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Theory of the Magnetron. III

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The general theory of the magnetron tubes, as developed in a former paper, showed that the magnetron should sustain oscillations in an outer circuit, when the frequency was nearly $\sqrt{2}$ times the Larmor's frequency ω_H. The present paper contains a more detailed study of the behavior of a magnetron with one cylindrical anode, when the radius a of the filament and b of the anode are both taken into account. It is shown that the magnetron is able to sustain oscillations on the frequency

$$\omega=\omega_H(2+2a^4/b^4)^{\frac{1}{2}}$$

which lies between $\sqrt{2}\omega_H$ and $2\omega_H$, according to the dimensions of the electrodes. The agreement of this theory with some interesting results obtained by Blewett and Ramo is shown, and the limiting case of the plane magnetron is briefly discussed.

1. INTRODUCTION. PROPER OSCILLATIONS OF AN ELECTRON CLOUD IN A MAGNETIC FIELD

The aim of the present paper is to give a more precise and accurate discussion of some results obtained in a former paper[1] and to show their agreement with a similar theory developed by Blewett and Ramo[2] and with the very interesting experimental proofs brought forth by these authors.

All the known effects, as the apparent change in dielectric power, and the possibility of sustaining oscillations depend on the fact that the rotating electron cloud has a proper frequency of oscillations (provided the radius of the filament can be neglected) given by

$$\omega=\sqrt{2}\omega_H, \tag{1}$$

where ω_H = Larmor frequency, as noticed in the previous paper. It can be shown by a very direct calculation how such oscillations of cylindrical symmetry will take place in the electron cloud. Equations have been obtained [see Eq. (12) of reference 1] for motions of this type

$$\dot{\eta}=C/r^2, \quad \ddot{r}/r=(e/m)2\pi\rho+\dot{\eta}^2-\omega_H^2, \tag{2}$$

where $\dot{\eta}$ is the angular velocity with respect to an axis rotating with Larmor angular velocity. Critical conditions, as observed in a magnetron, correspond to

$$C=0, \quad \dot{\eta}=0, \quad \rho_0=(m/2\pi e)\omega_H^2. \tag{3}$$

Let us now assume small oscillations of cylindrical symmetry to be excited in this cloud. An electron, originally located, at the instant t, at a distance r_0 will be displaced to

$$r=r_0(1+\xi) \text{ in which } \xi \text{ is small.} \tag{4}$$

This means an expansion of the whole cloud, the density of which becomes

$$\rho=\rho_0(1-2\xi) \tag{5}$$

so that the total electric charge ρr^2 remains constant. Introducing (4) and (5) in the equation of motion (2) one obtains

$$\ddot{r}/r=\ddot{\xi}/(1+\xi)=\omega_H^2(1-2\xi)-\omega_H^2$$

or

$$\ddot{\xi}=-2\omega_H^2\xi, \tag{6}$$

which yields very directly the proper frequency (1) of these cylindrical oscillations. It is now easy to understand how a medium possessing such a proper frequency of oscillations will show an abnormal behavior of its dielectric power,

$$\epsilon=1-(2\omega_H^2/\omega^2) \text{ e.s.u.} \tag{7}$$

as was proved both theoretically and experimentally by Blewett and Ramo.

Turning back to the problem of the magnetron and assuming these elementary calculations to yield a first approximation we come to the following conclusion: A magnetron, with a

[1] L. Brillouin, "Theory of the magnetron. I," Phys. Rev. **60**, 385 (1941) and Elec. Commun. **20**, 112 (1941).
[2] J. P. Blewett and S. Ramo, J. App. Phys. **12**, 856 (1941).

filament of radius a and an anode of radius b should behave like a cylindrical condenser filled with a medium of dielectric power ϵ [Eq. (7)]. Its capacity would then amount to

$$C=\frac{\epsilon}{2\log(b/a)} \tag{8}$$

per unit of length of the filament. This yields an impedance, at the frequency ω

$$Z=-\frac{i}{C\omega}=-i\frac{2\log(b/a)}{\omega\epsilon}=i\frac{2\omega\log(b/a)}{2\omega_H{}^2-\omega^2}. \tag{9}$$

Z would be positive for $\omega<\sqrt{2}\omega_H$ and negative for $\omega>\sqrt{2}\omega_H$, so that the magnetron would behave like a self-inductance below its frequency of resonance $\sqrt{2}\omega_H$, and like a capacity above this proper frequency.

Such a treatment is, however, just a crude first approximation; a more accurate discussion shows that the internal frequencies of oscillation are not constant throughout the electron cloud, but vary from $2\omega_H$ near the filament to $\sqrt{2}\omega_H$ at a distance. This will result in a modification of formula (9) as shall be shown in the next sections, where the problem of the positive or negative internal resistance will be discussed simultaneously.

2. INTERNAL IMPEDANCE OF A MAGNETRON

The theory developed in "Magnetron. I" for a one-anode magnetron is correct so far as Eq. (57) is concerned, but some approximations made in the deduction of Eq. (58) proved to be unreliable, and the calculations have to start again from that point.

