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Strong Coupling Mesotron Theory of Nuclear Forces

R. SERBER AND S. M. DANCOFF
University of Illinois, Urbana, Illinois
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A theory of nuclear forces is developed, based on the hypothesis that the interaction between a nuclear particle and the mesotron field is strong. Two types of mesotron fields are considered, the charged scalar and the neutral pseudoscalar. The latter, for large enough separation, gives forces between two nuclear particles of the same type as those obtained from perturbation theory and, hence, with the spin dependence and exchange properties required to fit experiment. However, at closer approach the forces become ordinary (non-spin-dependent). It is found impossible to obtain spin dependent forces which, at the same time, extend to small separations and are of sufficient strength to account for the properties of the deuteron.

I. INTRODUCTION

IT is well known that the interaction between mesotrons and nuclear particles cannot be treated as small; such perturbation treatments are not only inconsistent—they lead to inacceptable nuclear forces and to a much too large scattering cross section for the mesotron.* Oppenheimer and Schwinger[1] have shown that the strong coupling theory affords an explanation of the smallness of the scattering cross section; the question of nuclear forces was left open by these authors.

It is not difficult to see that in the limit of very strong coupling, the nuclear forces will not be right. For this there are two related reasons:

(1) For strong enough coupling, the mesotron field due to the interaction of two nuclear particles becomes large compared to the field of a single mesotron; field fluctuations become unimportant, and the equations of the field may be treated classically. The classical solutions of the field equations of course exhibit the same singularities as arise in the lowest order perturbation theory. (2) At the same time, the interaction energy becomes large compared to the excitation energy of proton and neutron isobars, so that it is energetically favorable to excite many isobaric states; the total spin of each particle (intrinsic spin or isotopic spin, plus spin or isotopic spin of the associated mesotron field) becomes effectively very large, and behaves also in an essentially classical manner. The spins are thus free to orient themselves in such a way as to reduce the potential energy to a minimum. It follows that the interaction energy is an ordinary potential, non-exchange and spin independent, with singularities as bad as those of the perturbation theory. In the following sections it will be shown that these expectations are fully borne out by the detailed quantum-mechanical calculations.

It is, of course, true that with fixed magnitude

* See W. Pauli, Abstract No. 25, Bull. Am. Phys. Soc. **18**, No. 1 (1943). Professor Pauli considers the possibility of removing divergent terms associated with a point source by a subtraction formalism, while reducing the scattering cross section by means of radiation reaction.

[1] J. R. Oppenheimer and J. Schwinger, Phys. Rev. **60**, 150 (1941).

of the coupling constant, the interaction energy between two nuclear particles becomes small as the distance between them increases; for sufficiently large separation, the interaction energy becomes smaller than the energy of excitation of isobars, and beyond this point the forces change over to the type predicted by perturbation theory. The question that now arises, and which is the subject of our investigation, is whether a theory with an intermediate coupling strength can give a satisfactory account of the forces and whether the coupling constant can be so chosen that the region of spin dependent exchange forces is sufficiently extensive and the forces in this region sufficiently strong.

As we have just seen, for a theory with spin dependent coupling the nuclear forces remain spin dependent only for separations large enough to make

$$g^2e^{-\kappa r}/\kappa r<(\kappa a)/g^2,$$

where g is the coupling constant, a the radius of a nuclear particle, and $(\kappa)^{-1}$ the range of the forces. The observed scattering cross section demands $\kappa a<0.1$. It should be remarked that the condition for the validity of a perturbation treatment is that the self-energy of a nucleon due to its interaction with the field be small compared to the mesotron rest energy, $g^2/(\kappa a)^3<1$. Weak coupling would thus require $g^2<(\kappa a)^3\sim 0.001$. Forces derived from these constants are 100 times weaker than actual nuclear forces. Thus we have only to consider $g^2\gg(\kappa a)^3$, or strong coupling.

The second decisive point in our investigation is this: Although for $g^2e^{-\kappa r}/\kappa r<(\kappa a)/g^2$, the forces given by the strong coupling theory are of the same type and radial dependence as those of perturbation theory, they are reduced by a numerical factor f : $\frac{1}{4}$ for the charged scalar, $\frac{1}{9}$ for neutral pseudoscalar, charged and symmetrical pseudoscalar.[2] These factors arise because the spin and charge of a nucleon oscillate with high frequency in their own fields even in the absence of a second nucleon; their components responsible for the nuclear interaction have smaller expectation values.

[2] We are indebted to Professor Pauli for the results in the charged and symmetrical pseudoscalar cases.

If now $g^2<1$, the spin dependent forces are too small; if $g^2>1$, the forces are spin dependent only for $\kappa r>\sim 2$, and their maximum depth will be less than $\sim 10^{-2}\mu c^2$ or 1 Mev. Thus we see that with $\kappa a\leqslant 0.1$, no value of g^2 gives spin dependent forces large enough to agree with experience.[3]

It may be remarked that values of a of the order $(\kappa)^{-1}$ not only conflict with the data on mesotron scattering, but essentially render nugatory a field theory of forces, since for $r\sim a$ these forces are entirely determined by the nature of the source, and not by that of the field.

II. CLASSICAL THEORY

As has been pointed out in the introduction, the main features of the strong coupling theory of nuclear forces can already be seen in a classical theory. We shall, therefore, begin with an investigation of the symmetrical scalar theory in the classical limit.

The heavy particle will be supposedly spread over a finite region of radius a with a density $U(\mathbf{r})$, $\int d\mathbf{r}U(\mathbf{r})=1$. In accordance with our above remarks concerning the size of the source we suppose $\kappa a\ll 1$ where $\kappa=\mu c/\hbar$ and μ is the mesotron's rest mass. The Hamiltonian of the symmetrical scalar theory when two heavy particles are present is

$$H=\frac{1}{2}\int d\mathbf{r}[\{\boldsymbol{\pi}(\mathbf{r})\}^2+\boldsymbol{\phi}(\mathbf{r})\cdot\omega^2\boldsymbol{\phi}(\mathbf{r})]$$

$$-g(4\pi)^{\frac{1}{2}}\int d\mathbf{r}[U_a(\mathbf{r})\boldsymbol{\tau}_a\cdot\boldsymbol{\phi}(\mathbf{r})+U_b(\mathbf{r})\boldsymbol{\tau}_b\cdot\boldsymbol{\phi}(\mathbf{r})].\quad(1)$$

Here U_a and U_b are the source functions of the two heavy particles, $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ their isotopic spins. The wave function $\boldsymbol{\phi}$ is a vector in $\boldsymbol{\tau}$ space; its x and y components are the real and imaginary parts of the charged field, its z component is the neutral field. The operator $\omega^2=(\kappa^2-\Delta)$, whereas the coupling constant g is related to the dimensionless parameter g' through $g=g'(\hbar c)^{\frac{1}{2}}$.[4] The

[3] According to investigations of Nelson and Oppenheimer (unpublished) strong coupling pair theories give a mesotron scattering cross section equal to the square of the range of the forces, and forces which are therefore inacceptable. In addition, in these theories it is impossible to combine spin dependence with saturation.

[4] We adopt rational units throughout: $\hbar=c=1$.

equations of motion are:

$$\ddot{\phi}+\omega^2\phi=g(4\pi)^{\frac{1}{2}}(\boldsymbol{\tau}_a U_a+\boldsymbol{\tau}_b U_b), \quad (2)$$

$$\hbar\dot{\boldsymbol{\tau}}_a=-g(4\pi)^{\frac{1}{2}}\boldsymbol{\tau}_a\times\int d\mathbf{r}\phi(\mathbf{r})U_a(\mathbf{r}), \quad (3)$$

$$\hbar\dot{\boldsymbol{\tau}}_b=-g(4\pi)^{\frac{1}{2}}\boldsymbol{\tau}_b\times\int d\mathbf{r}\phi(\mathbf{r})U_b(\mathbf{r}).$$

These equations will be treated classically; that is, we shall ignore all commutators of the quantities which appear.

For strong coupling, $g\gg 1$, we can find a class of solutions which are non-radiating (i.e., involve only frequencies smaller than κ, and thus no wave zone field). We take

$$\boldsymbol{\tau}_a=\boldsymbol{\tau}_b=\boldsymbol{\tau}_0+\boldsymbol{\tau}_1 e^{ivt}, \quad \boldsymbol{\tau}_a^2=\boldsymbol{\tau}_b^2=\boldsymbol{\tau}_0^2+\boldsymbol{\tau}_1^2=1.$$

The real and imaginary parts of the complex vector $\boldsymbol{\tau}_1 e^{ivt}$ are the components of $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ along two axes which form, with $\boldsymbol{\tau}_0$, an orthogonal axis system. The solution of (2) is then

$$\phi(\mathbf{r})=g(4\pi)^{\frac{1}{2}}\boldsymbol{\tau}_0\int Y(\mathbf{r},\mathbf{r}')[U_a(\mathbf{r}')+U_b(\mathbf{r}')]d\mathbf{r}'$$

$$+g(4\pi)^{\frac{1}{2}}\boldsymbol{\tau}_1 e^{ivt}\int Y_v(\mathbf{r},\mathbf{r}')[U_a(\mathbf{r}')+U_b(\mathbf{r}')]d\mathbf{r}'. \quad (4)$$

Where Y is the Yukawa potential:

$$Y(\mathbf{r},\mathbf{r}')=1/4\pi\frac{\exp(-\kappa|r-r'|)}{|\mathbf{r}-\mathbf{r}'|}$$

and

$$Y_v(\mathbf{r},\mathbf{r}')=1/4\pi\frac{\exp[-(\kappa^2-v^2)^{\frac{1}{2}}|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}.$$

This expression for ϕ substituted in (3) gives

$$iv\boldsymbol{\tau}_1=4\pi g^2(\boldsymbol{\tau}_0\times\boldsymbol{\tau}_1)\int\int d\mathbf{r}d\mathbf{r}'[Y(\mathbf{r},\mathbf{r}')-Y_v(\mathbf{r},\mathbf{r}')][U_a(\mathbf{r}')+U_b(\mathbf{r}')]U_a(\mathbf{r}) \quad (5)$$

or, expanding Y_v in powers of v to order v^2:

$$iv\boldsymbol{\tau}_1=g^2(\boldsymbol{\tau}_0\times\boldsymbol{\tau}_1)(v^2/2\kappa)(R+S)$$

with

$$R=\int\int d\mathbf{r}d\mathbf{r}'U_a(\mathbf{r})U_a(\mathbf{r}')\exp(-\kappa|\mathbf{r}-\mathbf{r}'|)\sim 1, \quad (\kappa a\ll 1)$$

$$S=\int\int d\mathbf{r}d\mathbf{r}'U_a(\mathbf{r})U_b(\mathbf{r}')\exp(-\kappa|\mathbf{r}-\mathbf{r}'|)\sim e^{-\kappa d},$$

d being the separation of the two sources. Hence we obtain a solution if

$$v=2\kappa/g^2(R+S)\tau_0. \quad (6)$$

Solutions with $\boldsymbol{\tau}_a$ not parallel to $\boldsymbol{\tau}_b$ give high frequencies of order $4\pi g^2\int\int d\mathbf{r}d\mathbf{r}'U_a(\mathbf{r})U_b(\mathbf{r}')Y(\mathbf{r},\mathbf{r}')$ in contrast to the quasi-static solution just found.

The energy associated with our quasi-static fields can be calculated from (1) which gives, to order $1/g^2$:

$$H=\frac{2\kappa}{g^2(R+S)}\frac{\boldsymbol{\tau}_1^2}{\boldsymbol{\tau}_0^2}-g^2(I+J), \quad (7)$$

$$I=4\pi\int\int d\mathbf{r}d\mathbf{r}'U_a(\mathbf{r})U_a(\mathbf{r}')Y(\mathbf{r},\mathbf{r}')\sim 1/a, \quad (\kappa a\ll 1),$$

$$J=4\pi\int\int d\mathbf{r}d\mathbf{r}'U_a(\mathbf{r})U_b(\mathbf{r}')Y(\mathbf{r},\mathbf{r}')\sim e^{-\kappa d}/d, \quad (d\gg a).$$

The dominant g^2 term contains the self-energy of the two heavy particles, $-g^2I$, and the interaction energy between them, $-g^2J$. The small term in $1/g^2$ may be rewritten in terms of the charge vector $\mathbf{Q}=\int(\phi\times\pi)d\mathbf{r}$, whose z component gives the mesotron charge in units e. Using (4) we find $\mathbf{Q}=-2\boldsymbol{\tau}_1 e^{ivt}$, in agreement with the constancy of the total charge vector $\mathbf{Q}+\boldsymbol{\tau}_a+\boldsymbol{\tau}_b$. The first term in (7) is thus

$$\frac{\kappa}{2g^2(R+S)}\frac{Q^2}{1-Q^2}$$

and represents the classical analogue of the heavy particle isobar energy discussed by Wentzel.[5] This non-radiating solution can be obtained only for values of the charge $Q<1$.

