THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

VOL. 62, NOS. 3 AND 4 AUGUST 1 AND 15, 1942 SECOND SERIES

The Pseudoscalar Meson Field with Strong Coupling

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(Received June 23, 1942)

The present paper treats the symmetrical and charged pseudoscalar theories of the meson field, using the strong coupling approximation; it restricts itself to the case of a single source. The energy levels of the excited states of the heavy particle and the scattering cross section for free mesons are computed by wave mechanical methods. An expression is also obtained for the magnetic moment of the proton or neutron. While the scattering cross section can, with reasonable assumptions, be brought into agreement with experimental values, the results for the magnetic moment are qualitatively at variance with the known values in that equal and opposite moments are predicted for proton and neutron.

1. INTRODUCTION

THE perturbation theoretic treatment of the coupling of the meson field to the heavy particle, based on the assumption that the coupling is weak, encounters several fundamental difficulties in its application. The divergences which arise from the treatment of the heavy particle as an infinite point mass have been in some instances arbitrarily removed by cut-off methods, but it can be shown that such methods cannot be consistently formulated in such a way as to satisfy the criterion for weak coupling.[1] Moreover, calculations of the scattering cross section for mesons in this theory lead to values which are generally much too high to agree with cross sections observed for cosmic-ray mesons. The results vary somewhat depending on the particular type of meson field assumed. If we assume that the spin of the meson is zero[2] (charged or symmetrical pseudoscalar), perturbation theory leads to the following result[3] for the scattering of a meson by a nucleon.[4]

$$dq = g^4\kappa^{-2}(p/\kappa)^2(1+\cos\theta)d\Omega \quad \text{for} \quad \hbar p \gg Mc,$$
$$dq = g^4\kappa^{-2}(p^2/\kappa E)^2 d\Omega \quad \text{for} \quad \hbar p \ll Mc, \quad M\to\infty.$$

Here g is the dimensionless constant which expresses the magnitude of the coupling; $\kappa=\mu c/\hbar$ where μ is the rest mass of the meson; p is the momentum of the incident meson divided by $\hbar$; E the total energy divided by $\hbar c$; M the rest mass of the nucleon; and θ the scattering angle. These cross sections refer to processes in which

[1] This point will be more fully discussed in a subsequent paper in which the two-source problem will be considered. We are indebted to Professor R. Serber for valuable discussions on this point as well as on several others considered in this paper.

[2] This is one of the two possibilities left open by a consideration of the electromagnetic radiation processes of the meson. See R. F. Christy and S. Kusaka, Phys. Rev. **59**, 405 and 414 (1941). The other possible value of the spin ($\frac{1}{2}$) will be discussed by J. R. Oppenheimer and E. Nelson in a forthcoming publication.

[3] H. Yukawa and Y. Tanikawa, Proc. Phys. Math. Soc. Japan **23**, 445 (1941).

[4] Nucleon is equivalent to "proton-neutron."

the meson is scattered with the original charge. In the case of the symmetrical theory there exists also a scattering process in which the meson changes its charge. If a negative (positive) meson collides with a proton (neutron) it can be scattered as a neutral meson, while the nucleon changes into a neutron (proton). According to computations of F. Adler, not yet published, the corresponding cross section is

$$dq = g^4\kappa^{-2}(p/\kappa)^2(1-\cos\theta/2)^2 d\Omega \quad \text{for} \quad hp \gg Mc,$$
$$dq = g^4\kappa^{-2}(p^2/\kappa E)^2 2\sin^2\theta d\Omega \text{ for } hp \ll Mc,\ M\rightarrow\infty.$$

A numerical estimate of the total cross section depends upon an assignment of the magnitude of g. This is ordinarily done by considering the nuclear forces predicted by the theory and adjusting g to fit the properties of the deuteron. This involves a cut-off or other readjustment of the radial dependence of the interaction. As remarked above, such a procedure cannot be carried through in a manner consistent with the weak coupling hypothesis. Nevertheless, the value of g so determined ($g^2\sim 0.1$) will be used to give an order of magnitude estimate of the scattering cross section. We obtain a total cross section $\sim 2\times 10^{-26}$ cm^2, as compared with the upper limit of 5×10^{-28} cm^2 determined from experiments on cosmic-ray mesons.[5]

Two apparently different theories have been proposed to explain the smallness of the meson scattering cross section. Heisenberg pointed out that the reaction of the eigenfield of the nucleon to the motion of its spin—and particularly the terms proportional to $1/a$ which one can consider as an inertia of the spin—are considerable.[6] As an example, he computed by classical methods, the scattering in a neutral theory with spin 1 mesons; he found a scattering cross section proportional to $a^2(p/E)^4$ if $g^2 \gg \kappa a$.[7]

The second theory is due to Heitler[8] and Bhabha[9] and based on the assumption of the existence of excited states (isobars) of the nucleon with higher values of the charge (Bhabha) and of the spin (Heitler). In this theory the energy difference ΔE between consecutive isobaric states is not derived, but arbitrarily assumed. On the basis of these assumptions, one finds a considerably reduced value for the cross section for scattering of a meson by a nucleon (of infinite mass) at rest:

$$q_1 = q_0(\Delta E/E)^2.$$

E is the meson energy and q_0 is the total cross section (given above) for scattering as calculated by perturbation theory.

It was shown by Oppenheimer and Schwinger[10] that the two explanations are in reality one and the same. One can indeed derive from Heisenberg's assumptions the existence of excited states of the nucleon, corresponding to the degree of freedom associated with the reaction of the nucleon spin. It is easy to see that if we assume this excitation energy proportional to

$$\Delta E \sim \kappa^2 a/g^2$$

and combine this with Heisenberg's result

$$q_1 \sim a^2(p/E)^4$$

we obtain Heitler and Bhabha's formula for q_1/q_0. The above formula for ΔE, presumably valid for vector mesons, is identical with the corresponding formula that will be rigorously derived for the pseudoscalar case (see below).

2. RESULTS AND CONCLUSIONS

We give here the principal results of this investigation, including the excitation of the isobaric states, the cross section for meson scattering, and the resultant magnetic moments of the proton and neutron. As the condition for the validity of the strong coupling approximation we obtain $g \gg \kappa a$, in the case of small source size

[5] For instance, see R. P. Shutt, Phys. Rev. **61**, 6 (1942). Older literature is reviewed.