It should be first noticed that all potentials $V(r)$ are with reference to the filament, taken as zero potential point. The current I per unit of length of the filament has been defined by a formula (38) and is thus taken as positive if running from the filament to the anode. This is the opposite of the usual assumption in electricity, where a current running from a point at potential zero to a point at a positive potential $V(b)$ should be counted as a negative current.[3] We shall now revert to these usual definitions and change the sign of the currents, writing

$$I_a=-J_ae^{i\omega t}$$

$$\int_{t_0}^{t} I_a dt=i(J_a/\omega)e^{i\omega t}[1-e^{-i\omega\tau}] \qquad t=t_0+\tau \tag{10}$$

instead of formula (54) in "Magnetron. I." As already noticed in Section 7 [Eq. (60)] of the earlier paper it is necessary, for physical and mathematical reasons, to add in Eq. (49) a small damping term s. This will result in the following small change in formula (57):

$$\frac{\partial V_a}{\partial r}=\frac{m}{e}\omega^2 r_a=J_ae^{i\omega t}\frac{2i\omega}{\omega^2(r)-\omega^2+is\omega}\,\frac{1-e^{-i\omega\tau}}{r}. \tag{11}$$

This formula gives the alternative potential $V_a(r)$ at a distance r; it contains the transit time τ from the filament ($r=a$) to r. The damping coefficient s is assumed to be very small, and its absolute value will play a very small role

The assumption is that the direct current I_c flowing through the magnetron is small enough not to perturb appreciably the potential distribution $V(r)$ inside the tube. The average potential should thus be the one obtained under critical conditions

$$V_c(r)=-\frac{m}{2e}\omega_H{}^2r^2\left(1-\frac{a^2}{r^2}\right)^2 \tag{12}$$

as shown in "Magnetron. I," Eq. (51), while the space charge density is

$$\rho_c(r)=\frac{m\omega_H{}^2}{2\pi e}\left(1+\frac{a^4}{r^4}\right) \tag{13}$$

and the proper frequency, Eq. (52),

$$\omega^2(r)=2\omega_H{}^2\left(1+\frac{a^4}{r^4}\right). \tag{14}$$

The value of $\omega(r)$ thus decreases from $2\omega_H$ near the filament down to $\sqrt{2}\omega_H$ at a distance. We furthermore need the relation between transit time τ and distance r, which we obtain from Eq. (43), corrected for the change in the sign of the current:

$$\frac{2}{r}I_c\tau=-E(r)=\frac{\partial V_c}{\partial r}=-\frac{m}{e}\omega_H{}^2r\left(1-\frac{a^4}{r^4}\right).$$

[3] The same correction was also applied in "Magnetron. II," Phys. Rev. 62, 166 (1942), p. 175, before Eq. (77).

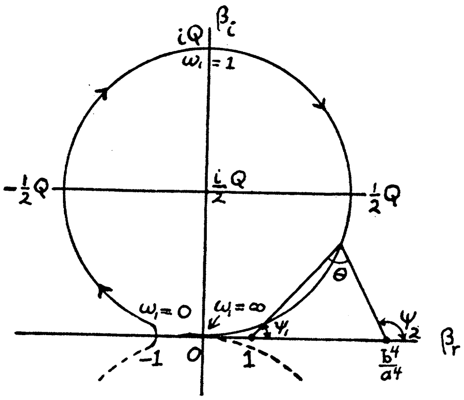

FIG. 1. Curve described by β in the imaginary plane, when ω increases.

Hence

$$\tau=-\frac{m\omega_H{}^2}{2I_ce}r^2\left(1-\frac{a^4}{r^4}\right)=\frac{r^2}{2\omega_H L^2}\left(1-\frac{a^4}{r^4}\right), \quad (15)$$

where L is the characteristic length defined in "Magnetron. I," Eq. (33).

$$L^2=-eI_c/m\omega_H{}^3.$$

This enables us to give a more precise statement of our assumptions: It is assumed that the direct current I_c is small enough to make L smaller than the radius of the filament:

$$L\ll a, \quad \xi_0=\omega a^2/2\omega_H L^2\gg 1. \quad (16)$$

Expressions (14) and (15) for ω^2 and τ must be used in formula (11) from which we obtain the potential $V_a(r)$:

$$V_a(r)=2i\omega J_a e^{i\omega t}$$
$$\times\int_a^r\left(\frac{1-e^{-i\xi}}{2\omega_H{}^2-\omega^2+is\omega+2\omega_H{}^2(a^4/r^4)}\right)\frac{dr}{r} \quad (17)$$

with

$$\xi(r)=\omega\tau(r)=\xi_0\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right).$$

The problem is now to perform the integration, and to obtain the potential $V_a(b)$ on the anode, from which we deduce the internal impedance of the magnetron:

$$Z=\frac{V_a(b)}{J_a e^{i\omega t}}=2i\omega\int_a^b(\cdots)\frac{dr}{r}. \quad (18)$$

It is convenient to use a new variable ζ and to put

$$\zeta=r^2/a^2$$

$$\beta=\frac{-2\omega_H{}^2}{2\omega_H{}^2-\omega^2+is\omega}=2\omega_H{}^2\frac{\omega^2-2\omega_H{}^2+is\omega}{(2\omega_H{}^2-\omega^2)^2+s^2\omega^2}, \quad (19)$$

which yields

$$Z=\frac{i\omega}{2\omega_H{}^2-\omega^2+is\omega}$$
$$\times\int_{\zeta=1}^{\zeta=b^2/a^2}\frac{1-\exp\left[-i\xi_0(\zeta-1/\zeta)\right]}{\zeta^2-\beta}\zeta d\zeta. \quad (20)$$

This is the integral to discuss, and it should be stated that no approximations have been used, but for (15). These equations differ from those of "Magnetron. I," only for the introduction of the damping term s. The physical necessity for this damping term was already emphasized in "Magnetron. I," but one more word of explanation should be added: There is no damping term for a magnetron of infinite length, which does not emit any radiation (no more than an infinite solenoid or a toroidal one) but for a magnetron of finite length, damping is unavoidable, as any motion of the electrons inside the magnetron will result in electromagnetic fields emitted from both ends, and radiative dissipation of energy. There is also a possibility for energy dissipation inside the magnetron itself resulting from the non-linear terms which have been neglected in the equations.