The most striking feature of the classical calculation is that, for the solution of minimum energy, there is a strong coupling between the

[5] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940).

directions of the isotopic spins of the heavy particles causing them to line up parallel to each other. This has as a consequence that the interaction energy $-g^2J$ is an "ordinary" potential, completely independent of isotopic spin. It will be observed also that for $a\to 0$ the singularity in the potential is of the same type as yielded by perturbation theoretic treatment.

This treatment of the classical theory is due to Professor J. R. Oppenheimer. We are very grateful to him for permission to quote his results.

III. CHARGED SCALAR THEORY

We shall begin our consideration of quantum field theory with a discussion of the charged scalar field, the mesotron theory originally proposed by Yukawa. Although this simple theory cannot be expected to yield the observed nuclear forces, all the cogent results can already be obtained—no essentially new factors appear in the more complex fields which have been invoked to explain nuclear forces.

A. One Nuclear Particle

A treatment of a single heavy particle coupled to the charged scalar field has been given by Schwinger.[6] However, we wish to introduce a somewhat different mathematical treatment than was used by Schwinger, or by the present authors in their original work.[7] The change consists in expanding the fields in terms of the Yukawa function $\omega^{-2}U(\mathbf{r})$, instead of in terms of the source function $U(\mathbf{r})$ itself. This more convenient treatment was suggested by Professor W. Pauli and it is with deep appreciation that we acknowledge his permission to use it in this paper.

The Hamiltonian of the heavy particle plus mesotron field is

$$H=\int d\mathbf{r}[\bar{\pi}(\mathbf{r})\pi(\mathbf{r})+\bar{\phi}(\mathbf{r})\omega^2\phi(\mathbf{r})] +g(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\,U(\mathbf{r})[\tau_-\bar{\phi}(\mathbf{r})+\tau_+\phi(\mathbf{r})]. \quad (8)$$

The notation is the same as that used in Section II except that $\phi(\mathbf{r})$ is a complex mesotron field amplitude and $\pi(\mathbf{r})$ its canonical conjugate.

$$[\pi(\mathbf{r}),\,\phi(\mathbf{r}')]=-i\delta(\mathbf{r}-\mathbf{r}'). \quad (9)$$

The operators τ_- and τ_+ change the projection of the heavy particle's isotopic spin from proton to neutron and from neutron to proton, respectively. We introduce also

$$N=4\pi\int d\mathbf{r}[U(\mathbf{r})]^2,$$

$$\mathrm{X}(\mathbf{r})=4\pi\omega^{-2}U(\mathbf{r}),$$

that is

$$(\kappa^2-\Delta)\mathrm{X}(\mathbf{r})=4\pi U(\mathbf{r}),$$

$$I=\int d\mathbf{r}\mathrm{X}(\mathbf{r})\,U(\mathbf{r}), \quad (10)$$

$$\xi(\mathbf{r})=\mathrm{X}(\mathbf{r})/I,$$

so that

$$\int d\mathbf{r}\xi(\mathbf{r})\,U(\mathbf{r})=1.$$

It is convenient to split ϕ and π into "coupled" and "uncoupled" components as follows:

$$\phi(\mathbf{r})=(4\pi)^{-\frac{1}{2}}\phi_0\xi(\mathbf{r})+\phi_1(\mathbf{r}),$$
$$\pi(\mathbf{r})=(4\pi)^{\frac{1}{2}}\pi_0 U(\mathbf{r})+\pi_1(\mathbf{r}), \quad (11)$$

where we require of the functions $\phi_1(\mathbf{r})$ and $\pi_1(\mathbf{r})$ that

$$\int d\mathbf{r}\phi_1(\mathbf{r})\,U(\mathbf{r})=\int d\mathbf{r}\pi_1(\mathbf{r})\xi(\mathbf{r})=0. \quad (12)$$

ϕ_0 and π_0 are canonically conjugate expansion amplitudes. They can be determined by

$$\phi_0=(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\phi(\mathbf{r})\,U(\mathbf{r}),$$
$$\pi_0=(4\pi)^{-\frac{1}{2}}\int d\mathbf{r}\pi(\mathbf{r})\xi(\mathbf{r}). \quad (13)$$

$\pi_1(\mathbf{r})$ and $\phi_1(\mathbf{r})$, however, are not exact canonical conjugates, but satisfy the commutation relation

$$[\pi_1(\mathbf{r}),\,\phi_1(\mathbf{r}')]=-i[\delta(\mathbf{r}-\mathbf{r}')-U(\mathbf{r})\xi(\mathbf{r}')]. \quad (14)$$

[6] J. Schwinger, unpublished.
[7] S. M. Dancoff and R. Serber, Phys. Rev. **61**, 394 (1942).

In terms of these variables

$$H=N\bar{\pi}_0\pi_0+\bar{\phi}_0\phi_0/I+g(\tau_-\bar{\phi}_0+\tau_+\phi_0)$$
$$+\int d\mathbf{r}[\bar{\phi}_1(\mathbf{r})\omega^2\phi_1(\mathbf{r})+\bar{\pi}_1(\mathbf{r})\pi_1(\mathbf{r})]$$
$$+(4\pi)^{\frac{1}{2}}\bar{\pi}_0\int d\mathbf{r}\pi_1(\mathbf{r})U(\mathbf{r})$$
$$+(4\pi)^{\frac{1}{2}}\pi_0\int d\mathbf{r}\bar{\pi}_1(\mathbf{r})U(\mathbf{r}). \quad (15)$$

Introduce polar coordinates:

$$\begin{aligned}\phi_0&=q_0e^{-i\theta},\\ \pi_0&=\tfrac{1}{2}e^{i\theta}[p_0+(i/q_0)(p_\theta+\tfrac{1}{2}-\Sigma)],\\ \bar{\phi}_0&=q_0e^{i\theta},\\ \bar{\pi}_0&=\tfrac{1}{2}e^{-i\theta}[p_0-(i/q_0)(p_\theta-\tfrac{1}{2}-\Sigma)],\end{aligned} \quad (16)$$

$$\begin{aligned}\phi_1(\mathbf{r})&=q_1(\mathbf{r})e^{-i\theta}, & \pi_1(\mathbf{r})&=p_1(\mathbf{r})e^{i\theta},\\ \bar{\phi}_1(\mathbf{r})&=\bar{q}_1(\mathbf{r})e^{i\theta}, & \bar{\pi}_1(\mathbf{r})&=\bar{p}_1(\mathbf{r})e^{-i\theta},\end{aligned} \quad (17)$$

where

$$\Sigma=-i\int d\mathbf{r}[\pi_1(\mathbf{r})\phi_1(\mathbf{r})-\bar{\pi}_1(\mathbf{r})\bar{\phi}_1(\mathbf{r})].$$

Here (p_0, q_0) and (p_θ, θ) are real canonically conjugate variables. The total mesotronic charge is given by

$$-i\int d\mathbf{r}[\pi(\mathbf{r})\phi(\mathbf{r})-\bar{\pi}(\mathbf{r})\bar{\phi}(\mathbf{r})]=p_\theta. \quad (18)$$

The complex variables $p_1(\mathbf{r})$, $q_1(\mathbf{r})$ satisfy the same commutation relation as do $\pi_1(\mathbf{r})$, $\phi_1(\mathbf{r})$, namely, (14). The Hamiltonian becomes

$$H=(N/4)[p_0{}^2+(1/q_0{}^2)(p_\theta{}^2-1/4)]$$
$$+q_0{}^2/I+gq_0(\tau_-e^{i\theta}+\tau_+e^{-i\theta})$$
$$+\int d\mathbf{r}[\bar{q}_1(\mathbf{r})\omega^2q_1(\mathbf{r})+\bar{p}_1(\mathbf{r})p_1(\mathbf{r})]$$
$$+(4\pi)^{\frac{1}{2}}(p_0/2)\int d\mathbf{r}\,U(\mathbf{r})[p_1(\mathbf{r})+\bar{p}_1(\mathbf{r})]$$
$$+(4\pi)^{\frac{1}{2}}(i/2q_0)p_\theta\int d\mathbf{r}\,U(\mathbf{r})$$
$$\times[\bar{p}_1(\mathbf{r})-p_1(\mathbf{r})].^{8} \quad (19)$$

We transform to a representation in which the interaction energy is diagonal. The eigenstates of the heavy particle for this term are not proton and neutron, respectively, but are derived from the latter by a unitary transformation $F'=SF$ where S is the unitary operator $\exp(i\tau_z\theta/2)$. When the states of the heavy particle are represented as proton and neutron τ_-, τ_+, and τ_z are given by the matrices

$$\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

After the S transformation[9] the Hamiltonian becomes $H'=SHS^{-1}$. In particular the expression $(\tau_-e^{i\theta}+\tau_+e^{-i\theta})$ is transformed into $(\tau_-+\tau_+)=\tau_x$. If we choose for the states of the heavy particle a representation in which τ_x is diagonal, the interaction energy is found to be diagonalized with eigenvalues $\pm gq_0$. On the other hand, off diagonal terms are introduced elsewhere in H, since $p_\theta\rightarrow p_\theta+\tau_z/2$ and in this representation τ_z is off diagonal. In the strong coupling limit, we may choose the lowest value of the interaction energy and neglect terms linear in τ_z, as coupling widely separated states. The condition implied is

$$g/p_\theta\gg 1. \quad (20)$$

The Hamiltonian becomes

$$H'=(N/4)(p_0{}^2+p_\theta{}^2/q_0{}^2)+q_0{}^2/I-gq_0+\cdots. \quad (21)$$

The term linear in q_0 is eliminated by a shift in origin:

$$Q_0=q_0-gI/2. \quad (22)$$

Since $gI/2$ is a constant, p_0 is conjugate to Q_0. We get

$$H'=(N/4)(p_0{}^2+p_\theta{}^2/q_0{}^2)$$
$$-g^2I/4+Q_0{}^2/I+\cdots. \quad (23)$$

The leading term, $-g^2I/4$, gives the energy, to order g^2, of the static field bound to the heavy particle.

It is convenient to resolve the complex components of the field into real and imaginary parts:

$$\begin{aligned}p_1(\mathbf{r})&=(1/\sqrt{2})[p_x(\mathbf{r})+ip_y(\mathbf{r})],\\ q_1(\mathbf{r})&=(1/\sqrt{2})[q_x(\mathbf{r})-iq_y(\mathbf{r})].\end{aligned} \quad (24)$$

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[8] $p_0{}^2$ is a symbolic expression for $(1/q_0)p_0q_0p_0$. We have neglected terms of the type Σ/q_0 because, as the later work shows, Σ vanishes in the absence of free (unbound) mesotrons. Such terms are of higher order in $1/g$ and are negligible if $g/p_\theta\gg 1$.

[9] For previous use of the S transformation, see W. Pauli, Helv. Phys. Acta **12**, 147 (1930); also reference 12.

Here again (p_x, q_x) and (p_y, q_y) are real pseudo-conjugate pairs satisfying the commutation relations (14). The Hamiltonian separates into two commuting parts:

$$H' = -g^2 I/4 + H_x + H_y,$$

$$H_x = \tfrac{1}{4} N p_0^2 + q_0^2/I + \tfrac{1}{2}\int d\mathbf{r}\, q_x(\mathbf{r})\omega^2 q_x(\mathbf{r}) + (4\pi)^{\frac{1}{2}} p_0/\sqrt{2}\int d\mathbf{r}\, p_x(\mathbf{r})\, U(\mathbf{r}) + \tfrac{1}{2}\int d\mathbf{r}[p_x(\mathbf{r})]^2, \quad (25)$$

$$H_y = N p_\theta^2/4q_0^2 + \tfrac{1}{2}\int d\mathbf{r} \times [q_y(\mathbf{r})\omega^2 q_y(\mathbf{r}) + p_y(\mathbf{r})p_y(\mathbf{r})] + (p_\theta/\sqrt{2}q_0)(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\, p_y(\mathbf{r})\, U(\mathbf{r}).$$

H_x gives the energy of free oscillations about the equilibrium position $q_0 = gI/2$. For, if we construct the functions,

$$\phi_x(\mathbf{r}) = (4\pi)^{-\frac{1}{2}} Q_0 \xi(\mathbf{r}) + (1/\sqrt{2}) q_x(\mathbf{r}),$$

$$\pi_x(\mathbf{r}) = \sqrt{\pi}\, p_0 U(\mathbf{r}) + (1/\sqrt{2}) p_x(\mathbf{r}).$$

Then ϕ_x and π_x are canonically conjugate and

$$H_x = \int d\mathbf{r}\{[\pi_x(\mathbf{r})]^2 + \phi_x(\mathbf{r})\omega^2\phi_x(\mathbf{r})\}. \quad (26)$$

H_x is diagonalized in plane waves and represents a field of unscattered mesotrons.