[6] The quantity a measures the radius of the nucleon. A precise definition is given by (4).

[7] W. Heisenberg, Zeits. f. Physik **113**, 61 (1939). The problem of interaction of a dipole with its own electromagnetic or meson field is treated in detail by H. J. Bhabha and H. C. Corben, Proc. Roy. Soc. **A178**, 273 (1941) and H. J. Bhabha, Proc. Roy. Soc. **A178**, 314 (1941). A classical but relativistically invariant method is used. The terms in the Hamiltonian proportional to $1/a$ are replaced by terms containing undetermined constants. Such a procedure may approximate an exact treatment more closely than the one used in this paper, but it is not suitable for a quantum-mechanical treatment of the problem.

[8] Compare the detailed paper of W. Heitler and S. T Ma, Proc. Roy. Soc. **A176**, 368 (1940).

[9] H. J. Bhabha, Phys. Rev. **59**, 100 (1941). We refer to this paper for the discussion of the influence of the Coulomb forces on the probability of the generation of a doubly charged proton.

[10] J. R. Oppenheimer and J. Schwinger, Phys. Rev. **60**, 150 (1941).

($\kappa a \ll 1$). For the energy of excitation of the isobars we find

$$\frac{\Delta E}{\mu c^2}=\frac{3}{2}\frac{\kappa a}{g^2}\left[j(j+1)-\frac{3}{4}\right] \text{ in the symmetrical theory,}$$

$$\frac{\Delta E}{\mu c^2}=\frac{3}{2}\frac{\kappa a}{g^2}\left[2j(j+1)-n^2-\frac{5}{4}\right] \text{ in the charged theory.}$$

Here the half-integer j is the total angular momentum, and the half-integer quantum number n is the electric charge (in units e) minus one-half. In both cases, the value of n is restricted to $-j \leqq n \leqq j$, otherwise the isobar energy in the first case is independent of n. The definition of the radius a of the nucleon is given by (4). The proton and neutron are identified as the two lowest states of this system, corresponding to $j=\frac{1}{2}$, $n=\pm\frac{1}{2}$. Higher states will be stable against "meson decay" to these ground states only if $\Delta E \ll \mu c^2$ or $g^2/\kappa a \gg 1$.

For the total scattering cross section (elastic plus inelastic scattering with and without change of the charge) we find under the conditions $g \gg \kappa a$, $\kappa a \ll 1$, $pa \ll 1$:

$$dq=\tfrac{3}{4}(p/E)^4 a^2(1+\cos^2\theta)d\Omega,$$
$$q=4\pi(p/E)^4 a^2 \text{ in the symmetrical theory;}$$
$$dq=\tfrac{3}{4}(p/E)^4 a^2(1+3\cos^2\theta)d\Omega,$$
$$q=6\pi(p/E)^4 a^2 \text{ in the charged theory.}$$

As has already been shown by Oppenheimer and Schwinger, one gets agreement with the experimental cross section for meson scattering, in the symmetrical theory for instance, by assuming $a \sim \hbar/Mc = 2\times 10^{-14}$ cm which means $\kappa a \sim 0.1$. Referring to the condition for strong coupling, $g \gg \kappa a$, we see that values of $g^2 \gg 0.01$ satisfy this condition.

The magnetic moment of the system nucleon $+$meson field is found to be

$$\pm\left[\frac{10}{36}\frac{g^2}{\kappa a}+\frac{1}{6}\right] \text{ proton magnetons}$$

the $\pm$ symbol referring to proton and neutron, respectively. As regards order of magnitude, this is not inconsistent with the known magnitudes. A moment of 1.93 proton magnetons, such as has been observed for the neutron, can be obtained by taking $g^2/\kappa a = 6.3$; using $\kappa a = 0.1$ we find $g^2 = 0.63$, well into the strong coupling domain. However, the prediction that the neutron and proton moments are equal and opposite is in contradiction with experience. The second term in the bracket above represents the contribution of the nucleon itself. This contribution is somewhat uncertain because of our ignorance of the fundamental properties of the "bare" proton or neutron. Consequently the discrepancy cannot be taken as an unambiguous disproof of the strong coupling hypothesis. However, the hope that the anomalous moments would follow in a simple way from the theory has not been justified.

It is to be noted that the results obtained above are quantitatively different from the results of the application of semi-classical methods employed by Oppenheimer and Schwinger and described in detail by Schwinger.[11] The latter method treats the spin $\boldsymbol{\sigma}$ and the isotopic spin $\boldsymbol{\tau}$ as classical unit vectors. In the charged scalar and neutral pseudoscalar theories, it gave a quantitative agreement with the wave mechanical results in the limit of strong coupling. This is, however, not true in the present case; in particular the restriction $-j \leqq n \leqq j$ does not appear. The discrepancy cannot be considered surprising in view of the neglect, in the semi-classical method, of the commutation relations between the various components of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. One sees in the wave mechanical treatment that these relations play an essential part in determining the minimum eigenvalue of the interaction energy (see Section 4). Once this part of the problem has been solved, however, there is no difference between the two methods of treatment.

The remaining sections are devoted to the derivation of the results quoted above. In Sections 3 to 6 we introduce such new field variables for the symmetrical pseudoscalar field as to express the energy in terms of unbound mesons and mesons bound to the nucleon. In 6 we find the dependence of the energy on the charge and angular momentum of the bound meson cloud, and discuss in detail the conditions for validity of the various strong coupling approximations introduced. In 7 we calculate the scattering cross sections. Section 8 contains a

[11] J. Schwinger, publication in preparation.

specialization of all these calculations as applied to the charged pseudoscalar case. Section 9 consists of the magnetic moment computation and an appendix describes an alternate method for expanding field quantities, with the use of Euler angles.