3. INTEGRATION AND DISCUSSION

It is essential to discuss the variation of β and $\beta^{\frac{1}{2}}$ as functions of the frequency ω. The last expression must be of importance in the calculation of the integral (20), which has two poles

$$\zeta_P=\pm\beta^{\frac{1}{2}}, \quad \zeta_P{}^2=\beta. \quad (21)$$

This condition has a clear physical meaning, as it corresponds to the resonance on one of the internal frequencies of the electron cloud. If we neglect, for one moment, the damping term s (supposed to be very small) then condition (21) means

$$\omega^2=\omega^2(r)=2\omega_H{}^2(1+a^4/r^4) \quad (21a)$$

which indicates resonance on the layer at dis-

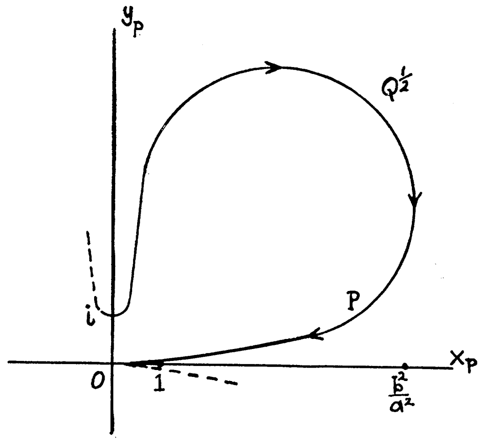

FIG. 2. Curve described by the poles P for increasing ω_1. $\zeta_P=\beta^{\frac{1}{2}}=x_P+iy_P$.

tance r. Here is the place where the approximations made in Section 1 prove unreliable: The radius a of the filament may be extremely small, but nevertheless, the inner proper frequencies always range from $\sqrt{2}\omega_H$ (r large) to $2\omega_H$ ($r=a$).

Figure 1 shows the motion of the point β in the complex plane, when ω varies from zero to infinity. According to Eq. (19)

$$\beta=\frac{1}{\omega_1{}^2-1-i(\omega_1/Q)}$$

with

$$\omega_1=\omega/\sqrt{2}\omega_H=y/\sqrt{2}, \quad Q=\sqrt{2}\omega_H/s. \tag{21b}$$

The curve starts from $\beta=-1$ for $\omega_1=0$ with a vertical tangent, and then runs upwards, with an almost circular shape. The upper point $\beta=iQ$ corresponds to $\omega_1=1$ while $\omega_1=\infty$ brings the representative point near the origin. The dotted curve corresponds to negative ω_1 and has no meaning in our problem. The damping term s is supposed to be very small, so that the Q factor will be extremely large. Figure 2 shows how the position of the poles $\beta^{\frac{1}{2}}$ varies with the frequency ω. For very low frequency the poles are in $\pm i$, then describe large loops, reach a very large distance $Q^{\frac{1}{2}}$ in the 45° direction (for $\omega_1=1$), and come back to 0 along the real axis, for ω_1 infinite.

The integral (20) is taken along the real axis, from $\zeta=1$ to b^2/a^2 and this path runs below the trajectory followed by the pole. The pole may, however, be very near to it (just above it) when its abscissa x lies between 1 and b^2/a^2 which means approximately

$$2\omega_H \geqslant \omega \geqslant \omega_H(2+2a^4/b^4)^{\frac{1}{2}}. \tag{22}$$

In case b is much greater than a the lower limit is very nearly $\sqrt{2}\omega_H$.

The integral (20) may conveniently be split into two parts

$$\int_{\zeta=1}^{\zeta=b^2/a^2}\cdots d\zeta=I_1+I_2,$$

$$\begin{aligned} I_1&=\tfrac{1}{2}\int_1^{b^2/a^2}\frac{2\zeta d\zeta}{\zeta^2-\beta}=\tfrac{1}{2}\left|\log\,(\zeta^2-\beta)\right|_1^{b^2/a^2},\\ I_2&=-\int_1^{b^2/a^2}\frac{\exp\left[-i\xi_0(\zeta-1/\zeta)\right]}{\zeta^2-\beta}\zeta d\zeta. \end{aligned} \tag{23}$$

Integral I_1 is the logarithm of imaginary expressions

$$2I_1=\log\left[(b^4/a^4)-\beta\right]-\log\,(1-\beta). \tag{24}$$

Now, as well known, for an imaginary $\varphi=\varphi_r+i\varphi_i$

$$\log\varphi=\log\,|\varphi|+i\,\mathrm{arctg}\,\varphi_i/\varphi_r, \quad |\varphi|^2=\varphi_r{}^2+\varphi_i{}^2.$$

Both terms in (24) are of the following type, with X real (1 or b^4/a^4) and $\beta=\beta_r+i\beta_i$ as in (21b),

$$\begin{aligned} \log\,(X-\beta)&=\log\,|X-\beta|+i\,\mathrm{arctg}\frac{\beta_i}{\beta_r-X}\\ &=\log\,|X-\beta|+i\psi. \end{aligned}$$