Meanwhile H_y can be transformed into a similar form if we first eliminate the term linear in $p_y(r)$ by applying a unitary transformation.

$$F = e^{iU} F' e^{-iU} = F' + i[U, F] + \frac{i^2}{2!}[U, [U, F]] + \cdots, \quad (27)$$

where

$$U = \frac{(4\pi)^{\frac{1}{2}}\sqrt{2}\, p_\theta}{g\int d\mathbf{r}[\mathrm{X}(\mathbf{r})]^2}\int d\mathbf{r}\,\mathrm{X}(\mathbf{r}') q_y'(\mathbf{r}'). \quad (28)$$

This produces a canonical transformation from the variables (p_y, q_y) to (p_y', q_y'), each pair satisfying (14). To first order in $1/g$:

$$p_y(\mathbf{r}) = p_y'(\mathbf{r}) - (4\pi)^{\frac{1}{2}}\sqrt{2}(p_\theta/g)\left[U(\mathbf{r})/I - \mathrm{X}(\mathbf{r})\Big/\int d\mathbf{r}[\mathrm{X}(\mathbf{r})]^2\right] + \cdots, \quad (29)$$

$$q_y(\mathbf{r}) = q_y'(\mathbf{r}) + \cdots.$$

It can be shown that the criterion for neglect of succeeding terms in (29) is, once again, $g/p_\theta \gg 1$. If (29) is substituted into (25), the linear term is eliminated, and

$$H_y = 4\pi p_\theta^2\Big/ g^2\int d\mathbf{r}\{\mathrm{X}(\mathbf{r})\}^2 + \tfrac{1}{2}\int d\mathbf{r}[\{p_y'(\mathbf{r})\}^2 + q_y'(\mathbf{r})\omega^2 q_y'(\mathbf{r})]. \quad (30)$$

Here we have substituted for q_0 the equilibrium value $gI/2$ in accordance with the strong coupling approximation, a procedure which can be proved valid for $g \gg 1$. The latter two terms in (30) would, in analogy to H_x, respesent the energy of a set of free mesotrons except that p_y' and q_y' are not exact canonical conjugates. We will not here calculate the eigenstates of H_y; we will only remark that they can be represented at large distances as plane+scattered wave, leading to a scattering of free mesotrons by the heavy particle.[10] Since we are interested only in the heavy particle in the absence of unbound mesotrons, we may assume that none of the unscattered "x" waves nor any of the scattered "y" waves are present. The total energy then reduces to

$$H = -g^2 I/4 + 4\pi p_\theta^2\Big/ g^2\int d\mathbf{r}\{\mathrm{X}(\mathbf{r})\}^2. \quad (31)$$

The latter term thus appears as a correction to the static self-energy of order $1/g^2$. Since for the states under consideration the expectation value of $\tau_z = 0$, it follows that the expectation value of the charge of the heavy particle core $= \frac{1}{2}$. Hence, if Q is the total charge of the system $p_\theta = Q - \frac{1}{2}$. For small sources, $(\kappa a \gg 1)$,

$$(1/4\pi)\int d\mathbf{r}\{\mathrm{X}(\mathbf{r})\}^2 = 1/2\kappa.$$

[10] J. Schwinger, unpublished.

Hence the isobar energy takes the form

$$2\mu c^2(Q-\tfrac{1}{2})^2/g^2. \tag{32}$$

We are restricted by our approximation to isobars for which $(Q-\frac{1}{2})/g\ll 1$.

B. Two Nuclear Particles

The Hamiltonian is

$$H=\int d\mathbf{r}[\bar{\pi}(\mathbf{r})\pi(\mathbf{r})+\bar{\phi}(\mathbf{r})\omega^2\phi(\mathbf{r})]$$
$$+g(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\{U^A(\mathbf{r})[\tau_-{}^A\bar{\phi}(\mathbf{r})+\tau_+{}^A\phi(\mathbf{r})]$$
$$+U^B(\mathbf{r})[\tau_-{}^B\bar{\phi}(\mathbf{r})+\tau_+{}^B\phi(\mathbf{r})]\}. \tag{33}$$

The procedure parallels closely that for a single heavy particle. We introduce

$$\mathrm{X}^A(\mathbf{r})=4\pi\omega^{-2}U^A(\mathbf{r});$$
$$\mathrm{X}^B(\mathbf{r})=4\pi\omega^{-2}U^B(\mathbf{r});$$
$$N=4\pi\int d\mathbf{r}[U^A(\mathbf{r})]^2=4\pi\int d\mathbf{r}[U^B(\mathbf{r})]^2,$$
$$M=4\pi\int d\mathbf{r}[U^A(\mathbf{r})U^B(\mathbf{r})],$$

and define

$$I=\int d\mathbf{r}\mathrm{X}^A(\mathbf{r})U^A(\mathbf{r})=\int d\mathbf{r}\mathrm{X}^B(\mathbf{r})U^B(\mathbf{r}),$$
$$J=\int d\mathbf{r}\mathrm{X}^A(\mathbf{r})U^B(\mathbf{r})=\int d\mathbf{r}\mathrm{X}^B(\mathbf{r})U^A(\mathbf{r}).$$

We now introduce functions orthogonal to the U's:

$$\int d\mathbf{r}\xi^A(\mathbf{r})U^B(\mathbf{r})=\int d\mathbf{r}\xi^B(\mathbf{r})U^A(\mathbf{r})=0,$$
$$\int d\mathbf{r}\xi^A(\mathbf{r})U^A(\mathbf{r})=\int d\mathbf{r}\xi^B(\mathbf{r})U^B(\mathbf{r})=1.$$

Such functions are

$$\xi^A=\frac{I\mathrm{X}^A-J\mathrm{X}^B}{I^2-J^2};\quad \xi^B=\frac{I\mathrm{X}^B-J\mathrm{X}^A}{I^2-J^2}. \tag{34}$$

We expand ϕ, π into coupled and uncoupled components:

$$\begin{aligned}\phi(\mathbf{r})&=(4\pi)^{-\frac{1}{2}}\{\phi_0{}^A\xi^A(\mathbf{r})+\phi_0{}^B\xi^B(\mathbf{r})\}+\phi_1(\mathbf{r}),\\ \pi(\mathbf{r})&=(4\pi)^{\frac{1}{2}}\{\pi_0{}^AU^A(\mathbf{r})+\pi_0{}^BU^B(\mathbf{r})\}+\pi_1(\mathbf{r}),\end{aligned} \tag{35}$$

where we require

$$\begin{aligned}&\int d\mathbf{r}\phi_1(\mathbf{r})U^A(\mathbf{r})=\int d\mathbf{r}\phi_1(\mathbf{r})U^B(\mathbf{r})=0,\\ &\int d\mathbf{r}\pi_1(\mathbf{r})\xi^A(\mathbf{r})=\int d\mathbf{r}\pi_1(\mathbf{r})\xi^B(\mathbf{r})=0.\end{aligned} \tag{36}$$

The amplitudes ϕ_0 and π_0 are determined by

$$\begin{aligned}\phi_0{}^A&=(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\phi(\mathbf{r})U^A(\mathbf{r});\\ \phi_0{}^B&=(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\phi(\mathbf{r})U^B(\mathbf{r}),\\ \pi_0{}^A&=(4\pi)^{-\frac{1}{2}}\int d\mathbf{r}\pi(\mathbf{r})\xi^A(\mathbf{r});\\ \pi_0{}^B&=(4\pi)^{-\frac{1}{2}}\int d\mathbf{r}\pi(\mathbf{r})\xi^B(\mathbf{r}).\end{aligned} \tag{37}$$

$(\phi_0{}^A,\ \pi_0{}^A)$ and $(\phi_0{}^B,\ \pi_0{}^B)$ are conjugate pairs. However,

$$[\pi_1(\mathbf{r}),\phi_1(\mathbf{r}')]=-i[\delta(\mathbf{r}-\mathbf{r}')-U^A(\mathbf{r})\xi^A(\mathbf{r}')-U^B(\mathbf{r})\xi^B(\mathbf{r}')]. \tag{38}$$

In terms of these variables

$$\begin{aligned}H=&N(\bar{\pi}_0{}^A\pi_0{}^A+\bar{\pi}_0{}^B\pi_0{}^B)+M(\bar{\pi}_0{}^A\pi_0{}^B+\bar{\pi}_0{}^B\pi_0{}^A)\\ &+[1/(I^2-J^2)][I(\bar{\phi}_0{}^A\phi_0{}^A+\bar{\phi}_0{}^B\phi_0{}^B)\\ &-J(\bar{\phi}_0{}^A\phi_0{}^B+\bar{\phi}_0{}^B\phi_0{}^A)]\\ &+g[\tau_-{}^A\bar{\phi}_0{}^A+\tau_+{}^A\phi_0{}^A+\tau_-{}^B\bar{\phi}_0{}^B+\tau_+{}^B\phi_0{}^B]\\ &+(4\pi)^{\frac{1}{2}}\pi_0{}^A\int d\mathbf{r}U^A(\mathbf{r})\bar{\pi}_1(\mathbf{r})\\ &+(4\pi)^{\frac{1}{2}}\bar{\pi}_0{}^A\int d\mathbf{r}U^A(\mathbf{r})\pi_1(\mathbf{r})\\ &+(4\pi)^{\frac{1}{2}}\pi_0{}^B\int d\mathbf{r}U^B(\mathbf{r})\bar{\pi}_1(\mathbf{r})\\ &+(4\pi)^{\frac{1}{2}}\bar{\pi}_0{}^B\int d\mathbf{r}U^B(\mathbf{r})\pi_1(\mathbf{r})\\ &+\int d\mathbf{r}[\bar{\pi}_1(\mathbf{r})\pi_1(\mathbf{r})+\bar{\phi}_1(\mathbf{r})\omega^2\phi_1(\mathbf{r})].\end{aligned} \tag{39}$$

We introduce polar coordinates for $\phi_0{}^A$, $\phi_0{}^B$:

$$\phi_0{}^A = q_0{}^A \exp(-i\theta_A); \quad \phi_0{}^B = q_0{}^B \exp(-i\theta_B).$$

It is more convenient, however, to use instead of θ_A, θ_B the quantities

$$\theta = (\theta_A + \theta_B)/2; \quad \psi = (\theta_A - \theta_B)/2,$$
$$p_\theta = p_\theta{}^A + p_\theta{}^B; \quad p_\psi = p_\theta{}^A - p_\theta{}^B.$$

The canonical transformation takes the form:

$$\begin{aligned}
\phi_0{}^A &= q_0{}^A e^{-i(\theta+\psi)};\\
\pi_0{}^A &= [e^{i(\theta+\psi)}/2][p_0{}^A + (i/q_0{}^A)\\
&\quad \times\{(p_\theta + p_\psi)/2 + \tfrac{1}{2} - \tfrac{1}{2}\textstyle\sum\}],\\
\bar{\phi}_0{}^A &= q_0{}^A e^{i(\theta+\psi)};\\
\bar{\pi}_0{}^A &= [e^{-i(\theta+\psi)}/2][p_0{}^A - (i/q_0{}^A)\\
&\quad \times\{(p_\theta + p_\psi)/2 - \tfrac{1}{2} - \tfrac{1}{2}\textstyle\sum\}],\\
\phi_0{}^B &= q_0{}^B e^{-i(\theta-\psi)};\\
\pi_0{}^B &= [e^{i(\theta-\psi)}/2][p_0{}^B + (i/q_0{}^B)\\
&\quad \times\{(p_\theta - p_\psi)/2 + \tfrac{1}{2} - \tfrac{1}{2}\textstyle\sum\}],\\
\bar{\phi}_0{}^B &= q_0{}^B e^{i(\theta-\psi)};\\
\bar{\pi}_0{}^B &= [e^{-i(\theta-\psi)}/2][p_0{}^B - (i/q_0{}^B)\\
&\quad \times\{(p_\theta - p_\psi)/2 - \tfrac{1}{2} - \tfrac{1}{2}\textstyle\sum\}],\\
\phi_1(\mathbf{r}) &= q_1(\mathbf{r})e^{-i\theta}; \quad \pi_1(\mathbf{r}) = p_1(\mathbf{r})e^{+i\theta},\\
\bar{\phi}_1(\mathbf{r}) &= \bar{q}_1(\mathbf{r})e^{i\theta}; \quad \bar{\pi}_1(\mathbf{r}) = \bar{p}_1(\mathbf{r})e^{-i\theta},\\
\textstyle\sum &= -i\int d\mathbf{r}[\pi_1(\mathbf{r})\phi_1(\mathbf{r}) - \bar{\pi}_1(\mathbf{r})\bar{\phi}_1(\mathbf{r})].
\end{aligned} \tag{40}$$