3. SPLIT OF THE FIELD INTO A ZERO STATE AND UPPER STATES

In the symmetrical pseudoscalar theory it is convenient to describe the field by three *real* quantities $\varphi_\alpha(x)$, $\alpha=1, 2, 3$. The Hamiltonian H consists of the part H_0 of free particles and the interaction energy H_1. The first part is given by[12]

$$H_0=\sum_{\alpha=1}^{3}\tfrac{1}{2}\int[\pi_\alpha^2+(\nabla\varphi_\alpha)^2+\kappa^2\varphi_\alpha^2]dV$$

$$=\sum_{\alpha=1}^{3}\tfrac{1}{2}\int[\pi_\alpha^2+\varphi_\alpha(-\Delta+\kappa^2)\varphi_\alpha]dV, \quad (1)$$

where the π_α are the momenta conjugate to φ_α and fulfill, at a given instant of time, the commutation rules

$$i[\pi_\alpha(\mathbf{x}),\ \varphi_\beta(\mathbf{x}')]=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'). \quad (2)$$

In the interaction energy we suppose the heavy particle, which we shall call nucleon, to be at rest and characterized by a spherically symmetrical source function $K(\mathbf{x})=K(|x|)$ which is normalized according to

$$\int K(x)dV=1 \quad (3)$$

and determines a radius a of the nucleon:

$$a^{-1}=\int\int K(x)(1/r)K(x')dVdV', \quad (4)$$

where here and in the following, r has the meaning

$$r=|\mathbf{x}-\mathbf{x}'|. \quad (5)$$

Furthermore, the nucleon is capable of existing in two states corresponding to proton and neutron, and moreover to two states corresponding to different directions of the spin. The first degree of freedom is described by the isotopic spin matrices τ_1, τ_2, τ_3, or τ_α with $\alpha=1, 2, 3$; the second by the spin matrices σ_x, σ_y, σ_z, or $\boldsymbol{\sigma}$. Both of them fulfill the same kind of relations

$$\tau_1\tau_2=i\tau_3,\ \cdots \quad \tau_1^2=1,\ \cdots, \quad (6)$$

$$\sigma_x\sigma_y=i\sigma_z,\ \cdots \quad \sigma_x^2=1,\ \cdots, \quad (7)$$

where the $\cdots$ denotes similar relations derived by cyclic permutations. The interaction energy is then given by

$$H=-(g/\kappa)(4\pi)^{\frac{1}{2}}/\sqrt{2}\sum_\alpha\int K(x)\tau_\alpha(\boldsymbol{\sigma}\cdot\nabla)\varphi_\alpha dV$$

$$=(g/\kappa)(4\pi)^{\frac{1}{2}}/\sqrt{2}\sum_\alpha\int\nabla K\tau_\alpha\cdot\boldsymbol{\sigma}\varphi_\alpha dV. \quad (8)$$

The factor $(4\pi)^{\frac{1}{2}}$ is introduced in order to measure g in ordinary units, not in Heaviside units; the factor $1/\sqrt{2}$ in order to bring it into accordance with the notation in the theories which introduce charged particles only; and the factor $1/\kappa$ in order to make g dimensionless.[13] The total charge of the system meson field+nucleon, measured in the unit e of the electron charge is given by

$$\epsilon=\int(\varphi_1\pi_2-\varphi_2\pi_1)dV+\tfrac{1}{2}(1+\tau_3). \quad (9)$$

The symmetrical theories are characterized by the fact that the charge is only one component of the isotopic spin $T_{\alpha\beta}=-T_{\beta\alpha}$ (α, $\beta=1, 2, 3$) which is a more general integral of motion and given by

$$T_{\alpha\beta}=\int(\varphi_\alpha\pi_\beta-\varphi_\beta\pi_\alpha)dV+\tfrac{1}{2}\tau_{\alpha\beta} \quad (10)$$

with $\tau_{12}=-\tau_{21}=\tau_3$, $\cdots$. The charge is then the 12 component of the isotopic spin, $\epsilon=T_{12}+\frac{1}{2}$.

The form (8) of the interaction energy suggests the definition[14]

$$\overset{0}{\varphi}_{\alpha k}=-(4\pi)^{\frac{1}{2}}\int K(x)\frac{\partial\varphi_\alpha}{\partial x_k}dV$$

$$=4\pi\int\frac{\partial K}{\partial x_k}\varphi_\alpha dV, \quad (11)$$

which gives

$$H_1=\frac{g}{\kappa\sqrt{2}}\sum_{\alpha,k}\tau_\alpha\sigma_k\overset{0}{\varphi}_{\alpha k}. \quad (8')$$

Moreover, we split the field $\varphi_\alpha(x)$ into two parts

[12] We always use here natural units where $\hbar=c=1$ or, in other words, energies are divided by $\hbar c$, momenta by $\hbar$, angular momenta by $\hbar$, electric charges by $(\hbar c)^{\frac{1}{2}}$; κ is the rest mass μ of the free meson divided by $\hbar c$.

[13] The connection of our constant g with the constant g_Y introduced by Yukawa is

$$g=g_Y(\hbar c)^{-\frac{1}{2}}.$$

[14] Small roman indices i, k, $\cdots$ run from 1 to 3 and denote vector components in the ordinary space.

of which one, $\varphi_\alpha'(x)$, is orthogonal to the gradient of the source function

$$\int \varphi_\alpha' \nabla K dV = 0 \tag{12}$$

and the other proportional to the gradient of the potential $X(x)$ which the source function generates according to

$$(-\Delta+\kappa^2)X = 4\pi K,$$
$$X(x) = \int K(x')e^{-\kappa r}/r dV'. \tag{13}$$

If we define

$$\delta_{ik} I = \int \frac{\partial X}{\partial x_i}\frac{\partial K}{\partial x_k} dV = \int\int \frac{\partial K}{\partial x_i}\frac{e^{-\kappa r}}{r}\frac{\partial K}{\partial x_k} dVdV', \tag{14}$$

$$\xi(x) = \frac{X(x)}{I}, \quad \delta_{ik} = \int \frac{\partial \xi}{\partial x_i}\frac{\partial K}{\partial x_k} dV, \tag{15}$$

then it follows from (11) and (12) that

$$\varphi_\alpha(x) = 1/(4\pi)^{\frac{1}{2}} \sum_k \overset{0}{\varphi}_{\alpha k}\frac{\partial \xi}{\partial x_k} + \varphi_\alpha'(x), \tag{16}$$

hence

$$\varphi_\alpha'(x) = \varphi_\alpha(x) - \nabla\xi \cdot \int \nabla K \varphi_\alpha dV. \tag{17}$$