The ψ angles are indicated on Fig. 1, hence,

$$2I_1=\log\left|\frac{b^4/a^4-\beta}{1-\beta}\right|+i\theta, \quad \theta=\psi_2-\psi_1.$$

As noticed before, the damping coefficient s is very small, which means that Q is a *very large quantity*, and we assume

$$Q\gg b^4/a^4; \quad \beta\approx\beta_r, \quad \beta_i\approx 0. \tag{24a}$$

Figure 1 shows then that the angle θ is very small, except when the point P comes very near the integration path (conditions 22), in

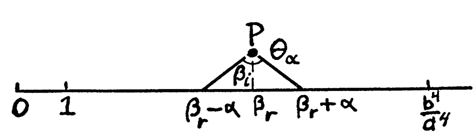

FIG. 3.

which case θ is practically equal to π. Finally

$$2I_1=\log\left|\frac{b^4/a^4-\beta_r}{1-\beta_r}\right|+i[\pi]; \tag{24b}$$

$[\pi]$ for conditions (22) only.

Turning now to the I_2 integral, in (23), we notice that the coefficient ξ_0 is large, so that the exponential oscillates very rapidly, which will practically result in cancelling the integration wherever the $\zeta/(\zeta^2-\beta)$ factor is a slowly varying function of ζ. Hence the most important part of the integral comes from the neighborhood of the poles $\zeta_P=\pm\beta^{\frac{1}{2}}$. As shown on Fig. 2, the pole with the plus sign may come near the integration path, which runs from 1 to b^2/a^2 along the real axis.

Two cases must be distinguished:

A—pole near the integration path, but outside of it.

$$\beta=\beta_r+i\beta_i \quad \text{and} \quad \beta_r\leqslant 1 \quad \text{or} \quad \beta_r\geqslant b^4/a^4.$$

B—pole near the integration path, and inside of it, which corresponds to conditions (22).

$$1<\beta_r<b^4/a^4.$$

Case B needs a special discussion before it can be reduced to a similar situation as in case A. We shall, in case B, divide the integration into 3 parts:

$$\begin{aligned}&\text{I.} && 1\leqslant\zeta^2\leqslant\beta_r-\alpha && \beta_i \text{ negligible,}\\ &\text{II.} && \beta_r-\alpha\leqslant\zeta^2\leqslant\beta_r+\alpha && \text{exponential practically constant,}\\ &\text{III.} && \beta_r+\alpha\leqslant\zeta^2\leqslant b^4/a^4 && \beta_i \text{ negligible.}\end{aligned} \tag{25}$$

α is a small quantity, which is chosen such as to satisfy the two following conditions:

$$\beta_i\ll\alpha\ll\frac{2\surd\beta_r}{\xi_0(1+1/\beta_r)}. \tag{25a}$$

The first inequality $\beta_i\ll\alpha$ enables one to neglect β_i in both intervals I and III where one has

$$\begin{aligned}\zeta^2-\beta=\zeta^2-\beta_r-i\beta_i\approx\zeta^2-\beta_r\\ |\zeta^2-\beta_r|\geqslant\alpha\gg\beta_i.\end{aligned} \tag{26}$$

The second inequality results in the fact that the exponential is practically constant inside the second interval where

$$\begin{aligned}&\zeta=\surd\beta_r(1+\epsilon),\quad 1/\zeta=(1-\epsilon)/\surd\beta_r,\\ &\zeta^2\approx\beta_r(1+2\epsilon),\\ &\xi_0[\zeta-(1/\zeta)]=\xi_0[\surd\beta_r-(1/\surd\beta_r)]\\ &\qquad+\xi_0\epsilon[\surd\beta_r+(1/\surd\beta_r)].\end{aligned} \tag{27}$$

The exponential is practically constant and the ϵ term can be dropped if

$$\epsilon\xi_0[\surd\beta_r+(1/\surd\beta_r)]\ll 1,$$

but the interval II is defined by

$$|\zeta^2-\beta_r|\leqslant\alpha \quad \text{or} \quad 2\beta_r|\epsilon|\leqslant\alpha,$$

and the inequality on ϵ is certainly satisfied if

$$\xi_0[\surd\beta_r+(1/\surd\beta_r)](\alpha/2\beta_r)\ll 1,$$

which is the second condition stated in (25a). These conditions (25a) can be satisfied only if

$$\frac{\beta_i}{\surd\beta_r}\ll\frac{2}{\xi_0[1+(1/\beta_r)]} \tag{28}$$

and this will result in a condition to be fulfilled by the damping coefficient s or the Q factor. According to (21b) and (16)

$$\beta_r=\frac{\omega_1^2-1}{(\omega_1^2-1)^2+\omega_1^2/Q^2}\approx\frac{1}{\omega_1^2-1},$$

$$\beta_i=\frac{\omega_1/Q}{(\omega_1^2-1)^2+\omega_1^2/Q^2}\approx\frac{\omega_1}{Q(\omega_1^2-1)},$$

in which $1/Q^2$ is neglected.

$$\xi_0=\frac{\omega}{2\omega_H}\frac{a^2}{L^2}=\frac{\omega_1 a^2}{\sqrt{2}L^2}\gg 1.$$

Hence condition (28) yields

$$Q\gg\frac{\omega_1^4 a^2}{(\omega_1^2-1)^{\frac{1}{2}}2\sqrt{2}L^2}. \tag{29}$$

When condition B (that the pole be inside the interval of integration) is satisfied, the reduced frequency ω_1 lies between 1 and $\sqrt{2}$. On the other hand, ξ_0 is large, which means $L^2\ll a^2$, and condition (29) proves that Q must be very large in order to make our approximations reasonable.