The independent conjugate pairs are $(p_0{}^A, q_0{}^A)$, $(p_0{}^B, q_0{}^B)$, (p_θ, θ), and (p_ψ, ψ). $p_1(\mathbf{r})$ and $q_1(\mathbf{r})$ commute with the other quantities, but

$$[p_1(\mathbf{r}), q_1(\mathbf{r}')] = -i[\delta(\mathbf{r}-\mathbf{r}') - U^A(\mathbf{r})\xi^A(\mathbf{r}') - U^B(\mathbf{r})\xi^B(\mathbf{r}')].$$

The total charge is

$$\begin{aligned}
-i\int d\mathbf{r}[\pi(\mathbf{r})\phi(\mathbf{r}) - \bar{\pi}(\mathbf{r})\bar{\phi}(\mathbf{r})]
&= -i[\pi_0{}^A\phi_0{}^A - \bar{\pi}_0{}^A\bar{\phi}_0{}^A] - i[\pi_0{}^B\phi_0{}^B - \bar{\pi}_0{}^B\bar{\phi}_0{}^B] + \textstyle\sum\\
&= \tfrac{1}{2}(p_\theta + p_\psi - \textstyle\sum) + \tfrac{1}{2}(p_\theta - p_\psi - \textstyle\sum) + \textstyle\sum = p_\theta.
\end{aligned}$$

Before writing down H, we shall consider just the leading, or interaction term. This becomes

$$g[q_0{}^A(\tau_-{}^A e^{i(\theta+\psi)} + \tau_+{}^A e^{-i(\theta+\psi)}) + q_0{}^B(\tau_-{}^B e^{i(\theta-\psi)} + \tau_+{}^B e^{-i(\theta-\psi)})].$$

Again apply an S transformation, where

$$S = \exp[i\tau_z{}^A(\theta+\psi)/2]\exp[i\tau_z{}^B(\theta-\psi)/2].$$

The interaction energy becomes

$$g[q_0{}^A\tau_x{}^A + q_0{}^B\tau_x{}^B].$$

This is diagonal in a representation in which $\tau_x{}^A$ and $\tau_x{}^B$ are diagonal. The lowest eigenvalue is $-g[q_0{}^A + q_0{}^B]$. Off diagonal terms coupling widely separated states are introduced elsewhere in H, since under the S transformation

$$p_\theta \to p_\theta + \tfrac{1}{2}(\tau_z{}^A + \tau_z{}^B) \quad \text{and} \quad p_\psi \to p_\psi + \tfrac{1}{2}(\tau_z{}^A - \tau_z{}^B).$$

As before, we neglect terms linear in $\tau_z{}^A$ and $\tau_z{}^B$ since these quantities have no diagonal matrix elements for the lowest eigenstate of the interaction energy. The Hamiltonian becomes (neglecting $\sum$ as before):

$$\begin{aligned}
H^{11} &= (N/4)[(p_0{}^A)^2 + (p_\theta + p_\psi)^2/4(q_0{}^A)^2]\\
&\quad + (N/4)[(p_0{}^B)^2 + (p_\theta - p_\psi)^2/4(q_0{}^B)^2]\\
&\quad + \tfrac{1}{2}M\cos 2\psi[p_0{}^A p_0{}^B + (p_\theta{}^2 - p_\psi{}^2)/4q_0{}^A q_0{}^B]\\
&\quad + (M/4)\sin 2\psi[p_0{}^A(p_\theta - p_\psi)/q_0{}^B\\
&\quad - p_0{}^B(p_\theta + p_\psi)/q_0{}^A] + [1/(I^2 - J^2)]\\
&\quad \times[I\{(q_0{}^A)^2 + (q_0{}^B)^2\} - 2Jq_0{}^A q_0{}^B\cos 2\psi]\\
&\quad - g(q_0{}^A + q_0{}^B) + (4\pi)^{\frac{1}{2}}(p_0{}^A/2)\int d\mathbf{r}\,U^A(\mathbf{r})\\
&\quad \times[p_1(\mathbf{r})e^{-i\psi} + \bar{p}_1(\mathbf{r})e^{i\psi}] + (4\pi)^{\frac{1}{2}}(p_0{}^B/2)\\
&\quad \times\int d\mathbf{r}\,U^B(\mathbf{r})[p_1(\mathbf{r})e^{i\psi} + \bar{p}_1(\mathbf{r})e^{-i\psi}]\\
&\quad + i(4\pi)^{\frac{1}{2}}\{(p_\theta + p_\psi)/4q_0{}^A\}\\
&\quad \times\int d\mathbf{r}\,U^A(\mathbf{r})[\bar{p}_1(\mathbf{r})e^{i\psi} - p_1(\mathbf{r})e^{-i\psi}]\\
&\quad + (4\pi)^{\frac{1}{2}}i\{(p_\theta - p_\psi)/4q_0{}^B\}\\
&\quad \times\int d\mathbf{r}\,U^B(\mathbf{r})[\bar{p}_1(\mathbf{r})e^{-i\psi} - p_1(\mathbf{r})e^{i\psi}]\\
&\quad + \int d\mathbf{r}[\bar{p}_1(\mathbf{r})p_1(\mathbf{r}) + \bar{q}_1(\mathbf{r})\omega^2 q_1(\mathbf{r})].
\end{aligned} \tag{41}$$

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[11] The argument for the neglect of terms involving Σ is the same as in note (6). Where products of the form $p_\psi f(\psi)$ appear, they will always be considered symmetrized.

1. *Freezing of the Isotopic Spin*

We first concern ourselves with the elimination of the linear term. This requires shifts in the origins of $q_0{}^A$ and $q_0{}^B$:

$$q_0{}^A = Q_0{}^A + (g/2)(I^2 - J^2)/(I - J\cos 2\psi),$$
$$q_0{}^B = Q_0{}^B + (g/2)(I^2 - J^2)/(I - J\cos 2\psi). \quad (42)$$

The part of H quadratic in $q_0{}^A$ and $q_0{}^B$ becomes

$$-(g^2/2)(I^2-J^2)/(I-J\cos 2\psi) + [1/(I^2-J^2)][I\{(Q_0{}^A)^2+(Q_0{}^B)^2\} - 2JQ_0{}^AQ_0{}^B\cos 2\psi]. \quad (43)$$

The first term is the static self-energy of the two-particle system and the dominant part of H in the limit of strong coupling. It is a function of the separation of the two sources through its dependence on J. ψ is the angle variable whose canonical conjugate is $p_\psi = p_{\theta_A} - p_{\theta_B}$, the difference of charges. In the limit of large g, ψ will adjust itself so as to make the energy a minimum, i.e., $\cos 2\psi = 1$, $\psi = 0$ or π. In this limit, the leading term becomes simply

$$-(g^2/2)(I+J), \quad (44)$$

a result analogous to that obtained in II for the classical symmetrical scalar theory. For $\kappa a \ll 1$, the part of (44) depending on the separation, $-g^2J/2$, reduces to $-g^2(\mu c^2)e^{-\kappa d}/2\kappa d$, where d is the separation. Because of its non-exchange character, this is an inadmissable potential for nuclear problems.

The condition $\psi=0$ or $\psi=\pi$ describes a "freezing" of the two isotopic spins with respect to each other. It does not follow that this freezing will occur for all values of d because the leading potential term—that of order g^2—falls off rapidly with increasing d and may eventually be dominated by other terms in the Hamiltonian whose minimizing does not require the freezing of the isotopic spin. That this is indeed the case may be seen by considering the approach of two heavy particles. At infinite separation they may be thought of simply as two independent systems such as described in Section IIIA. They have, together, an energy given by (31):

$$H = -g^2I/2 + 4\pi(p_{\theta_1}^2 + p_{\theta_2}^2)\Big/ g^2\int d\mathbf{r}\{X(\mathbf{r})\}^2,$$

which can also be written

$$H = -g^2I/2 + \mu c^2(p_\theta{}^2 + p_\psi{}^2)/g^2. \quad (45)$$

At this stage the value of ψ is completely arbitrary. As the particles approach each other, corrections appear to both terms in (45). By comparison with (43) one sees that the g^2 term gets a correction $-g^2J\cos 2\psi/2$. The isobar energy levels are also perturbed, but the correction is small down to separations of the order of κ^{-1}. For the purposes of this argument we will not be concerned with smaller separations and hence will use the second term in (45) unchanged. We can predict that freezing will set in when the separation becomes less than a certain critical value determined by

$$\kappa/g^2 = -g^2J/2,$$

or

$$2/g^4 = e^{-\kappa d}/\kappa d. \quad (46)$$

When the particles have approached to a separation equal to the range, $\kappa d = 1$, the isotopic spins will still be "liquid" if g is not larger than 1.2. For larger values of g, freezing will take place at larger separations, according to (46). An essentially identical, but somewhat more trustworthy estimate of the freezing radius will be given later by (55).

2. *Exchange Forces at Large Separation*

For separations larger than the critical one defined above, this theory does indeed lead to exchange forces. For here we may treat the expression $-g^2J\cos 2\psi/2$ as a perturbation to the zero-order energy, $\mu c^2 p_\psi{}^2/g^2$. The unperturbed stationary states are $\begin{cases}\sin n\psi\\ \cos n\psi\end{cases}$, where n gives the number of charge units by which the two systems differ. If the charge difference is zero, the perturbation vanishes in first order (proton-proton or neutron-neutron). For one unit charge difference, there are two independent states, symmetric and anti-symmetric in the isotopic spin coordinate, namely $\cos\psi$ and $\sin\psi$. The corresponding perturbation energies are

$$V_{\text{anti-s}} = +g^2J/4,$$
$$V_{\text{sym}} = -g^2J/4. \quad (47)$$

This differs only by the factor $\frac{1}{4}$ from the result obtained in perturbation theory. The occurrence of this factor is connected with the fact, which will be demonstrated presently, that half of the field [see, for comparison, H_y of (25)] consists of unbound mesotrons which contribute nothing in order g^2 to the self-field of the systems. The interaction (47) has numerous defects. If the antisymmetry of the total wave function of the deuteron is invoked, then the state $\sin\psi$ can be identified with the "triplet" state of the deuteron, and $\cos\psi$ with the "singlet." Equation (47) would then predict the singlet to lie lower, in contradiction to experience. Furthermore, it could not explain the quadrupole moment of the deuteron. The strong coupling theory has the additional weakness that the exchange character of the force may disappear inside a critical freezing radius. It will be useful, for comparison with the pseudoscalar theory which will be treated later, to investigate the freezing condition more thoroughly and without using the approximation introduced above.

3. *The "Almost Frozen" System*

We consider values of g and of the separation which are of such a magnitude that the minimum of H is determined, to first order, by minimizing the g^2 term in (43). In other words, we will consider ψ to be in the neighborhood of 0 or π, so that $\cos 2\psi \sim 1-2\psi^2$. In the remainder of the Hamiltonian, we keep only the leading terms in the expansion of $\cos 2\psi$, $\sin 2\psi$, etc. We obtain

$$\begin{aligned}H=&(N/4)[(p_0{}^A)^2+(p_\theta+p_\psi)^2/4(q_0{}^A)^2]\\&+(N/4)[(p_0{}^B)^2+(p_\theta-p_\psi)^2/4(q_0{}^B)^2]\\&+(M/2)[p_0{}^Ap_0{}^B+(p_\theta{}^2-p_\psi{}^2)/4q_0{}^Aq_0{}^B]\\&-g^2(I+J)/2+g^2J\psi^2(I+J)/(I-J)\\&+[1/(I^2-J^2)][I\{(Q_0{}^A)^2+(Q_0{}^B)^2\}\\&-2JQ_0{}^AQ_0{}^B]+(4\pi)^{\frac{1}{2}}(p_0{}^A/2)\int d\mathbf{r}\,U^A(\mathbf{r})\\&\times[p_1(\mathbf{r})+\bar{p}_1(\mathbf{r})]+(4\pi)^{\frac{1}{2}}(p_0{}^B/2)\\&\times\int d\mathbf{r}\,U^B(\mathbf{r})[p_1(\mathbf{r})+\bar{p}_1(\mathbf{r})]\\&+(4\pi)^{\frac{1}{2}}i[(p_\theta+p_\psi)/4q_0{}^A]\int d\mathbf{r}\,U^A(\mathbf{r})\\&\times[\bar{p}_1(\mathbf{r})-p_1(\mathbf{r})]+(4\pi)^{\frac{1}{2}}i[(p_\theta-p_\psi)/4q_0{}^B]\\&\times\int d\mathbf{r}\,U^B(\mathbf{r})[\bar{p}_1(\mathbf{r})-p_1(\mathbf{r})]\\&+\int d\mathbf{r}[\bar{p}_1(\mathbf{r})p_1(\mathbf{r})+\bar{q}_1(\mathbf{r})\omega^2q_1(\mathbf{r})].\qquad(48)\end{aligned}$$

