qquad (43)$$

For a forced closing-front, the analogs of Eqs. (32) and (33) can at once be written:

$$p - p_c = (1-\eta)\rho U_c(u_n - u_{cn}), \qquad (44)$$

$$\rho c^2(u_n - u_{cn}) = [\rho c^2\eta + (1-\eta)(p-p_c)]U_c. \qquad (45)$$

Here $p$ and $u$ denote pressure and particle velocity on the unbroken side of the front and the subscript $n$ denotes the component normal to the boundary, taken positive when directed toward the side of cavitation. Components of velocity tangential to the front are unaffected by its passage. Difficulty is encountered, however, in attempting to obtain a third equation in order to determine $p$, $u$, and $U_c$, if $\eta$ and $u_{cn}$ are assumed known. Even the condition for the distinction between resting boundaries and forced closing-fronts cannot easily be formulated in the general case. The disturbance in the unbroken liquid might, indeed, be resolved into an infinitude of plane waves; but the treatment of these waves at a curved boundary is rendered very complicated by the occurrence of diffraction. All that can readily be stated is that existing conditions in the unbroken liquid, which in turn depend in part upon conditions at distant boundaries, in combination with the existing distribution of the values of $\eta$ and of the particle velocity in the cavitated region, will determine whether the boundary advances or remains at rest; and, if the boundary advances, those conditions in the unbroken liquid combine with Eqs. (44) and (45) to fix the values of $p$, $u_n$, and $U_c$.

It is unfortunate that the three-dimensional theory cannot be carried further in simple form.

A concluding comment may be added, however, concerning the special case of spherical waves.

Let

$$p_1 = \frac{1}{r} f(ct - r)$$

represent the pressure in a train of waves diverging from a fixed center, $r$ being the distance from the center and $f$ any differentiable function. Let these waves be superposed upon other waves of small amplitude represented as a whole by $p_2(x, y, z, t)$.

In this case relation (38c) may conveniently be written, by means of (37b) and the relation, $p = p_1 + p_2$,

$$-\frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t} > c\left(\frac{\partial p_1}{\partial n} + \frac{\partial p_2}{\partial n}\right) \geqq 0.$$

If the normal along which $n$ is measured makes an angle $\theta$ with the direction of $r$,

$$\frac{\partial p_1}{\partial n} = \frac{\partial p_1}{\partial r}\cos\theta = -\left(\frac{1}{c}\frac{\partial p_1}{\partial t} + \frac{p_1}{r}\right)\cos\theta;$$

hence, the preceding equation, expressing the necessary condition for the propagation of a breaking-front, may be written

$$-(1 - \cos\theta)\frac{\partial p_1}{\partial t} + \frac{cp_1}{r}\cos\theta > \frac{\partial p_2}{\partial t} + c\frac{\partial p_2}{\partial n}. \qquad (46)$$

From this result it is clear that a breaking-front cannot follow a diverging spherical wave of tension into quiescent liquid ($p_1 < 0$, $p_2 = 0$); for in such a case $\theta = 0$, since over the front $p = p_b = p_1$. To cause cavitation, a spherical wave of tension must overrun other waves of such character that the right-hand member of Eq. (46) has sufficiently large negative values.

## 7. ENERGETIC AND THERMODYNAMICAL CONSIDERATIONS

The preceding equations have been based solely upon the conservation of matter and of momentum. An investigation, of which no details will be given, reveals, however, that, if $p_b < p_c$, the passage of either a breaking-front or a forced closing-front is likely to leave behind it a greater amount of energy than is accounted for by the elastic and kinetic energy of the cavitated water. Upon closer examination of the cavitation process, it appears probable that this excess energy is either dissipated or radiated away in consequence of local oscillations associated with the impulsive production or destruction of bubbles. Thus the theory involves no conflict with thermodynamics.

A difference between the breaking-pressure $p_b$ and the cavity pressure $p_c$ implies, however, an instability of the liquid which might seem to open the way for other types of action than those assumed above. There exists in fact, no hydrodynamic or thermodynamic reason why cavitation should not occur at any pressure below $p_c$. An analogy might be expected with the behavior of moderately supercooled liquid, in which freezing must be initiated by external causes but, once started, spreads rapidly until the temperature of the mass has been raised to the freezing point. It might be anticipated, therefore, that a breaking-front, if unable to advance further at the pressure of spontaneous cavitation, $p_b$, would continue to advance at higher pressure $p_b'$ until it encountered a pressure equal to $p_c$.

Against such an assumption the following objection may be raised. It can be shown that even on this alternative hypothesis a breaking-front must advance through the liquid at supersonic velocity. But the advance into regions of pressure higher than $p_b$ would necessarily occur, not as a consequence of automatic changes in the unbroken liquid ahead, but as an effect propagated out of the region already cavitated; and it seems unlikely, although certainly not impossible, that such an effect could be propagated at a speed exceeding that of sound.

Which assumption corresponds more nearly with the behavior of actual liquids remains to be discovered by experiment.

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# Thermal Diffusion with Ammonia

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With a "two-bulb" thermal diffusion experiment in which is used $NH_3$ having about a 15 percent $N^{15}$ content to increase the accuracy of the mass spectrometer analyses, it has been found that with decreasing temperature the thermal diffusion constant $\alpha$ of ammonia changes from $+$ to $-$ values at about room temperature. The value of $\alpha$ varies linearly with the logarithm of the absolute temperature, the rate of decrease being, however, nearly eight times that for neon and argon. A qualitative discussion is presented, attributing this effect largely to the strong first-order dipole-dipole intermolecular forces which are proportional to $1/R^4$.

IN general the phenomenon of thermal diffusion operates to produce an increased concentration of the heavier component in a binary gas mixture in the colder portion of the apparatus, while the lighter component tends to concentrate in the warmer part. For these cases there is a positive thermal diffusion constant $\alpha$ in the equation $D_T/D=\alpha c_1 c_2$, where $D_T$ is the coefficient of thermal diffusion, $D$ is the coefficient of ordinary diffusion, and $c_1$ and $c_2$ are the relative concentrations of the light and heavy components, respectively. In fact, until just recently no instance where thermal diffusion proceeds with a negative thermal diffusion constant had been reported. Grew[1] has now found a reversal in the sign of $\alpha$ with change in the composition of a neon-ammonia mixture. Between 0 and 75 percent neon, the heavier molecule, neon, tends to concentrate at the upper end of a Clusius-Dickel column. This indicates that the $\alpha$ is negative. Above 75 percent neon, the neon concentrates at the lower end, so that in this range $\alpha$ is positive. The theoretical possibility of such a reversal had been pointed out by Chapman.[2]

A definite temperature variation of the positive $\alpha$ for neon was found by Nier,[3] the value of $\alpha$ decreasing with the lowering of the mean temperature. Jones[4] has shown theoretically that a variation similar to that reported by Nier is to be expected for either the Sutherland or the Lennard-Jones 9,5 molecular models. This theoretical investigation indicated that, for tem-

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