This was always our general assumption. At any rate, conditions (24b) and (29) both postulate a very large Q value and extremely small damping coefficient s, which corresponds to actual conditions in magnetrons.

As a result of this discussion, we are left with the following integrals to calculate (case B, conditions 22):

$$I_2=-\int_1^{(\beta_r-\alpha)^{\frac{1}{2}}}-\int_{(\beta_r+\alpha)^{\frac{1}{2}}}^{b^2/a^2}\exp\left[-i\xi_0(\zeta-1/\zeta)\right]\frac{\zeta d\zeta}{\zeta^2-\beta_r}$$
$$-\exp\left[-i\xi_0(\sqrt{\beta_r}-1/\sqrt{\beta_r})\right]\int_{(\beta_r-\alpha)^{\frac{1}{2}}}^{(\beta_r+\alpha)^{\frac{1}{2}}}\frac{\zeta d\zeta}{\zeta^2-\beta_r-i\beta_i}. \quad (30)$$

The first two integrals correspond to cases I and III and the last one to case II. Let us first notice that we can add to the first two integrals an additional one, taken from $(\beta_r-\alpha)^{\frac{1}{2}}$ to $(\beta_r+\alpha)^{\frac{1}{2}}$ (on the interval II) and which is practically zero.

$$\int_{(\beta_r-\alpha)^{\frac{1}{2}}}^{(\beta_r+\alpha)^{\frac{1}{2}}}\exp\left[-i\xi_0(\zeta-1/\zeta)\right]\frac{\zeta d\zeta}{\zeta^2-\beta_r}\approx\exp\left(-i\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]\right)\int_{(\beta_r-\alpha)^{\frac{1}{2}}}^{(\beta_r+\alpha)^{\frac{1}{2}}}\frac{\zeta d\zeta}{\zeta^2-\beta_r}$$
$$=\tfrac{1}{2}\exp\left(-i\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]\right)\log|\alpha/\alpha|.$$

According to (27), the exponential is constant in this interval, hence the integral reduces to a logarithm, which is now taken on real quantities and yields $\log|\alpha/\alpha|=\log 1=0$. Integral (30) can thus be written

$$I_2=-\int_1^{b^2/a^2}\exp\left(-i\xi_0[\zeta-(1/\zeta)]\right)\frac{\zeta d\zeta}{\zeta^2-\beta_r}-\exp\left(-i\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]\right)\int_{(\beta_r-\alpha)^{\frac{1}{2}}}^{(\beta_r+\alpha)^{\frac{1}{2}}}\frac{\zeta d\zeta}{\zeta^2-\beta_r-i\beta_i}. \quad (31)$$

The last integral is again a logarithm, but contains an imaginary part β_i; it is similar to I_1 [Eq. (23)] but with different limits and yields

$$\int_{(\beta_r-\alpha)^{\frac{1}{2}}}^{(\beta_r+\alpha)^{\frac{1}{2}}}\frac{\zeta d\zeta}{\zeta^2-\beta_r-i\beta_i}=\tfrac{1}{2}\log\left|\frac{\alpha-i\beta_i}{\alpha+i\beta_i}\right|+\frac{i}{2}\theta_\alpha.$$

The angle θ_α, as shown on Fig. 3, is given by

$$tg(\theta_\alpha/2)=\alpha/\beta_i\gg 1,\quad \tfrac{1}{2}\theta_\alpha\approx\tfrac{1}{2}\pi,$$

according to the inequalities (25a), while the first log is zero. Summarizing these results we may write

$$I_2=-\int_1^{b^2/a^2}\exp\left(-i\xi_0[\zeta-(1/\zeta)]\right)\frac{\zeta d\zeta}{\zeta^2-\beta_r}-\left[\frac{i\pi}{2}\exp\left(-i\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]\right)\right], \quad (32)$$

where the last term inside the brackets has to be used only when conditions (22) (case B) are fulfilled.

We are left now with the problem of estimating the contribution of the first integral, where all the difficulties related with the imaginary part in the denominator have been excluded. This first integral now keeps the same type in both cases A and B.

As already noticed, the exponential oscillates very rapidly, and this practically cancels most of the integral, but for the neighborhood of the pole $\zeta_P=\beta_r^{\frac{1}{2}}$. Near this point we may conveniently use the following expansions:

$$\zeta=\sqrt{\beta_r}(1+\epsilon),\quad \zeta-1/\zeta=\sqrt{\beta_r}(1+\epsilon)-(1-\epsilon)/\sqrt{\beta_r},$$

$$I_2 = -\tfrac{1}{2}\exp\left(-i\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]\right)\int_{\epsilon_1}^{\epsilon_2}\exp\left(-i\xi_0\epsilon[\sqrt{\beta_r}+(1/\sqrt{\beta_r})]\right)\frac{d\epsilon}{\epsilon}$$

$$= -\tfrac{1}{2}\exp\left(-i\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]\right)\int_{\eta_1}^{\eta_2} e^{-i\eta}\frac{d\eta}{\eta} \quad (33)$$

with $\eta=\xi_0\epsilon[\sqrt{\beta_r}+(1/\sqrt{\beta_r})]=\xi_0[1+(1/\beta_r)](\zeta-\sqrt{\beta_r})$.