Eigenstates for oscillations of the system about the position of minimum energy may once again be obtained by resolving p_1 and q_1 into real and imaginary parts:

$$p_1(\mathbf{r})=(1/\sqrt{2})[p_x(\mathbf{r})+ip_y(\mathbf{r})],$$
$$q_1(\mathbf{r})=(1/\sqrt{2})[q_x(\mathbf{r})-iq_y(\mathbf{r})],$$
$$H=-g^2(I+J)/2+H_x+H_y,$$

where

$$\begin{aligned}H_x=&(N/4)[(p_0{}^A)^2+(p_0{}^B)^2]+(M/2)p_0{}^Ap_0{}^B\\&+[I\{(Q_0{}^A)^2+(Q_0{}^B)^2\}-2JQ_0{}^AQ_0{}^B]/(I^2-J^2)\\&+(4\pi)^{\frac{1}{2}}(p_0{}^A/\sqrt{2})\int d\mathbf{r}\,U^A(\mathbf{r})p_x(\mathbf{r})\\&+(4\pi)^{\frac{1}{2}}(p_0{}^B/\sqrt{2})\int d\mathbf{r}\,U^B(\mathbf{r})p_x(\mathbf{r})\\&+\tfrac{1}{2}\int d\mathbf{r}[\{p_x(\mathbf{r})\}^2+q_x(\mathbf{r})\omega^2q_x(\mathbf{r})].\qquad(49)\end{aligned}$$

Introduce the canonically conjugate variables

$$\phi_x(\mathbf{r})=(4\pi)^{-\frac{1}{2}}\{Q_0{}^A\xi^A(\mathbf{r})+Q_0{}^B\xi^B(\mathbf{r})\}+(1/\sqrt{2})q_x(\mathbf{r}),$$
$$\pi_x(\mathbf{r})=(4\pi)^{\frac{1}{2}}\{\tfrac{1}{2}p_0{}^AU^A(\mathbf{r})+\tfrac{1}{2}p_0{}^BU^B(\mathbf{r})\}+(1/\sqrt{2})p_x(\mathbf{r}),$$
$$H_x=\int d\mathbf{r}[\{\pi_x(\mathbf{r})\}^2+\phi_x(\mathbf{r})\omega^2\phi_x(\mathbf{r})],$$

representing a system of unscattered mesotrons. Meanwhile

$$\begin{aligned}H_y=&(N/16)[(p_\theta+p_\psi)^2/(q_0{}^A)^2+(p_\theta-p_\psi)^2/(q_0{}^B)^2]\\&+(M/8)(p_\theta{}^2-p_\psi{}^2)/q_0{}^Aq_0{}^B+g^2J\psi^2(I+J)/(I-J)\\&+(4\pi)^{\frac{1}{2}}\{\sqrt{2}(p_\theta+p_\psi)/4q_0{}^A\}\int d\mathbf{r}\,p_y(\mathbf{r})\,U^A(\mathbf{r})\\&+(4\pi)^{\frac{1}{2}}\{\sqrt{2}(p_\theta-p_\psi)/4q_0{}^B\}\int d\mathbf{r}\,p_y(\mathbf{r})\,U^B(\mathbf{r})\\&+\tfrac{1}{2}\int d\mathbf{r}[\{p_y(\mathbf{r})\}^2+q_y(\mathbf{r})\omega^2q_y(\mathbf{r})].\qquad(50)\end{aligned}$$

The terms linear in p_y are to be eliminated by a unitary transformation of the type (27), (29).

To first order, q_y is unchanged and

$$p_y(\mathbf{r}) = p_y'(\mathbf{r}) - ((4\pi)^{\frac{1}{2}}/\sqrt{8q_0}) \times[(p_\theta + p_\psi)(U^A(\mathbf{r}) - cX^A(\mathbf{r}) - dX^B(\mathbf{r})) + (p_\theta - p_\psi)(U^B(\mathbf{r}) - dX^A(\mathbf{r}) - cX^B(\mathbf{r}))]. \quad (51)$$

We have replaced $q_0{}^A$ and $q_0{}^B$ by their equilibrium values

$$q_0{}^A = q_0{}^B = q_0 = \frac{g}{2}(I+J).$$

The constants c and d [which may be determined, e.g., by the condition that $p_y'(\mathbf{r})$ be orthogonal to $\xi^A(\mathbf{r})$ and $\xi^B(\mathbf{r})$, a condition that is already fulfilled by $p_y(\mathbf{r})$] are given by

$$c = \frac{IR - SJ}{R^2 - S^2}; \quad d = \frac{JR - SI}{R^2 - S^2},$$

where

$$R = \int d\mathbf{r}[X^A(\mathbf{r})]^2 = \int d\mathbf{r}[X^B(\mathbf{r})]^2,$$

$$S = \int d\mathbf{r} X^A(\mathbf{r}) X^B(\mathbf{r}).$$

The condition for the convergence of the expansion (51) is $p_\theta/g \ll 1$ and $p_\psi/g \ll 1$. H_y becomes

$$H_y = \tfrac{1}{2}\int d\mathbf{r}[\{p_y'(\mathbf{r})\}^2 + q_y'(\mathbf{r})\omega^2 q_y'(\mathbf{r})] + \frac{4\pi p_\theta^2}{8q_0^2}\frac{(I+J)^2}{R+S} + \frac{4\pi p_\psi^2}{8q_0^2}\frac{(I-J)^2}{R-S} + g^2 J\psi^2 \frac{I+J}{I-J}. \quad (52)$$

The first two terms give the energy of a set of scattered waves, p_y' and q_y' being quasi-conjugate quantities satisfying (38). The isobar energy is

$$H_I = \frac{4\pi p_\theta^2}{2g^2(R+S)} + \frac{4\pi p_\psi^2}{2g^2(R-S)}\left(\frac{I-J}{I+J}\right)^2 + g^2 J\psi^2 \frac{I+J}{I-J}. \quad (53)$$

Since $\tau_z{}^A$ and $\tau_z{}^B$ are diagonal, the average heavy particle charge is the expectation value of $(1+\tau_z{}^A)/2 + (1+\tau_z{}^B)/2 = 1$. Consequently, if M is the total charge, the first term can be written

$$\frac{4\pi(M-1)^2}{2(R+S)g^2},$$

giving a parabolic dependence on the total charge of the two particle system, the minimum lying at $M=1$.

The rest of H_I describes harmonic oscillations of ψ about the "frozen" position. The zero-point amplitude of these oscillations is

$$\bar{\psi} = \frac{1}{g}\left[\left(\frac{I-J}{I+J}\right)^3 \frac{4\pi}{8J(R-S)}\right]^{\frac{1}{4}}. \quad (54)$$

For sources separated by d and of radius $a \ll d$ with a defined by

$$a^{-1} = 4\pi \int d\mathbf{r}\, U^A(\mathbf{r}) \frac{1}{\omega^2} U^A(\mathbf{r}) = \int\int d\mathbf{r} d\mathbf{r}'\, U^A(\mathbf{r}) \frac{U^B(\mathbf{r}') \exp[-\kappa|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|};$$

the various constants are given (approximately) by:

$$I = 1/a,$$
$$J = e^{-\kappa d}/d,$$
$$R/4\pi = 1/2\kappa,$$
$$S/4\pi = e^{-\kappa d}/2\kappa.$$

Since $J/I \ll 1$, (54) simplifies to

$$\bar{\psi} = \frac{1}{g}\left[\frac{4\pi}{8J(R-S)}\right]^{\frac{1}{4}}. \quad (55)$$

We may take $\bar{\psi} = 1$ as the maximum amplitude for which the isotopic spins can be regarded as even approximately frozen. This determines a value for g corresponding to any given value for d. For $\kappa d = 1$, this gives $g = 1.0$ as a minimum value below which the system will start to thaw and exchange forces will make an appearance. But since $g > 1$ is a condition for the validity of the above calculations, it is clear that with the strong coupling hypothesis, it is not possible to obtain exchange forces in the charged scalar theory for separations as small as $(\kappa)^{-1}$.

IV. THE NEUTRAL PSEUDOSCALAR THEORY

This theory, in the perturbation approximation, gives charge independent, but spin dependent forces through a coupling of the heavy particle spin with the gradient of the mesotron wave function. We shall treat first the one-source, then the two-source problem.

A. One Nuclear Particle

The Hamiltonian is:

$$H=\tfrac{1}{2}\int d\mathbf{r}[\{\pi(\mathbf{r})\}^2+\phi(\mathbf{r})\omega^2\phi(\mathbf{r})]$$
$$-\{g(4\pi)^{\frac{1}{2}}/\kappa\}\int d\mathbf{r}\,U(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{grad}\;\phi(\mathbf{r}). \quad (56)$$

$\boldsymbol{\sigma}$ is the heavy particle's spin vector. All quantities are defined as before with the exception of the following:

$$N=4\pi\int d\mathbf{r}\left[\frac{\partial}{\partial x_i}U(\mathbf{r})\right]^2, \qquad i=1,\,2,\,3$$

$$\mathrm{X}(\mathbf{r})=4\pi\omega^{-2}U(\mathbf{r}),$$

$$I=\int d\mathbf{r}\frac{\partial \mathrm{X}(\mathbf{r})}{\partial x_i}\frac{\partial U(\mathbf{r})}{\partial x_i}, \qquad i=1,\,2,\,3$$

$$\xi=\mathrm{X}/I \text{ so that } \int d\mathbf{r}\frac{\partial\xi(\mathbf{r})}{\partial x_i}\frac{\partial U(\mathbf{r})}{\partial x_j}=\delta_{ij}.$$

We split ϕ, π as follows:

$$\begin{aligned}\phi(\mathbf{r})&=(4\pi)^{-\frac{1}{2}}\boldsymbol{\phi}_0\cdot\mathbf{grad}\;\xi(\mathbf{r})+\phi_1(\mathbf{r}),\\ \pi(\mathbf{r})&=(4\pi)^{\frac{1}{2}}\boldsymbol{\pi}_0\cdot\mathbf{grad}\;U(\mathbf{r})+\pi_1(\mathbf{r})\end{aligned} \quad (57)$$

with auxiliary conditions

$$\int d\mathbf{r}\phi_1(\mathbf{r})\;\mathbf{grad}\;U(\mathbf{r})=0,$$

$$\int d\mathbf{r}\pi_1(\mathbf{r})\;\mathbf{grad}\;\xi(\mathbf{r})=0.$$

It follows that

$$\boldsymbol{\phi}_0=(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\phi(\mathbf{r})\;\mathbf{grad}\;U(\mathbf{r});$$

$$\boldsymbol{\pi}_0=(4\pi)^{-\frac{1}{2}}\int d\mathbf{r}\pi(\mathbf{r})\;\mathbf{grad}\;\xi(\mathbf{r}). \quad (58)$$

$(\pi_{0i},\,\phi_{0i})$ are canonical pairs, whereas

$$[\pi_1(\mathbf{r}),\,\phi_1(\mathbf{r}')]=-i[\delta(\mathbf{r}-\mathbf{r}')-\mathbf{grad}\;U(\mathbf{r})\cdot\mathbf{grad}\;\xi(\mathbf{r}')].$$

We obtain

$$H=\tfrac{1}{2}N(\pi_0)^2+(\phi_0)^2/2I+\frac{g}{\kappa}\boldsymbol{\sigma}\cdot\boldsymbol{\phi}_0$$
$$+(4\pi)^{\frac{1}{2}}\boldsymbol{\pi}_0\cdot\int d\mathbf{r}\pi_1(\mathbf{r})\;\mathbf{grad}\;U(\mathbf{r})$$
$$+\tfrac{1}{2}\int d\mathbf{r}[\{\pi_1(\mathbf{r})\}^2+\phi_1(\mathbf{r})\omega^2\phi_1(\mathbf{r})]. \quad (59)$$