The circumstance that we choose the first part of the field as a linear function of the $\partial X/\partial x_k$ (and not of the $\partial K/\partial x_k$) has the consequence that in virtue of (12) no cross terms between $\overset{0}{\varphi}_{\alpha k}$ and $\varphi_\alpha'(x)$ occur in the potential energy H_0. Indeed, one has, using (13), (15),

$$\frac{1}{2}\sum_\alpha \int \varphi_\alpha(-\Delta+\kappa^2)\varphi_\alpha dV = \frac{1}{2}\sum_{\alpha,k}\frac{(\overset{0}{\varphi}_{\alpha k})^2}{I} + \frac{1}{2}\sum_\alpha \int \varphi_\alpha'(-\Delta+\kappa^2)\varphi_\alpha' dV. \tag{18}$$

The conjugate decomposition of the momentum $\pi(x)$ is given by

$$\pi_\alpha(x) = (4\pi)^{\frac{1}{2}} \sum_k \overset{0}{\pi}_{\alpha k}\frac{\partial K}{\partial x_k} + \pi_\alpha'(x), \tag{16'}$$

$$\int \pi_\alpha' \nabla \xi dV = 0, \tag{12'}$$

$$\overset{0}{\pi}_{\alpha k} = 1/(4\pi)^{\frac{1}{2}} \int (\partial\xi/\partial x_k)\pi_\alpha dV, \tag{11'}$$

$$\pi_\alpha' = \pi_\alpha(x) - \nabla K \cdot \int \nabla \xi \pi_\alpha dV, \tag{17'}$$

which leads, in accordance with (2), to the commutation rules

$$i[\overset{0}{\pi}_{\alpha i}, \overset{0}{\varphi}_{\beta k}] = \delta_{\alpha\beta}\delta_{ik}, \tag{19}$$

$$[\overset{0}{\pi}_{\alpha k}, \varphi_\beta'(x)] = [\pi_\alpha'(x), \overset{0}{\varphi}_{\beta k}] = 0, \tag{20}$$

$$i[\pi_\alpha'(\mathbf{x}_1), \varphi_\beta'(\mathbf{x}_2)] = \delta_{\alpha\beta}(\delta(\mathbf{x}_1-\mathbf{x}_2) - \nabla_1 K\cdot\nabla_2\xi). \tag{21}$$

The latter relation is also in direct agreement with the orthogonality conditions (12), (12′).

The isotopic spin (10) decomposes simply into

$$T_{\alpha\beta} = \int (\varphi_\alpha'\pi_\beta' - \varphi_\beta'\pi_\alpha')dV + \sum\nolimits_k (\overset{0}{\varphi}_{\alpha k}\overset{0}{\pi}_{\beta k} - \overset{0}{\varphi}_{\beta k}\overset{0}{\pi}_{\alpha k}) + \tfrac{1}{2}\tau_{\alpha\beta}. \tag{22}$$

The kinetic energy however gets cross terms in $\overset{0}{\pi}_{\alpha i}$ and $\pi_\alpha'(x)$, and if we use the abbreviation

$$\delta_{ik}N = 4\pi \int \frac{\partial K}{\partial x_i}\frac{\partial K}{\partial x_k} dV$$

or

$$N = \frac{4\pi}{3}\int (\nabla K)^2 dV, \tag{23}$$

it is given by

$$\frac{1}{2}\sum_\alpha \int \pi_\alpha^2 dV = \frac{1}{2}N\sum_{\alpha,k}(\overset{0}{\pi}_{\alpha k})^2 + (4\pi)^{\frac{1}{2}}\sum_{\alpha,k}\overset{0}{\pi}_{\alpha k}\int \frac{\partial K}{\partial x_k}\pi_\alpha'(x)dV + \frac{1}{2}\sum_\alpha \int (\pi_\alpha')^2 dV. \tag{24}$$

One gets the total Hamiltonian by collecting (24), (18), (8′):

$$H = \frac{1}{2}N\sum_{\alpha,k}(\overset{0}{\pi}_{\alpha k})^2 + \frac{1}{2}\sum_{\alpha,k}(\overset{0}{\varphi}_{\alpha k})^2/I + g/\kappa\sqrt{2}\sum_{\alpha,k}\tau_\alpha\sigma_k\overset{0}{\varphi}_{\alpha k} + (4\pi)^{\frac{1}{2}}\sum_{\alpha,i}\overset{0}{\pi}_{\alpha i}\int(\partial K/\partial x_i)\pi_\alpha'(x)dV + \frac{1}{2}\sum_\alpha\int(\pi_\alpha')^2 dV + \frac{1}{2}\sum_\alpha\int\varphi_\alpha'(-\Delta+\kappa^2)\varphi_\alpha' dV. \tag{I}$$

Just as for the isotopic spin there exists in virtue of the spherical symmetry of the source function $K(x)$ the integral of the angular momentum $L_{ik} = -L_{ki}$ given by

$$L_{ik} = -\sum_\alpha \int \pi_\alpha\left(x_i\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_i}\right)\varphi_\alpha dV + \tfrac{1}{2}\sigma_{ik}. \tag{10'}$$

It decomposes as follows:

$$L_{ik}=\tfrac{1}{2}\sigma_{ik}+\sum_\alpha (\overset{0}{\varphi}_{\alpha i}\overset{0}{\pi}_{\alpha k}-\overset{0}{\varphi}_{\alpha k}\overset{0}{\pi}_{\alpha i})$$

$$-(4\pi)^{\frac{1}{2}}\sum_{\alpha,l}\overset{0}{\pi}_{\alpha l}\int\frac{\partial K}{\partial x_l}\left(x_i\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_i}\right)\varphi_\alpha' dV$$

$$-1/(4\pi)^{\frac{1}{2}}\sum_{\alpha,l}\overset{0}{\varphi}_{\alpha l}\int\pi_\alpha'\left(x_i\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_i}\right)\frac{\partial\xi}{\partial x_l}dV$$

$$-\sum_\alpha\int\pi_\alpha'\left(x_i\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_i}\right)\varphi_\alpha' dV. \quad (22')$$

The orthogonality relations (12), (12′) do not make the cross terms disappear in this case. We emphasize, however, the simple form of the angular momentum of the zero field, given by the first sum in (22′).