The new integral in η is

$$\int_{\eta_1}^{\eta_2} e^{-i\eta}\frac{d\eta}{\eta}=\int_{\eta_1}^{\eta_2}(\cos\eta-i\sin\eta)\frac{d\eta}{\eta}=-Ci\eta_1+Ci\eta_2+i\,si\eta_1-i\,si\eta_2 \quad (34)$$

according to the usual definitions[4] of integral sines and cosines for positive x

$$Cix=-\int_x^\infty \cos t\frac{dt}{t},\quad six=-\int_x^\infty \sin t\frac{dt}{t}. \quad (35)$$

When x is negative, the same definition may be used for six as the point $x=0$ is no pole for this integral, and the relation holds

$$si(-x)=-\pi-six. \quad (36)$$

For $Ci(-x)$, our integration path runs through the pole, which means that we must take (see reference 4, p. 4)

$$Ci(-x)=Ci(x). \quad (37)$$

Here we shall notice that the additional $-\pi$ term in (36) comes into play only when the lower limit η_1 is negative, which means that conditions (22) (case B) are realized. But then we have also an additional term in (32), and these two terms exactly cancel each other, hence, we obtain the following formula, as a good approximation *for all cases:*

$$I_2=-\tfrac{1}{2}\exp\left(-i\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]\right)\{-Ci|\eta_1|+Ci|\eta_2|\pm isi|\eta_1|\mp isi|\eta_2|\}. \quad (38)$$

the lower signs, for the si functions, correspond to the case of negative arguments η_1 or η_2.

As well known, the Ci and si functions become rapidly very small for large values of the argument η

$$\eta\gg 1,\quad Ci\eta\approx\frac{\sin\eta}{\eta},\quad si\eta\approx-\frac{\cos\eta}{\eta}. \quad (39)$$

These asymptotic expressions can be used as soon as η is larger than 2π. Now we must keep in mind that the approximate expression (38) for I_2 has been obtained only for small values of ϵ. But a small ϵ may mean a rather large η, according to (33), because ξ_0 is assumed to be large. Furthermore, we must notice that both I_2 and formula (38) become very small for large η and so we may, for practical purposes, assume formula (38) to hold for any value of η, even large. The limits are

$$\begin{aligned}\zeta_1&=1, & \eta_1&=\xi_0[1+(1/\beta_r)](1-\sqrt{\beta_r}),\\ \zeta_2&=b^2/a^2, & \eta_2&=\xi_0[1+(1/\beta_r)][(b^2/a^2)-\sqrt{\beta_r}].\end{aligned} \quad (40)$$

As soon as η_1 and η_2 are none too small (larger than 10), the asymptotic expansions (39) may be used

$$Ci|\eta|\mp isi|\eta|\approx(1/|\eta|)(\sin|\eta|\pm i\cos|\eta|)=(1/\eta)(\sin\eta+i\cos\eta)=(i/\eta)e^{-i\eta}. \quad (41)$$

Hence, the following result may be used only when η_1 or η_2 are not too small.

$$I_2\approx\frac{i}{2}\left\{\frac{1}{\eta_1}\exp\left[-i\xi_0\left(\sqrt{\beta_r}-\frac{1}{\sqrt{\beta_r}}\right)-i\eta_1\right]-\frac{1}{\eta_2}\exp\left[-i\xi_0\left(\sqrt{\beta_r}-\frac{1}{\sqrt{\beta_r}}\right)-i\eta_2\right]\right\}. \quad (42)$$

[4] E. Jahnke and F. Emde, *Tables of Functions* (Teubner, Berlin, 1938), third edition, p. 3.

But, according to the approximations used in (33),

$$\begin{aligned}\xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]+\eta_2 &\approx \xi_0[\zeta_2-(1/\zeta_2)]=\xi_0[(b^2/a^2)-(a^2/b^2)],\\ \xi_0[\sqrt{\beta_r}-(1/\sqrt{\beta_r})]+\eta_1 &\approx \xi_0[\zeta_1-(1/\zeta_1)]=0.\end{aligned} \tag{43}$$

Hence,

$$I_2\approx\frac{i}{2\xi_0[1+(1/\beta_r)]}\left[\frac{1}{1-\sqrt{\beta_r}}-\frac{\exp\left[-i\xi_0\{(b^2/a^2)-(a^2/b^2)\}\right]}{(b^2/a^2)-\sqrt{\beta_r}}\right]. \tag{44}$$

Returning now to Z from Eq. (20) and using (24b) and (44) we finally obtain

$$Z\approx\frac{\frac{1}{2}\omega}{2\omega_H{}^2-\omega^2+is\omega}\left\{i\log\left|\frac{b^4/a^4-\beta_r}{1-\beta_r}\right|-\frac{1}{\xi_0[1+(1/\beta_r)]}\times\left[\frac{1}{1-\sqrt{\beta_r}}-\frac{\exp\{-i\xi_0[(b^2/a^2)-(a^2/b^2)]\}}{(b^2/a^2)-\sqrt{\beta_r}}-[\pi]\right]\right\}. \tag{45}$$

The $-[\pi]$ term comes in only when conditions (22) are fulfilled. One should be reminded that formula (45) is valid only for large $\eta_1\eta_2$ which means $\sqrt{\beta_r}$ is very different from 1 or b^2/a^2. Let us here notice the following:

1. s may now be neglected, as very small, its introduction was only necessary for the calculation of residues near the pole. Q is practically infinite, hence,
2. $\beta=\beta_r=-2\omega_H{}^2/(2\omega_H{}^2-\omega^2)$, (46)
3. $\xi_0[(b^2/a^2)-(a^2/b^2)]=\omega\tau_{ab}$.