The cross term in $\boldsymbol{\pi}_0\pi_1(\mathbf{r})$ may be eliminated by a unitary transformation of the type (27), with

$$U=+(4\pi)^{\frac{1}{2}}\boldsymbol{\pi}_0\cdot\int d\mathbf{r}\phi_1(\mathbf{r})\;\mathbf{grad}\;\xi(\mathbf{r})/R, \quad (60)$$

with

$$R=\int d\mathbf{r}[\partial\xi(\mathbf{r})/\partial x_i]^2, \qquad i=1,\,2,\,3$$

$$\pi_1(\mathbf{r})=\pi_1'(\mathbf{r})-(4\pi)^{\frac{1}{2}}\boldsymbol{\pi}_0\cdot[\mathbf{grad}\;U(\mathbf{r})-\mathbf{grad}\;\xi(\mathbf{r})/R]$$

$$\phi_1(\mathbf{r})=\phi_1'(\mathbf{r}).$$

Substitution gives

$$H=\frac{4\pi}{2R}(\pi_0)^2+\frac{1}{2I}(\phi_0)^2+\frac{g}{\kappa}\boldsymbol{\sigma}\cdot\boldsymbol{\phi}_0$$
$$+\tfrac{1}{2}\int d\mathbf{r}[\{\pi_1'(\mathbf{r})\}^2+\phi_1'(\mathbf{r})\omega^2\phi_1'(\mathbf{r})]. \quad (61)$$

It is to be observed that (60) also generates a change in ϕ_0 which is not taken account of here. A more detailed analysis, parallel to that carried out for the scalar field, shows that this correction is negligible in the limit of strong coupling. The effect of the additional terms which would appear in H would be to spoil the exact separation of (61) into a set $(\pi_1',\,\phi_1')$ of unbound waves and a set $(\boldsymbol{\pi}_0,\,\boldsymbol{\phi}_0)$ of bound states. The condition governing the validity of this approximation is $(\kappa a/g)L\ll 1$, where L is the total angular momentum of the bound mesotron field defined below.[12]

[12] A more complete discussion of the strong coupling condition in this case is to be found in W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

We assume that none of the (π_1', ϕ_1') states are excited, and proceed to treat the system defined by

$$H = \frac{4\pi}{2R}(\boldsymbol{\pi}_0)^2 + \frac{1}{2I}(\boldsymbol{\phi}_0)^2 + \frac{g}{\kappa}\boldsymbol{\sigma}\cdot\boldsymbol{\phi}_0. \quad (62)$$

It is convenient to introduce an orthogonal system of unit vectors, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, defined as follows: $\mathbf{n}_3$ is parallel to $\boldsymbol{\phi}_0$ and $\mathbf{n}_1$ has the direction of the intersection with the xy plane of the plane perpendicular to $\mathbf{n}_3$. The co-latitude and azimuth of $\mathbf{n}_3$ are α and β, respectively. The components of the unit vectors are

$$\mathbf{n}_1: (-\sin\beta;\ \cos\beta;\ 0),$$
$$\mathbf{n}_2: (-\cos\alpha\cos\beta;\ -\cos\alpha\sin\beta;\ \sin\alpha),$$
$$\mathbf{n}_3: (\sin\alpha\cos\beta;\ \sin\alpha\sin\beta;\ \cos\alpha).$$

It follows that

$$\begin{aligned}\boldsymbol{\phi}_0 &= q_0\mathbf{n}_3,\\ \boldsymbol{\pi}_0 &= p_0\mathbf{n}_3 - \mathbf{n}_2 p_\alpha/q_0 + \mathbf{n}_1 p_\beta \Big/ q_0 \sin\alpha.\end{aligned} \quad (63)$$

Canonical pairs are (p_0, q_0), (p_α, α), and (p_β, β). The angular momentum of the mesotron field is

$$\mathbf{L} = \int d\mathbf{r}\pi(\mathbf{r})\mathfrak{L}\phi(\mathbf{r}),$$

where $\mathfrak{L}$ is the angular momentum operator. For our case, this becomes

$$\begin{aligned}\mathbf{L} &= [\boldsymbol{\phi}_0 \times \boldsymbol{\pi}_0]\\ &= p_\alpha\mathbf{n}_1 + p_\beta\mathbf{n}_2/\sin\alpha.\end{aligned}$$

We can also write

$$\boldsymbol{\pi}_0 = p_0\mathbf{n}_3 + (\mathbf{L}\times\mathbf{n}_3)/q_0.$$

The energy becomes

$$H = (4\pi/2R)[p_0^2 + (L^2-1)/q_0^2] + q_0^2/2I + (g/\kappa)q_0\boldsymbol{\sigma}\cdot\mathbf{n}_3, \quad (64)$$

where $L^2 = p_\alpha^2 + p_\beta^2/\sin^2 a$.

We transform to a representation in which the interaction energy is diagonal. The corresponding unitary operator is $S = \exp[i\sigma_y\alpha/2]\exp[i\sigma_z\beta/2]$. The transformation $F' = SF$ is applied to the wave function and $H' = SHS^{-1}$ to the Hamiltonian. In particular, $\boldsymbol{\sigma}\cdot\mathbf{n}_3$ is transformed into σ_z. By choosing for the states of the heavy particle a representation in which σ_z is diagonal, the interaction energy is diagonalized with eigenvalues $\pm(g/\kappa)q_0$. The angular momentum becomes

$$\mathbf{L} \rightarrow \mathbf{L} - \tfrac{1}{2}[\mathbf{n}_1\sigma_y + (\mathbf{n}_2/\sin\alpha) \times(\sigma_z\cos\alpha - \sigma_x\sin\alpha)], \quad (65)$$

and its square is transformed into

$$L^2 \rightarrow p_\alpha^2 + p_\beta^2/\sin^2\alpha + \tfrac{1}{4} + 1/4\sin^2\alpha - p_\beta\sigma_z\cos\alpha/\sin^2\alpha. \quad (66)$$

Terms linear in σ_x and σ_y have been neglected as they couple widely separated states. Aside from the additive constant, $\frac{1}{4}$, (66) is simply the square of the total angular momentum of a completely symmetric rigid rotator, the angular momentum corresponding to the third Euler angle being $\sigma_z/2$. The eigenvalues of (66) are therefore $j(j+1)-\frac{3}{4}$, where j is the quantum number for the total angular momentum of the heavy particle plus mesotron field.

We choose the lowest eigenvalue of the interaction energy, and get

$$H = \frac{4\pi}{2R}\left[p_0^2 + \frac{J^2-\frac{3}{4}}{q_0^2}\right] + q_0^2/2I - gq_0/\kappa. \quad (67)$$

The linear term is eliminated by the shift

$$Q_0 = q_0 - gI/\kappa,$$

with the result:

$$H = -g^2I/2\kappa^2 + 4\pi p_0^2/2R + Q_0^2/2I + (J^2-\tfrac{3}{4})/2Rq_0^2. \quad (68)$$

The leading term in (68) is the static self-energy. The two succeeding terms may be described as the energy of oscillation of the variable q_0 about its position of equilibrium, gI/κ. We treat the case where this degree of freedom is unexcited, i.e., when there are no positive energy mesotrons in the field. Hence, in the last term of (68) we substitute for q_0 its equilibrium value, and obtain for the isobar energy

$$H_I = (4\pi\kappa^2/2g^2I^2R)[j(j+1)-\tfrac{3}{4}].$$

For $\kappa a \ll 1$,

$$I^2R=\int d\mathbf{r}\left[\frac{\partial X(\mathbf{r})}{\partial x}\right]^2 \sim 4\pi/3a,$$
$$H_I=\frac{3\mu c^2}{2}(\kappa a/g^2)[j(j+1)-\tfrac{3}{4}], \tag{69}$$

in conventional units. The lowest state of this system is identified as the proton-neutron ($j=\frac{1}{2}$).

B. Two Nuclear Particles

The Hamiltonian is the following:

$$H=\tfrac{1}{2}\int d\mathbf{r}[\{\pi(\mathbf{r})\}^2+\phi(\mathbf{r})\omega^2\phi(\mathbf{r})]$$
$$-(g(4\pi)^{\frac{1}{2}}/\kappa)\int d\mathbf{r}[U(\mathbf{r}^A)\boldsymbol{\sigma}^A\cdot\mathbf{grad}\,\phi(\mathbf{r})$$
$$+U(\mathbf{r}^B)\boldsymbol{\sigma}^B\cdot\mathbf{grad}\,\phi(\mathbf{r})]. \tag{70}$$

Introduce

$$X(\mathbf{r}^A)=4\pi\omega^{-2}U(\mathbf{r}^A),$$
$$X(\mathbf{r}^B)=4\pi\omega^{-2}U(\mathbf{r}^B),$$
$$N=4\pi\int d\mathbf{r}\left[\frac{\partial U(\mathbf{r}^A)}{\partial x_i}\right]^2=4\pi\int d\mathbf{r}\left[\frac{\partial U(\mathbf{r}^B)}{\partial x_i}\right]^2,$$
$$M_i=4\pi\int d\mathbf{r}\frac{\partial U(\mathbf{r}^A)}{\partial x_i}\frac{\partial U(\mathbf{r}^B)}{\partial x_i}, \qquad i=1,2,3$$
$$I=\int d\mathbf{r}\frac{\partial X(\mathbf{r}^A)}{\partial x_i}\frac{\partial U(\mathbf{r}^A)}{\partial x_i}=\int d\mathbf{r}\frac{\partial X(\mathbf{r}^B)}{\partial x_i}\frac{\partial U(\mathbf{r}^B)}{\partial x_i},$$
$$J_i=\int d\mathbf{r}\frac{\partial X(\mathbf{r}^A)}{\partial x_i}\frac{\partial U(\mathbf{r}^B)}{\partial x_i}=\int d\mathbf{r}\frac{\partial X(\mathbf{r}^B)}{\partial x_i}\frac{\partial U(\mathbf{r}^A)}{\partial x_i}.$$

Note that all integrals vanish which involve derivatives with respect to two different x_i's. States orthogonal to the U's are defined as follows:

$$\int d\mathbf{r}\frac{\partial \xi_i(\mathbf{r}^A)}{\partial x_i}\frac{\partial U(\mathbf{r}^B)}{\partial x_i}=\int d\mathbf{r}\frac{\partial \xi_i(\mathbf{r}^B)}{\partial x_i}\frac{\partial U(\mathbf{r}^A)}{\partial x_i}=0,$$
$$\int d\mathbf{r}\frac{\partial \xi_i(\mathbf{r}^A)}{\partial x_i}\frac{\partial U(\mathbf{r}^A)}{\partial x_i}=\int d\mathbf{r}\frac{\partial \xi_i(\mathbf{r}^B)}{\partial x_i}\frac{\partial U(\mathbf{r}^B)}{\partial x_i}=1.$$

We find

$$\xi_i(\mathbf{r}^A)=\frac{IX(\mathbf{r}^A)-J_iX(\mathbf{r}^B)}{I^2-J_i^2};$$
$$\xi_i(\mathbf{r}^B)=\frac{IX(\mathbf{r}^B)-J_iX(\mathbf{r}^A)}{I^2-J_i^2}.$$

The fields are split as follows:

$$\phi(\mathbf{r})=(4\pi)^{-\frac{1}{2}}\left\{\sum_i \overset{A}{\phi}_{0i}\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^A)+\sum_i \overset{B}{\phi}_{0i}\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^B)\right\}+\phi_1(\mathbf{r}) \tag{71}$$
$$\pi(\mathbf{r})=(4\pi)^{\frac{1}{2}}\{\boldsymbol{\pi}_0{}^A\cdot\mathbf{grad}\,U(\mathbf{r}^A)+\boldsymbol{\pi}_0{}^B\cdot\mathbf{grad}\,U(\mathbf{r}^B)\}+\pi_1(\mathbf{r}),$$

with the auxiliary conditions:

$$\int d\mathbf{r}\phi_1(\mathbf{r})\frac{\partial}{\partial x_i}U(\mathbf{r}^A)=\int d\mathbf{r}\phi_1(\mathbf{r})\frac{\partial}{\partial x_i}U(\mathbf{r}^B)=0, \qquad i=1,2,3$$
$$\int d\mathbf{r}\pi_1(\mathbf{r})\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^A)=\int d\mathbf{r}\pi_1(\mathbf{r})\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^B)=0.$$

The amplitudes $\boldsymbol{\phi}_0$ and $\boldsymbol{\pi}_0$ are determined by

$$\overset{A}{\phi}_{0i}=(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\phi(\mathbf{r})\frac{\partial}{\partial x_i}U(\mathbf{r}^A);$$
$$\overset{B}{\phi}_{0i}=(4\pi)^{\frac{1}{2}}\int d\mathbf{r}\phi(\mathbf{r})\frac{\partial}{\partial x_i}U(\mathbf{r}^B),$$
$$\overset{A}{\pi}_{0i}=(4\pi)^{-\frac{1}{2}}\int d\mathbf{r}\pi(\mathbf{r})\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^A);$$
$$\overset{B}{\pi}_{0i}=(4\pi)^{-\frac{1}{2}}\int d\mathbf{r}\pi(\mathbf{r})\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^B),$$
$$[\overset{A}{\pi}_{0i},\overset{A}{\phi}_{0j}]=[\overset{B}{\pi}_{0i},\overset{B}{\phi}_{0j}]=-i\delta_{ij},$$
$$[\pi_1(\mathbf{r}),\phi_1(\mathbf{r}')]=-i\left[\delta(\mathbf{r}-\mathbf{r}')-\sum_i\frac{\partial}{\partial x_i}U(\mathbf{r}^A)\times\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^{A'})-\sum_i\frac{\partial}{\partial x_i}U(\mathbf{r}^B)\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^{B'})\right].$$