4. THE EIGENVALUES OF THE INTERACTION ENERGY. NEW VARIABLES FOR THE ZERO STATE

While in the usual treatment the interaction energy H_1 is considered as a small perturbation (weak coupling), we consider in this paper the opposite limit of strong coupling. In this case one has first to investigate the eigenvalues of the interaction energy H_1 given by (8′), and then to retain only the lowest of them under the assumption, whose validity will be investigated, that the next higher level of H_1 will not appreciably perturb a system in this lowest state.

In order to determine the four eigenvalues of the form

$$\sum_{\alpha,k}\tau_\alpha\sigma_k\overset{0}{\varphi}_{\alpha k}$$

in which the 9 coefficients $\overset{0}{\varphi}_{\alpha k}$ were real, we have to bear in mind that we are free to subject the τ's and σ's to independent real orthogonal transformations

$$\tau_\alpha=\sum_{r=1}^{3}\tau_r' B_{r\alpha},\quad \sigma_k=\sum_{s=1}^{3}A_{ks}\sigma_s', \quad (25)$$

where B and A fulfill orthogonality conditions which we can write in matrix notation

$$A\bar{A}=\bar{A}A=1,\quad B\bar{B}=\bar{B}B=1, \quad (26)$$

where $\bar{A}$ and $\bar{B}$ are the transposed matrices ($\bar{A}_{sk}=A_{ks}$, $\bar{B}_{\alpha r}=B_{r\alpha}$). The eigenvalues of our form are therefore the same as those of the other form

$$\sum_{r,s}\tau_r'(B\varphi^0A)_{rs}\sigma_s'. \quad (27)$$

We shall prove now that we can choose the orthogonal matrices A, B in such a way that the new matrix $B\varphi^0A$ becomes diagonal:

$$(B\varphi^0A)_{rs}=Q_r\delta_{rs} \quad (28)$$

or, according to (26),

$$\overset{0}{\varphi}_{\alpha k}=(\bar{B}Q\bar{A})_{\alpha k}=\sum_r B_{r\alpha}Q_rA_{kr}. \quad (29)$$

For this purpose we consider the two symmetrical forms

$$C=\bar{\varphi}\varphi \quad\text{or}\quad C_{ik}=C_{ki}=\sum_\alpha \overset{0}{\varphi}_{\alpha i}\overset{0}{\varphi}_{\alpha k}, \quad (30)$$

and

$$C'=\varphi\bar{\varphi} \quad\text{or}\quad C'_{\alpha\beta}=C'_{\beta\alpha}=\sum_k \overset{0}{\varphi}_{\alpha k}\overset{0}{\varphi}_{\beta k}. \quad (30')$$

We choose A as the orthogonal matrix which transforms C to principal axes (which is always possible because C is symmetric) and denote the eigenvalues of C by $(Q_r)^2$. These are certainly never negative because the form

$$\sum_{i,k}C_{ik}x_ix_k=\sum_\alpha(\sum_k\overset{0}{\varphi}_{\alpha k}x_k)^2$$

is definite. This means that we have

$$CA=AQ^2 \quad\text{or}\quad \bar{A}CA=Q^2. \quad (31)$$

We note that the matrix A is unique except for a sign, and except for the case where some of the Q_r's are zero or some of them are equal to each other. We now define the matrix B by

$$\varphi A=\bar{B}Q, \quad (32)$$

where Q_r is defined as the square root of the eigenvalue Q_r^2 of C, the sign of which we shall choose later. Multiplication by $\bar{\varphi}$ from the left gives, by virtue of (30), (31)

$$AQ^2=\bar{\varphi}\bar{B}Q,$$

or

$$AQ=\bar{\varphi}\bar{B},\quad \bar{A}AQ=Q=\bar{A}\bar{\varphi}\bar{B},$$

and by transposing the matrices, Q being diagonal

$$B\varphi A=Q$$

which proves Eq. (28). Now multiplying (32) by B from the left, we get

$$B\varphi A=Q=B\bar{B}Q.$$

Therefore $B\bar{B}=1$, which proves the orthogonality of B. (For the case of a multiple eigenvalue Q_r the orthogonality $B\bar{B}=1$ does not follow, but can easily be achieved.) Multiplying $AQ=\bar{\varphi}\bar{B}$

by $B\varphi$ from the left, we get

$$\begin{aligned} B\varphi A Q &= Q^2 = B\varphi\bar{\varphi}\bar{B} = BC'\bar{B}, \\ BC' &= Q^2 B \quad \text{or} \quad BC'\bar{B} = Q^2, \end{aligned} \tag{33}$$

which is analogous to (31) and shows that C' has the same eigenvalues as C.

If the eigenvalue Q_r is zero, the row $B_{r\alpha}$ of B is not defined by (32) and the preceding proof is incomplete. Let us assume first that there is a simple eigenvalue $Q_3=0$, $Q_1\neq 0$, $Q_2\neq 0$. From Eq. (32) we get for $r=3$

$$\textstyle\sum_k \overset{0}{\varphi}_{\alpha k} A_{k3} = 0.$$

The equation $AQ=\bar{\varphi}\bar{B}$ is in this case not a consequence of the proved equation $AQ^2=\bar{\varphi}\bar{B}Q$ but can be used for $r=3$ as a definition of the third row of the matrix B, namely

$$\textstyle\sum_\alpha \overset{0}{\varphi}_{\alpha k} B_{3\alpha} = 0.$$

This can always be fulfilled, the determinant of $\overset{0}{\varphi}_{\alpha k}$ being zero in this case. There is still an arbitrary factor independent of α undetermined in $B_{3\alpha}$. This factor can be adjusted to satisfy the equation $Q=B\bar{B}Q$; it can simultaneously be made to satisfy the equation

$$\textstyle\sum_\alpha B_{3\alpha}^2 = 1,$$

which does not follow directly from the above. Combining $Q=B\bar{B}Q$ and $\sum_\alpha B_{3\alpha}^2=1$, we obtain again $B\bar{B}=1$.