τ_{ab} represents the transit time for electrons running from the cathode a to the anode b.

4. PRACTICAL RESULTS

Let us first assume the magnetron to be operated below its critical conditions, with no current flowing to the anode, which makes ξ_0 infinite and I_2 naught. Formula (45) then reduces to

$$Z\approx\frac{1}{2}\frac{\omega}{2\omega_H{}^2-\omega^2}\left\{i\log\left|\frac{(b^4/a^4)-\beta}{1-\beta}\right|-[\pi]\right\}, \tag{47}$$

the $[\pi]$ term comes in only when conditions (22) are fulfilled, namely:

$$\omega_H[2+(2a^4/b^4)]^{\frac{1}{2}}\leqslant\omega\leqslant 2\omega_H. \tag{22}$$

This proves that such a magnetron behaves like a pure inductance for low or high frequencies, and like an inductance plus a positive resistance for frequencies defined in (22). The following cases may be considered:

$$\omega\ll\sqrt{2}\omega_H,\quad -\infty\ll\beta<-1. \tag{48a}$$

For such low frequencies the $[\pi]$ term does not come in and

$$Z\approx\frac{1}{2}\frac{i\omega}{2\omega_H{}^2-\omega^2}\log\left(\frac{(b^4/a^4)+|\beta|}{1+|\beta|}\right). \tag{49a}$$

$$\omega\gg2\omega_H,\quad 0<\beta\ll1. \tag{48b}$$

For very high frequencies the $[\pi]$ term also drops and

$$Z\approx\frac{1}{2}\frac{i\omega}{2\omega_H{}^2-\omega^2}\log\left(\frac{b^4}{a^4}\right). \tag{49b}$$

This later case corresponds to the result predicted in Eq. (9) and checks the conclusions of Blewett and Ramo as explained in Section 1.

The formula (47) needs also a special discussion of the behavior near $\sqrt{2}\omega_H$, when the denominator is zero, as shown below. Let us assume $\omega^2=2\omega_H{}^2+\epsilon$, $\beta=2\omega_H{}^2/\epsilon$, ϵ small.

$$\frac{(b^4/a^4)-\beta}{1-\beta}=\frac{1-(\epsilon b^4/2\omega_H{}^2a^4)}{1-(\epsilon/2\omega_H{}^2)}=1+\frac{\epsilon}{2\omega_H{}^2}\left(1-\frac{b^4}{a^4}\right).$$

Hence,

$$\log\left|\frac{(b^4/a^4)-\beta}{1-\beta}\right|=\frac{\epsilon}{2\omega_H{}^2}\left[1-\left(\frac{b^4}{a^4}\right)\right]$$

and

$$Z=\frac{i}{2}\frac{\omega}{(-\epsilon)}\frac{\epsilon}{2\omega_H{}^2}\left(1-\frac{b^4}{a^4}\right)=\frac{i}{2\sqrt{2}\omega_H}\left(\frac{b^4}{a^4}-1\right) \tag{50}$$

which remains finite. This shows that these conditions do not lead to any difficulty.

The resistance term in [π], which comes in when conditions (22) are fulfilled is positive, as the minus sign before π is compensated by the negative denominator $2\omega_H^2-\omega^2$. It should be explained by the following physical considerations, based on the remarks made at the end of Section 2: When the frequency lies between the limits (22), a certain layer of the electronic space charge oscillates very strongly on resonance conditions and radiates energy outside of the magnetron through both ends of the magnetron.

Let us now assume that the magnetron is operated just above critical conditions, with a small current flowing to the anode and a large but finite ξ_0 value. The complete expressions from (38) must now be taken into consideration but for practical discussions we may use the simplified formula (45). The internal impedance of the magnetron will yield a real part R or resistance, with the following approximate values:

Case I. $\dfrac{b^2}{a^2}<\surd\beta,\quad 2\omega_H<\omega<\omega_H\left[2+\left(2\dfrac{a^4}{b^4}\right)\right]^{\frac{1}{2}}$

β very large,

$$R\approx\frac{\omega}{2\omega_H^2-\omega^2}\frac{\cos\xi_0[(b^2/a^2)-(a^2/b^2)]}{2\xi_0[1+(1/\beta)][(b^2/a^2)-\surd\beta]}. \tag{51}$$

Case II. $1<\surd\beta<b^2/a^2$ (conditions 22),

$$R\approx\frac{\omega}{\omega^2-2\omega_H^2}\left\{\frac{\pi}{2}+\frac{1}{2\xi_0[1+(1/\beta)]}\right. \cdot$$
$$\left.\times\left[\frac{1}{1-\surd\beta}-\frac{\cos\xi_0[(b^2/a^2)-(a^2/b^2)]}{(b^2/a^2)-\surd\beta}\right]\right\}. \tag{52}$$

Case III. $\surd\beta<1,\quad \omega>2\omega_H,$

$$R\approx\frac{\omega}{\omega^2-2\omega_H^2}\frac{1}{2\xi_0[1+(1/\beta)](1-\surd\beta)}. \tag{53}$$