In terms of these variables

$$H=\frac{N}{2}\{(\pi_0^A)^2+(\pi_0^B)^2\}+\sum_i M_i\pi_{0i}^A\pi_{0i}^B$$
$$+\sum_i\frac{1}{I^2-J_i^2}\left[\frac{I}{2}\{(\phi_{0i}^A)^2+(\phi_{0i}^B)^2\}-J_i\phi_{0i}^A\phi_{0i}^B\right]$$
$$+(4\pi)^{\frac{1}{2}}\boldsymbol{\pi}_0^A\cdot\int d\mathbf{r}\pi_1(\mathbf{r})\ \mathbf{grad}\ U(\mathbf{r}^A)$$
$$+(4\pi)^{\frac{1}{2}}\boldsymbol{\pi}_0^B\cdot\int d\mathbf{r}\pi_1(\mathbf{r})\ \mathbf{grad}\ U(\mathbf{r}^B)$$
$$+\tfrac{1}{2}\int d\mathbf{r}[\{\pi_1(\mathbf{r})\}^2+\phi_1(\mathbf{r})\omega^2\phi_1(\mathbf{r})]$$
$$+(g/\kappa)(\boldsymbol{\sigma}^A\cdot\boldsymbol{\phi}_0^A+\boldsymbol{\sigma}^B\cdot\boldsymbol{\phi}_0^B). \quad (72)$$

The cross terms involving π_1 are eliminated by a unitary transformation like (27), (60), but with

$$U=(4\pi)^{\frac{1}{2}}\sum_i\left[\pi_{0i}^A\int d\mathbf{r}\phi_1(\mathbf{r})\frac{\partial}{\partial x_i}\{R_i\xi_i(\mathbf{r}^A)-S_i\xi_i(\mathbf{r}^B)\}+\pi_{0i}^B\int d\mathbf{r}\phi_1(\mathbf{r})\times\frac{\partial}{\partial x_i}\{R_i\xi_i(\mathbf{r}^B)-S_i\xi_i(\mathbf{r}^A)\}\right]\Big/(R_i^2-S_i^2), \quad (73)$$

where

$$R_i=\int d\mathbf{r}\left[\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^A)\right]^2=\int d\mathbf{r}\left[\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^B)\right]^2,$$
$$S_i=\int d\mathbf{r}\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^A)\frac{\partial}{\partial x_i}\xi_i(\mathbf{r}^B).$$

It follows that

$$\pi_1(\mathbf{r})=\pi_1'(\mathbf{r})-(4\pi)^{\frac{1}{2}}\sum_i\left[\pi_{0i}^A\times\frac{\partial}{\partial x_i}\left\{U(\mathbf{r}^A)-\frac{R_i\xi_i(\mathbf{r}^A)-S_i\xi_i(\mathbf{r}^B)}{R_i^2-S_i^2}\right\}+\pi_{0i}^B\frac{\partial}{\partial x_i}\left\{U(\mathbf{r}^B)-\frac{R_i\xi_i(\mathbf{r}^B)-S_i\xi_i(\mathbf{r}^A)}{R_i^2-S_i^2}\right\}\right],$$
$$\phi_1(\mathbf{r})=\phi_1'(\mathbf{r}).$$

We have, as a result

$$H=4\pi\sum_i\left[\{(\pi_{0i}^A)^2+(\pi_{0i}^B)^2\}\frac{R_i}{2}-\pi_{0i}^A\pi_{0i}^BS_i\right]\Big/(R_i^2-S_i^2)+\sum_i\left[\{(\phi_{0i}^A)^2+(\phi_{0i}^B)^2\}\frac{I}{2}-\phi_{0i}^A\phi_{0i}^BJ_i\right]\Big/(I^2-J_i^2)$$
$$+(g/\kappa)(\boldsymbol{\sigma}^A\cdot\boldsymbol{\phi}_0^A+\boldsymbol{\sigma}^B\cdot\boldsymbol{\phi}_0^B)$$
$$+\tfrac{1}{2}\int d\mathbf{r}[\{\pi_1'(\mathbf{r})\}^2+\phi_1'(\mathbf{r})\omega^2\phi_1'(\mathbf{r})]. \quad (74)$$

As in the one source case, changes generated by (73) in $\boldsymbol{\phi}_0^A$ and $\boldsymbol{\phi}_0^B$ are neglected here. Again, the (π_1', ϕ_1') system of states is taken empty so that its contribution to (74) vanishes. We introduce two systems of unit vectors $\mathbf{n}_1^A$, $\mathbf{n}_2^A$, $\mathbf{n}_3^A$ and $\mathbf{n}_1^B$, $\mathbf{n}_2^B$, $\mathbf{n}_3^B$ defined as above through two sets of angles α^A, β^A and α^B, β^B

$$\boldsymbol{\phi}_0^A=q_0^A\mathbf{n}_3^A,^{13}$$
$$\boldsymbol{\pi}_0^A=p_0^A\mathbf{n}_3^A-\mathbf{n}_2^Ap_\alpha^A/q_0^A+\mathbf{n}_1^Ap_\beta^A\Big/q_0^A\sin\alpha^A,$$
$$\boldsymbol{\phi}_0^B=q_0^B\mathbf{n}_3^B,$$
$$\boldsymbol{\pi}_0^B=p_0^B\mathbf{n}_3^B-\mathbf{n}_2^Bp_\alpha^B/q_0^B+\mathbf{n}_1^Bp_\beta^B\Big/q_0^B\sin\alpha^B.$$

Using

$$\mathbf{L}^A=\mathbf{n}_1^Ap_\alpha^A+\mathbf{n}_2^Ap_\beta^A/\sin\alpha^A,$$
$$\mathbf{L}^B=\mathbf{n}_1^Bp_\alpha^B+\mathbf{n}_2^Bp_\beta^B/\sin\alpha^B,$$

we find for the total angular momentum of the mesotron field:

$$\mathbf{L}=\mathbf{L}^A+\mathbf{L}^B.$$

We now obtain the minimum eigenvalue of the interaction energy by applying an S transformation, with

$$S=\exp[i\sigma_y^A\alpha^A/2]\exp[i\sigma_z^A\beta^A/2]\times\exp[i\sigma_y^B\alpha^B/2]\exp[i\sigma_z^B\beta^B/2].$$

The interaction energy becomes

$$(g/\kappa)(q_0^A\sigma_z^A+q_0^B\sigma_z^B). \quad (75)$$

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[13] Symmetrization with respect to p_α and functions of α is to be understood throughout.

We choose a representation in which both σ_z^A and σ_z^B are diagonal, and consider the state in which each has the eigenvalue -1. The effect of the S transformation on $\mathbf{L}^A$, $\mathbf{L}^B$ and their squares is indicated by (65), (66). For example, $(L^A)^2$ becomes $(J^A)^2+\frac{1}{4}$ where $\mathbf{J}^A$ is the total angular momentum of source A plus its associated field.

So far the orientation of the space axes is quite arbitrary and we may assume the z axis to lie in the direction of the line separating the two sources. Then we may write $R_y=R_x=R_\perp$, $S_x=S_y=S_\perp$, $J_x=J_y=J_\perp$; in general $J_\perp\neq J_z$, etc.

The separation of the static self-energy is accomplished by the elimination of the terms linear in the q's. The part of the Hamiltonian involved is

$$\sum_i\left[\{(\overset{A}{\phi}_{0i})^2+(\overset{B}{\phi}_{0i})^2\}\frac{I}{2}-\overset{A}{\phi}_{0i}\overset{B}{\phi}_{0i}J_i\right]\Big/(I^2-J_i^2)-(g/\kappa)(q_0{}^A+q_0{}^B). \quad (76)$$

It is not hard to show that the absolute minimum of (76) is attained when the vectors $\boldsymbol{\phi}_0{}^A$ and $\boldsymbol{\phi}_0{}^B$ coincide in magnitude and in the magnitude of their components. If $J_i>0$, then $\overset{A}{\phi}_{0i}=\overset{B}{\phi}_{0i}$, while if $J_i<0$, then $\overset{A}{\phi}_{0i}=-\overset{B}{\phi}_{0i}$. In any case, $q_0{}^A=q_0{}^B=q_0$. Equation (76) may be written

$$\sum_i\frac{(\overset{A}{\phi}_{0i})^2}{I+|J_i|}-2gq_0/\kappa. \quad (77)$$

The direction of $\boldsymbol{\phi}_0{}^A$ which minimizes (77) is determined by the relative magnitudes of J_z and $J_\perp$. In our case ($\kappa a\ll 1$), J_z is negative and has a greater magnitude than $J_\perp$. Consequently, both $\boldsymbol{\phi}_0{}^A$ and $-\boldsymbol{\phi}_0{}^B$ (i.e., $\mathbf{n}_3{}^A$ and $-\mathbf{n}_3{}^B$) point in the positive z direction for a minimum of (77). The minimizing value of q_0 is

$$q_0=-(g/\kappa)(I+|J_z|),$$

and (77) becomes

$$-(g^2/\kappa^2)(I+|J_z|). \quad (78)$$

1. Forces at Small Separation

If the sources are sufficiently close together, the minimum of the Hamiltonian is essentially given by the minimum of (76), namely (78). Both $\boldsymbol{\phi}_0{}^A$ and $\boldsymbol{\phi}_0{}^B$ are "frozen" and are equal to

$$\boldsymbol{\phi}_0{}^B=-(g/\kappa)(I+|J_z|)\mathbf{n}_z;$$

$$\boldsymbol{\phi}_0{}^A=(g/\kappa)(I+|J_z|)\mathbf{n}_z.$$

Equation (78) contains, in addition to the sum of the self-energies of two separated sources, the space dependent part $-g^2|J_z|/\kappa^2$ which in this theory is quite singular, varying as $1/d^3$ for small separations. Equation (78) is moreover an "ordinary" force, showing neither spin dependence nor exchange properties. We must, therefore, investigate the possibility of thawing of the spins due to the first term of (74). We expand the Hamiltonian about the frozen position; (74) becomes

$$\begin{aligned}&4\pi[\{(p_0{}^A)^2+(p_0{}^B)^2\}R_z/2-p_0{}^Ap_0{}^BS_z]/(R_z^2-S_z^2)+4\pi[\{(J^A)^2+(J^B)^2\}R_\perp/2-\mathbf{J}^A\cdot\mathbf{J}^BS_\perp]\Big/(R_\perp^2-S_\perp^2)q_0^2\\&+[\{(Q_0{}^A)^2+(Q_0{}^B)^2\}I/2-Q_0{}^AQ_0{}^B|J_z|]/(I^2-J_z^2)\\&+\left[\{(\alpha^A)^2+(\alpha^B)^2\}\frac{I|J_z|+J_\perp^2}{2(I+|J_z|)}-\alpha^A\alpha^B\cos(\beta^A-\beta^B)J_\perp\right]q_0^2/(I^2-J_\perp^2)\\&-(g^2/\kappa^2)(I+|J_z|). \quad (79)\end{aligned}$$

Here we have used

$$\mathbf{J}^A=\mathbf{n}_1{}^Ap_\alpha{}^A+\mathbf{n}_2{}^A(p_\beta{}^A+\tfrac{1}{2})/\sin\alpha^A,$$

$$(J^A)^2=(p_\alpha{}^A)^2+(p_\beta{}^A+\tfrac{1}{2})^2/\sin^2\alpha^A,$$

$$\sin\alpha^A\sim\alpha^A;\qquad \sin\alpha^B\sim\alpha^B,$$

$$\cos\alpha^A\sim 1-(\alpha^A)^2/2;\quad \cos\alpha^B\sim -1+(\alpha^B)^2/2,$$

$$q_0{}^A=q_0+Q_0{}^A$$

with analogous definitions for source B. $Q_0{}^A$, the displacement from the equilibrium amplitude, is canonically conjugate to $p_0{}^A$. We may separate off the first and third lines of (79) which we call

H_z and bring to principal axes by means of the canonical transformation:

$$\begin{aligned} p^+ &= (1/\sqrt{2})(p_0{}^A + p_0{}^B), \\ q^+ &= (1/\sqrt{2})(Q_0{}^A + Q_0{}^B), \\ p^- &= (1/\sqrt{2})(p_0{}^A - p_0{}^B), \\ q^- &= (1/\sqrt{2})(Q_0{}^A - Q_0{}^B), \end{aligned} \tag{80}$$

$$H_z = \frac{4\pi p_+{}^2}{2(R_z+S_z)} + \frac{q_+{}^2}{2(I+|J_z|)} + \frac{4\pi p_-{}^2}{2(R_z-S_z)} + \frac{q_-{}^2}{2(I-|J_z|)}. \tag{81}$$

As in the one-source case, all the states of these two oscillator systems are taken empty.