In Section 8 we shall meet the particular case

$$Q_3=0, \quad Q_1\neq 0, \quad Q_2\neq 0, \quad \overset{0}{\varphi}_{3k}=0.$$

In this case one has $B_{3\alpha}=0$ for $\alpha=1, 2$, which follows from

$$\textstyle\sum_\alpha \overset{0}{\varphi}_{\alpha k} B_{3\alpha} = 0.$$

Moreover, (32) leads to $B_{\alpha 3}=0$ for $\alpha=1, 2$. Finally, B_{33} is fixed by $B\bar{B}=1$ and is equal to one. The latter condition also expresses the orthogonality of the two-row and two-column matrix $B_{r\alpha}$ ($r=1, 2$; $\alpha=1, 2$).

In a similar way the case of a multiple eigenvalue zero of C and C' can be treated by considering all equations in the preceding proof which cannot be directly derived as possible additional defining postulates.

In order to have the determinants of A and B both $+1$, which is essential for the validity of the correct algebraic relations for the τ_r', σ_s' we get from (32) the condition

$$\text{det. } \varphi = Q_1 Q_2 Q_3. \tag{34}$$

Apart from this condition, the signs of the Q_r's are arbitrary and can be fixed by definition. This means that to every combination of signs of the Q_r which fulfills (34) there exist, for a given $\overset{0}{\varphi}_{\alpha k}$, matrices A and B which are orthogonal in the usual sense and which fulfill (28). We can for instance define all signs of the Q_r's as positive (negative) if det. φ is positive (negative).

Now inserting (28) in (27) we get

$$\sum_{\alpha, k} \tau_\alpha \sigma_k \overset{0}{\varphi}_{\alpha k} = \sum_r Q_r \tau_r' \sigma_r'. \tag{35}$$

The eigenvalues of every τ_r', σ_r' are ± 1, but the product of the three matrices $\tau_r'\sigma_r'$ is always -1. Therefore, the eigenvalues of H_1 for a given field $\overset{0}{\varphi}_{\alpha k}$ are

$$(g/\kappa\sqrt{2}) \textstyle\sum_r Q_r \epsilon_r \quad \text{with} \quad \epsilon_1\epsilon_2\epsilon_3 = -1 \tag{36}$$

or

$$\begin{aligned} E_1 &= -(g/\kappa\sqrt{2})(Q_1+Q_2+Q_3), \\ E_2 &= (g/\kappa\sqrt{2})(-Q_1+Q_2+Q_3), \\ E_3 &= (g/\kappa\sqrt{2})(Q_1-Q_2+Q_3), \\ E_4 &= (g/\kappa\sqrt{2})(Q_1+Q_2-Q_3). \end{aligned} \tag{36'}$$

The sum of all four E's is zero, but if E is an eigenvalue in general $-E$ is not an eigenvalue unless at least one of the three Q_r's is zero. We repeat that the Q_r's are the square roots of the eigenvalues of the symmetrical matrix C (or C'). From (31) we get immediately by evaluating the trace

$$\sum_{r=1}^{3} Q_r^2 = \sum_k C_{kk} = \sum_{\alpha, k} (\overset{0}{\varphi}_{\alpha k})^2. \tag{37}$$

The sum of the Q_r's themselves, however, cannot be expressed rationally by the field $\overset{0}{\varphi}_{\alpha k}$.

We now introduce into the kinetic energy of the zero field the variables corresponding to Q_r, A_{kr}, $B_{r\alpha}$. Instead of expressing the latter variables by Euler angles and using their conjugate momenta, we can also use the components of the angular momentum

$$\overset{0}{L}_{ik} = \textstyle\sum_\alpha (\overset{0}{\varphi}_{\alpha i}\overset{0}{\pi}_{\alpha k} - \overset{0}{\varphi}_{\alpha k}\overset{0}{\pi}_{\alpha i}), \tag{22_0}$$

and the isotopic spin

$$\overset{0}{T}_{\alpha\beta} = \textstyle\sum_k (\overset{0}{\varphi}_{\alpha k}\overset{0}{\pi}_{\beta k} - \overset{0}{\varphi}_{\beta k}\overset{0}{\pi}_{\alpha k}). \tag{$22_0'$}$$

As operators they correspond to infinitesimal rotations of the ordinary space and of the isotopic spin space, and fulfill commutation relations with the A_{kr} and $B_{r\alpha}$ analogous to those with the $\overset{0}{\varphi}_{k\alpha}$, namely

$$i[\overset{0}{L}_{ik},\, A_{lr}]=\delta_{kl}A_{ir}-\delta_{il}A_{kr},\qquad [\overset{0}{L}_{ik},\, B_{r\alpha}]=0, \quad (38)$$

$$i[\overset{0}{T}_{\alpha\beta},\, B_{\gamma r}]=\delta_{\beta\gamma}B_{\alpha r}-\delta_{\alpha\gamma}B_{\beta r},\qquad [\overset{0}{T}_{\alpha\beta},\, A_{kr}]=0, \quad (38')$$

while the components of $\overset{0}{L}_{ik}$ or $\overset{0}{T}_{\alpha\beta}$ fulfill with each other the well-known commutation relations

$$i[\overset{0}{L}_{ik},\, \overset{0}{L}_{lm}]=\delta_{kl}\overset{0}{L}_{im}+\delta_{im}\overset{0}{L}_{kl}-\delta_{il}\overset{0}{L}_{km}-\delta_{km}\overset{0}{L}_{il}, \quad (39)$$

or, with the vector notation $L_1{}^0$, $L_2{}^0$, $L_3{}^0$ for $\overset{0}{L}_{23}$, $\overset{0}{L}_{31}$, $\overset{0}{L}_{12}$,

$$i[L_1{}^0,\, L_2{}^0]=-L_3{}^0,\ \cdots. \quad (39')$$

As scalars the Q_r's commute with both the $\overset{0}{L}_{ik}$ and the $\overset{0}{T}_{\alpha\beta}$,

$$[\overset{0}{L}_{ik},\, Q_r]=[\overset{0}{T}_{\alpha\beta},\, Q_r]=0. \quad (40)$$

In the same way we have

$$i[\overset{0}{T}_{\alpha\beta},\, \overset{0}{T}_{\gamma\delta}]=\delta_{\beta\gamma}\overset{0}{T}_{\alpha\delta}+\delta_{\alpha\delta}\overset{0}{T}_{\beta\gamma}-\delta_{\alpha\gamma}\overset{0}{T}_{\beta\delta}-\delta_{\beta\delta}\overset{0}{T}_{\alpha\gamma}, \quad (39'')$$

the $\overset{0}{T}_{\alpha\beta}$ and the $\overset{0}{L}_{ik}$ commute.