Let us now discuss the sign of the resistance: In cases II and III the resistance is always positive. In case II, Eq. (52), ξ_0 is large and $\surd\beta$ lies between 1 and b^2/a^2 so that the first term in $\pi/2$, inside the brackets, is by far the most important. In case III, Eq. (53) the result is obviously positive. Hence the discussion is now centered on case I, which shows a certain number of frequency bands with negative resistance, according to the sign of the cosine. The denominator is positive, and we obtain

$$R<0\quad\text{when}\quad\cos\xi_0(b^2/a^2-a^2/b^2)<0. \tag{54}$$

The negative resistance may reach high values when the frequency is such as to make the denominator D very small. By use of (46.2) we have

$$D=2\xi_0(2\omega_H^2-\omega^2)[1+(1/\beta)][(b^2/a^2)-\surd\beta]$$
$$=\xi_0(2\omega_H^2-\omega^2)\frac{\omega^2}{\omega_H^2}\left[\left(\frac{b^2}{a^2}\right)-\left(\frac{2\omega_H^2}{\omega^2-2\omega_H^2}\right)^{\frac{1}{2}}\right].$$

This amounts to

$$D=\xi_0(\omega^2/\omega_H^2)(\omega^2-2\omega_H^2)^{\frac{1}{2}}$$
$$\times[\surd 2\omega_H-(b^2/a^2)(\omega^2-2\omega_H^2)^{\frac{1}{2}}] \tag{55}$$

which is zero for

$$\omega=\surd 2\omega_H \tag{56I}$$

or

$$\omega=\omega_H(2+2a^4/b^4)^{\frac{1}{2}}. \tag{56II}$$

The first case does not seem of real importance, first because the corresponding zero would be avoided by keeping the damping term s as in Eq. (45) and second because the calculations and approximations made in Eq. (33) and following are no more reliable, as for such frequencies the pole runs far away from the integration path, on the line at 45° (Fig. 2). Hence it does not seem that the negative resistance might reach any high value near the frequency (56I).

The second root (56II) is much more important as it means that the frequency ω is just the one which sets into resonance the space charge density near the anode. The conclusion of all this discussion is the following:

One anode magnetrons should yield a number of very narrow frequency bands with negative resistance [Eq. (54)] *for frequencies lying just below the limit* (56II).

This checks the general conclusion of a former paper (Magnetron. I). One should notice here that the negative resistance R does not become infinite on the limit (56II), as Eq. (51) seems to indicate. Near this limit, the approximate formula (45) should not be used any more, as it rests on the use of asymptotic expansions (39).

One should revert to the original formula (38) with integral sines and cosines, thus avoiding infinite R values.

5. PLANE MAGNETRON

As an example of application of the general theory, the case of the plane magnetron will be briefly discussed. This case is obtained as the limit of a cylindrical magnetron when the radius a of the filament is increased to infinity, while the distance $d=b-a$ between filament and anode is kept constant. Hence

$$a\rightarrow\infty,\quad b=a+d=a(1+p_b). \tag{57}$$

A point at a distance r from the center will be defined by a parameter p:

$$r=a(1+p) \tag{58}$$

in which $p=(r-a)/a$ is small. The angular velocity of the electrons is given by Eq. (15) of "Magnetron. I" which now reads

$$\dot{\theta}=\omega_H[1-(a^2/r^2)]=\omega_H(1-1/(1+p)^2)$$
$$\approx\omega_H(2p-3p^2\cdots). \tag{59}$$

This means that the tangential velocity becomes

$$\dot{x}=r\dot{\theta}=2a\omega_H p(1+p)(1-\tfrac{3}{2}p)\approx 2\omega_H a p(1-\tfrac{1}{2}p).$$

Denoting y the distance $r-a$ from the surface of the cathode, we obtain

$$\dot{x}=2\omega_H y[1-(y/2a)]. \tag{60}$$

For the plane magnetron ($a\rightarrow\infty$) the x velocity increases proportionately to the distance y from the cathode.

Under critical conditions (no current on the anode), the space charge density is given by Eq. (25) of "Magnetron. I":

$$\rho=\frac{m\omega_H^2}{2\pi e}\left(1+\frac{a^4}{r^4}\right)\approx\frac{m\omega_H^2}{\pi e}(1-2p), \tag{61}$$

which on the limit of a plane magnetron ($p=0$), is a constant. In a plane magnetron (of infinite length) operated under critical conditions, there is a uniform space charge density (61) between the plane cathode and the plane anode. These charges have no perpendicular component of velocity ($\dot{y}=0$) and move parallel to the electrodes with a velocity $\dot{x}$ given by Eq. (60). The voltage V is obtained from Eq. (23) of "Magnetron. I."

$$V_0(r)=-\frac{m\omega_H^2}{2e}\left[r-\left(\frac{a^2}{r}\right)\right]^2$$
$$=-\frac{m\omega_H^2}{2e}a^2\left(1+p-\frac{1}{1+p}\right)^2\approx\frac{2m\omega_H^2}{e}a^2p^2(1-p)$$

or

$$V_0(y)=\frac{2m\omega_H^2}{e}y^2\left(1-\frac{y}{a}\right). \tag{62}$$

This yields the anode voltage $V(b)$ when $y=d$ is inserted in the formula. When the total electric charge Q between the electrodes is computed, it comes out to be

$$Q=-V_0L/2\pi d, \tag{63}$$

L is the length measured along the x axis, and the calculation refers to a unit length of the filament. The plane magnetron thus yields a capacity twice as large as a plane condenser of same dimensions.

Such a plane magnetron can be used for sustaining oscillations, and the discussion of the preceding section shows that it should oscillate in the neighborhood of the frequency

$$\omega=2\omega_H,\quad \frac{b}{a}\rightarrow 1, \tag{64}$$

which corresponds to the proper frequency of vibration for the electronic layers near the filament, as emphasized in Section 1.

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