Lines 2 and 4 of (79) give the energy of excitation of the bound system above its minimum position, which we shall call $H_\perp$. First we substitute.

$$p_\beta + \tfrac{1}{2} = p_\beta{}', \quad \beta = \beta'$$

for both sources. Then introduce Cartesian coordinates

$$x = \sin\alpha\cos\beta \sim \alpha\cos\beta,$$

$$y = \sin\alpha\sin\beta \sim \alpha\sin\beta,$$

$$\begin{aligned} H_\perp = 4\pi\Big[&\{(p_x{}^A)^2 + (p_y{}^A)^2 \\ &+ (p_x{}^B)^2 + (p_y{}^B)^2\}R_\perp/2 \\ &- \{p_x{}^A p_x{}^B + p_y{}^A p_y{}^B\}S_\perp\Big] \Big/ (R_\perp{}^2 - S_\perp{}^2)q_0{}^2 \\ &+ \Big[\{(x^A)^2 + (y^A)^2 \\ &+ (x^B)^2 + (y^B)^2\}\frac{I|J_z| + J_\perp{}^2}{2(I+|J_z|)} \\ &- \{x^A x^B + y^A y^B\}J_\perp\Big] q_0{}^2/(I^2 - J_\perp{}^2) \end{aligned} \tag{82}$$

aside from a constant. This consists of two identical systems of which it will suffice to consider either: we shall treat that involving x^A, x^B. Normal coordinates are

$$\begin{aligned} v_I &= (x^A + x^B)/\sqrt{2}, & p_I &= (p^A + p^B)/\sqrt{2}, \\ v_{II} &= (x^A - x^B)/\sqrt{2}, & p_{II} &= (p^A - p^B)/\sqrt{2}, \end{aligned} \tag{83}$$

$$H_{\perp x} = \frac{4\pi p_I{}^2}{2q_0{}^2(R_\perp + S_\perp)} + \frac{4\pi p_{II}{}^2}{2q_0{}^2(R_\perp - S_\perp)} + \frac{v_I{}^2 q_0{}^2(|J_z| - J_\perp)}{2(I+|J_z|)(I+J_\perp)} + \frac{v_{II}{}^2 q_0{}^2(|J_z| + J_\perp)}{2(I+|J_z|)(I-J_\perp)}. \tag{84}$$

The integrals of interest are tabulated below for the limit $\kappa a \ll 1$ and also $a \ll d$ where d is the separation

$$I \sim 1/a^3,$$

$$J_\perp \sim \frac{e^{-\kappa d}}{d}(1/d^2 + \kappa/d),$$

$$J_z \sim -\frac{e^{-\kappa d}}{d}(2/d^2 + 2\kappa/d + \kappa^2),$$

$$R_\perp = R_z = 4\pi a^5/3.$$

$S_\perp$ and S_z are of order κa^6. In this limit J/I and S/R may be neglected. Consider the (p_{II}, v_{II}) oscillator. The amplitude of zero-point oscillations is

$$\bar{\psi} = (\kappa/g)\left[8\frac{RI^2}{4\pi}(|J_z| + J_\perp)\right]^{-\frac{1}{4}}.$$

Thawing takes place only when $\bar{\psi} > 1$,

$$\kappa d e^{+\kappa d} \Big/ \left(1 + \frac{3}{\kappa d} + \frac{3}{\kappa^2 d^2}\right) < \frac{8}{3}\frac{g^4}{\kappa a},$$

or, with $g \sim 1$, $\kappa a \sim 0.1$, when $\kappa r > 3$.

2. *Forces at Large Separation*

This problem is handled by a perturbation method. In zero order, one takes the sum of the energies of two infinitely separated sources [see (68)].

$$H_0 = -g^2 I/\kappa^2 + 4\pi[(J^A)^2 + (J^B)^2 - \tfrac{3}{2}]/2Rq_0{}^2, \tag{85}$$

where

$$(J^A)^2 = (p_\alpha{}^A)^2 + (p_\beta{}^A)^2/\sin^2\alpha^A + 1/4\sin^2\alpha^A + p_\beta\cos\alpha^A/\sin^2\alpha^A.$$

In this order, the angles α^A, β^A, α^B, β^B are completely unrestricted. As the sources approach, the leading modification of H_0 is the addition of the space dependent part of the interaction energy, which is obtainable from (74), and is

$$H_1=\{-J_z \cos\alpha^A \cos\alpha^B - J_\perp \sin\alpha^A \sin\alpha^B \cos(\beta^A-\beta^B)\}g^2/\kappa^2. \quad (86)$$

Other modifications of (85) contained in (74) are less important as long as the separation is appreciably greater than a.

We require first the eigenfunctions of (85) or, more simply, the eigenfunctions of $(J^A)^2$ and $(J^B)^2$. $(J)^2$ may be rewritten:

$$(J)^2=p_\alpha^2+p_\beta^2+(1/\sin^2\alpha)(p_\beta\cos\alpha+\tfrac{1}{2})^2. \quad (87)$$

We assume a solution of the form $A(\alpha)e^{in\beta}$. Single-valuedness of the wave function in β requires that n be a half integer, since the original wave function is related to the eigenfunction here obtained through the S transformation: $F=S^{-1}F'$, with

$$S=\exp[i\sigma_y\alpha/2]\exp[i\sigma_z\beta/2].$$

Since σ_z has the eigenvalue -1, the above requirement on n follows. The equation for A[14] is

$$\frac{d^2A}{d\alpha^2}+\frac{\cos\alpha}{\sin\alpha}\frac{dA}{d\alpha}-\frac{1}{\sin^2\alpha}\times(n\cos\alpha+\tfrac{1}{2})^2A+(E-n^2)A=0, \quad (88)$$

where E is the eigenvalue of (87). This can be transformed into the hypergeometric equation by the introduction of a new variable

$$z=(1+\cos\alpha)/2,$$

and a new dependent variable

$$A=Zz^{|n-\frac{1}{2}|/2}(z-1)^{|n+\frac{1}{2}|/2}$$

$$z(1-z)\frac{d^2Z}{dz^2}+[\gamma-(\alpha_0+\beta_0+1)z]\frac{dZ}{dz}-\alpha_0\beta_0Z=0, \quad (89)$$

where

$$\alpha_0=|n|+\tfrac{1}{2}+(E+\tfrac{1}{4})^{\frac{1}{2}},$$

$$\beta_0=|n|+\tfrac{1}{2}-(E+\tfrac{1}{4})^{\frac{1}{2}},$$

$$\gamma=|n-\tfrac{1}{2}|+1.$$

Z is the hypergeometric function, $F(\alpha_0, \beta_0, \gamma, z)$, and the solution for the eigenfunction of (87) is

$$e^{in\beta}\{z^{|n-\frac{1}{2}|}(z-1)^{|n+\frac{1}{2}|}\}^{\frac{1}{2}}F(\alpha_0,\beta_0,\gamma,z). \quad (90)$$

For this function to be regular at $\alpha=0$, π, the hypergeometric series must terminate, i.e., β_0 must be a negative integer or zero. This means that E is restricted to the values $(|n|+k)\times(|n|+k+1)$, where k is a positive integer or zero. Each value of $j=|n|+k$ corresponds to an isobaric level of the system, each level being multiply degenerate, according to the number of combinations of n and k which satisfy the above relation, namely $2j+1$.

We are concerned with the attraction between two systems which approach each other in their lowest isobaric levels, i.e., with $k=0$, $j=\frac{1}{2}$, $n=\pm\frac{1}{2}$. This level is a doublet whose wave functions may be written

$$\begin{aligned}&(\sin\alpha/2)e^{i\beta/2},\\ &(\cos\alpha/2)e^{-i\beta/2}.\end{aligned} \quad (91)$$

These two states, corresponding to $n=\pm\frac{1}{2}$, are to be thought of as representing the two projections of the "empirical" spin of the heavy particle (to be distinguished from $\boldsymbol{\sigma}$, the spin vector introduced above for the heavy particle core).

We now calculate the expectation value of (86). The term involving J_z has a finite matrix element only if the projections n^A and n^B both remain unchanged. Its expectation value is $\pm(-g^2J_z/9\kappa^2)$, corresponding to $n^A=\pm n^B$. It can be written $(-g^2J_z/9\kappa^2)s_z^As_z^B$ where $\mathbf{s}(s_x, s_y, s_z)$ is the spin projection operator for the doublet under consideration. On the other hand, the term in $J_\perp$ contributes only if the projection is increased by one unit for system and decreased by one unit for the other. The expectation value is $(-g^2J_\perp/9\kappa^2)(s_x^As_x^B+s_y^As_y^B)$. The energy of

[14] R. deL. Kronig and I. I. Rabi, Phys. Rev. 29, 262 (1927).

interaction is consequently

$$V_1=-\frac{g^2}{9\kappa^2}[J_z s_z^A s_z^B+J_\perp(s_x^A s_x^B+s_y^A s_y^B)].$$

Introduce the unit vector $\hat{\mathbf{r}}$ along the line joining the two systems (in the z direction). Then

$$V_1=\frac{-g^2}{9\kappa^2}[J_\perp \mathbf{s}^A\cdot\mathbf{s}^B+(J_z-J_\perp)(\mathbf{s}^A\cdot\hat{\mathbf{r}})(\mathbf{s}^B\cdot\hat{\mathbf{r}})].$$

The integrals J_z and $J_\perp$ are given by (84). Letting $\kappa d=x$, we have

$$V_1=\frac{g^2\kappa e^{-x}}{9(4\pi x)}\bigg[(1/3)\mathbf{s}^A\cdot\mathbf{s}^B+\{(\mathbf{s}^A\cdot\hat{\mathbf{r}})(\mathbf{s}^B\cdot\hat{\mathbf{r}})-(1/3)\mathbf{s}^A\cdot\mathbf{s}^B\}\left\{\frac{3}{x^2}+\frac{3}{x}+1\right\}\bigg]. \quad (92)$$

Aside from the factor $\frac{1}{9}$, this is precisely the interaction between two heavy particles derived from the same Hamiltonian by perturbation methods.

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Cloud-Chamber and Counter Studies of Cosmic Rays Underground

VOLNEY C. WILSON AND DONALD J. HUGHES
Ryerson Laboratory, The University of Chicago, Chicago, Illinois

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A counter controlled cloud chamber and two counter coincidence sets were used to study the nature of the cosmic rays observable underground. The experiments were performed in a copper mine at depths of 71, 141, 582, and 657 meters water equivalent. The data are easily interpreted, if one assumes that underground the primary rays are mesotrons and that the soft rays and showers are electronic secondaries produced by the penetrating mesotrons.

INTRODUCTION

At the time of the cosmic-ray symposium[1] held at Chicago in June, 1939, it was evident that there was as yet much unknown about the nature of cosmic rays underground as well as considerable disagreement concerning the known material. The shape of the total intensity *vs.* depth curve seemed to be quite well established. When plotted on a log log scale, the data fall on a line composed of two straight portions, the change in slope occurring at about 250 to 400 meters water equivalent. However, concerning the nature of the rays responsible for the two parts of the curve, there was a diversity of opinion. Several observers had found evidence for the non-ionizing character of the rays which carry the energy down to great depths; Wataghin and Santos[2] at 250 and 400 m,[3] Barnothy and Forro[4] at 1000 m. However, Wilson's[5] experiments at 30 m and 300 m indicated that the penetrating rays responsible for the observed intensity at both these depths are ionizing. Clay's[6] interpretation was that protons are predominant below the break in the intensity curve.

Neither data nor interpretation were clear-cut on the matter of shower production and the abundance of soft particles. The ratio of soft to hard components, as measured by counter absorption experiments, varied greatly, and it was quite evident that much of this disagreement

[1] V. C. Wilson, Rev. Mod. Phys. **11**, 230 (1939).

[2] G. Wataghin and M. Damy de Souza Santos, Ann. Acad. Brasil. Sci. **11** (March 11, 1939).

[3] m means meters water equivalent of rock calculated on a density basis.

[4] J. Barnothy and M. Forro, Phys. Rev. **55**, 870 (1939).

[5] V. C. Wilson, Phys. Rev. **55**, 6 (1939).

[6] J. Clay, Rev. Mod. Phys. **11**, 128 (1939).