It is useful to introduce the components of (L^0), (T^0) with respect to the axes defined by (A_{kr}), $(B_{r\alpha})$ which we define by

$$L_0{}^{rs}=\sum_{i,k} A_{ir}A_{ks}\overset{0}{L}_{ik}=\sum_{i,k}\overset{0}{L}_{ik}A_{ir}A_{ks},$$
$$T_0{}^{rs}=\sum_{\alpha,\beta} B_{r\alpha}B_{s\beta}\overset{0}{T}_{\alpha\beta}=\sum_{\alpha,\beta}\overset{0}{T}_{\alpha\beta}B_{r\alpha}B_{s\beta}. \quad (41)$$

We note that in spite of the non-commutativity of $\overset{0}{L}_{ik}$ with the A_{lr} and of $\overset{0}{T}_{\alpha\beta}$ with the $B_{r\gamma}$ the $L_0{}^{rs}$ and $T_0{}^{rs}$ are Hermitian operators like the $\overset{0}{L}_{ik}$ and $\overset{0}{T}_{\alpha\beta}$ as a consequence of the equalities indicated in (41) which follow from (38), (38'). The total square of L_0 and T_0 is given by

$$L_0{}^2\equiv\sum_{i<k}(\overset{0}{L}_{ik})^2=\sum_{r<s}(L_0{}^{rs})^2, \quad (42)$$

$$T_0{}^2\equiv\sum_{\alpha<\beta}(\overset{0}{T}_{\alpha\beta})^2=\sum_{r<s}(T_0{}^{rs})^2. \quad (42')$$

The commutation relations of the $L_0{}^{rs}$ with the A_{kr} and of $T_0{}^{rs}$ with the $B_{r\alpha}$ are given by

$$i[L_0{}^{rs},\, A_{kt}]=\delta_{rt}A_{ks}-\delta_{st}A_{kr}, \quad (43)$$

$$i[T_0{}^{rs},\, B_{t\alpha}]=\delta_{rt}B_{s\alpha}-\delta_{st}B_{r\alpha}, \quad (43')$$

and of the $L_0{}^{rs}$ or the $T_0{}^{rs}$ with each other

$$i[L_0{}^{rs},\, L_0{}^{tu}]=\delta_{rt}L_0{}^{su}+\delta_{su}L_0{}^{rt}-\delta_{st}L_0{}^{ru}-\delta_{ru}L_0{}^{st}, \quad (44)$$

$$i[T_0{}^{rs},\, T_0{}^{tu}]=\delta_{rt}T_0{}^{su}+\delta_{su}T_0{}^{rt}-\delta_{st}T_0{}^{ru}-\delta_{ru}T_0{}^{st}. \quad (44')$$

The signs in the relations (43), (44) or (43'), (44') are just opposite to those in the relations (38), (39) or (38'), (39'').

Inserting (29), (22_0), and ($22_0'$) into (41) we get

$$L_0{}^{rs}=\sum_{k,\alpha}(Q_rA_{ks}B_{r\alpha}\overset{0}{\pi}_{\alpha k}-Q_sA_{kr}B_{s\alpha}\overset{0}{\pi}_{\alpha k}), \quad (45)$$

$$T_0{}^{rs}=\sum_{k,\alpha}(Q_rA_{kr}B_{s\alpha}\overset{0}{\pi}_{\alpha k}-Q_sA_{ks}B_{r\alpha}\overset{0}{\pi}_{\alpha k}). \quad (45')$$

From this we get the important expression

$$\sum_{k,\alpha}A_{kr}B_{s\alpha}\overset{0}{\pi}_{\alpha k}=\frac{1}{2}\left\{\frac{L_0{}^{rs}+T_0{}^{rs}}{Q_r-Q_s}-\frac{L_0{}^{rs}-T_0{}^{rs}}{Q_r+Q_s}\right\}\ \text{for } r\neq s. \quad (46)$$

Finally, we get the variable canonically conjugate to Q_r, which we call P_r, by putting for $r=s$,

$$\sum_{k,\alpha}A_{kr}B_{r\alpha}\overset{0}{\pi}_{\alpha k}=P_r. \quad (47)$$

Therefore, we have finally

$$\overset{0}{\pi}_{\alpha k}=\sum_{r,s}A_{kr}B_{s\alpha}\times\left\{P_r\delta_{rs}+\frac{1}{2}\frac{L_0{}^{rs}+T_0{}^{rs}}{Q_r-Q_s}-\frac{1}{2}\frac{L_0{}^{rs}-T_0{}^{rs}}{Q_r+Q_s}\right\}, \quad (48)$$

where the second term is to be taken for $r\neq s$ only. Assuming

$$[P_r,\, A_{ks}]=[P_r,\, B_{s\alpha}]=[P_r,\, L_0{}^{st}]=[P_r,\, T_0{}^{st}]=0,\quad i[P_r,\, Q_s]=\delta_{rs}, \quad (49)$$

one can verify the canonical commutation relations (19) for the $\overset{0}{\pi}_{\alpha i}$ and $\overset{0}{\varphi}_{\beta k}$ with the help of (43) (44), (43'), (44'). It may be noted that the operator P_r defined by (47) is not Hermitian and that P_r in (48) has to be replaced by $P_r{}^+$ if the factor $A_{kr}B_{s\alpha}$ is put on the right side of the bracket. For the kinetic energy of the zero field (apart from the factor $N/2$) we get from (48), taking into account the anti-symmetry in r, s of

$(L_0^{rs}-T_0^{rs})/Q_r+Q_s$ and the symmetry in r, s of $(L_0^{rs}+T_0^{rs})/Q_r-Q_s$

$$\sum_{\alpha,k}(\overset{0}{\pi}_{\alpha,k})^2=\sum_r P_r^+P_r+\frac{1}{4}\sum_{r\neq s}\frac{(L_0^{rs}+T_0^{rs})^2}{(Q_r-Q_s)^2}+\frac{1}{4}\sum_{r,s}\frac{(L_0^{rs}-T_0^{rs})^2}{(Q_r+Q_s)^2}. \quad (50)$$

We conclude from (46) that if $Q_r=Q_s$ then it must also be true that $L_0^{rs}=-T_0^{rs}$. In view of the importance of this case it is sometimes convenient to use different coordinates. Indeed, if all three Q_r's are equal, which will be proved to be true in good approximation in the case of large coupling, according to (29) only the orthogonal matrix

$$e_{\alpha k}=\sum_r A_{kr}B_{r\alpha} \quad (51)$$

which fulfills

$$\sum_\alpha e_{\alpha i}e_{\alpha k}=\delta_{ik}, \quad \sum_k e_{\alpha k}e_{\beta k}=\delta_{\alpha\beta}, \quad (51a)$$

is uniquely determined by the field $\overset{0}{\varphi}_{\alpha k}$ and not the A_{kr} and $B_{r\alpha}$ themselves. Therefore, besides the three independent variables $e_{\alpha k}$, we can introduce the six symmetrical quantities

$$\overset{0}{q}_{\alpha\beta}=\sum_k \overset{0}{\varphi}_{\alpha k}e_{\beta k}=\sum_r B_{r\alpha}B_{r\beta}Q_r=\overset{0}{q}_{\beta\alpha} \quad (52)$$

to describe the field

$$\overset{0}{\varphi}_{\alpha k}=\sum_\beta e_{\beta k}\overset{0}{q}_{\alpha\beta}. \quad (53)$$

The components

$$L_0^{\alpha\beta}=\sum_{r,s}B_{r\alpha}B_{s\beta}L_0^{rs}=\sum_{i,k}e_{\alpha i}e_{\beta k}\overset{0}{L}_{ik} \quad (54)$$

commute with the $\overset{0}{q}_{\gamma\delta}$ and fulfill similar commutation relations with each other and with the $e_{\alpha i}$ as do the L_0^{rs} with each other and with the A_{kt},

$$i[L_0^{\alpha\beta}, e_{\gamma k}]=\delta_{\alpha\gamma}e_{\beta k}-\delta_{\beta\gamma}e_{\alpha k}, \quad (55)$$

$$i[L_0^{\alpha\beta}, L_0^{\gamma\delta}]=\delta_{\alpha\gamma}L_0^{\beta\delta}+\delta_{\beta\delta}L_0^{\alpha\gamma}-\delta_{\beta\gamma}L_0^{\alpha\delta}-\delta_{\alpha\delta}L_0^{\beta\gamma}. \quad (56)$$

Finally, we search for symmetrical quantities $\overset{0}{p}_{\alpha\beta}=\overset{0}{p}_{\beta\alpha}$ which are conjugate to the $\overset{0}{q}_{\alpha\beta}$ in the sense that they fulfill the commutation relations

$$i[\overset{0}{p}_{\alpha\beta}, \overset{0}{q}_{\gamma\delta}]=\tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}). \quad (57)$$

(This means, for instance, that $i[p_{11}{}^0, q_{11}{}^0]=1$, $i[p_{12}{}^0, q_{12}{}^0]=\frac{1}{2}$; this is more convenient than to have the value 1 for the latter bracket.) This is just the case if, with the help of the first two terms in the bracket in (48), we put

$$\overset{0}{p}_{\alpha\beta}=\sum_r B_{r\alpha}B_{r\beta}P_r+\sum_{r\neq s}{}' B_{r\alpha}B_{s\beta}\frac{1}{2}\frac{L_0^{rs}+T_0^{rs}}{Q_r-Q_s}. \quad (58)$$

From (48) we find also

$$\overset{0}{p}_{\alpha\beta}=\tfrac{1}{2}\sum_k(e_{\alpha k}\overset{0}{\pi}_{\beta k}+e_{\beta k}\overset{0}{\pi}_{\alpha k}). \quad (59)$$

The remark made about the non-Hermitian character of the P_r in (47) applies also to the $\overset{0}{p}_{\alpha\beta}$ defined here. One can prove (57) with the help of (54), (43), (43′), and moreover

$$[\overset{0}{p}_{\alpha\beta}, L_0^{\gamma\delta}]=0. \quad (60)$$

We have therefore three kinds of variables for the zero field:

(1) $\overset{0}{\varphi}_{\alpha k}$, $\overset{0}{\pi}_{\alpha k}$,

(2) Q_r, A_{kr}, $B_{r\alpha}$; P_r, L_0^{rs}, T_0^{rs},

(3) $\overset{0}{q}_{\alpha\beta}=\overset{0}{q}_{\beta\alpha}$, $e_{\alpha k}$; $\overset{0}{p}_{\alpha\beta}=\overset{0}{p}_{\beta\alpha}$, $L_0^{\alpha\beta}$.

While we were able to get an explicit expression for the kinetic energy in terms of the variables (2), this is not rigorously possible with the variables (3). From (48), (50) we get only

$$\sum_{\alpha,k}(\overset{0}{\pi}_{\alpha k})^2=\sum_{\alpha,\beta}\overset{0\dagger}{p}_{\alpha\beta}\overset{0}{p}_{\alpha\beta}+\frac{1}{4}\sum_{r,s}\frac{(L_0^{rs}-T_0^{rs})^2}{(Q_r+Q_s)^2}. \quad (61)$$

In the next section we shall see, however, that in the case of small deviations of the Q_r from a large common value $Q_r{}^0=D$, further simplifications are possible, and that the variables (3) are the most convenient ones in this case.

5. MINIMUM OF THE POTENTIAL ENERGY. STRONG COUPLING APPROXIMATION

We determine first the minimum of the total potential energy of the zero field which is, according to (18), (36), (37), equal to

$$\overset{0}{E}_{\text{pot}}=\frac{1}{2I}\sum_{r=1}^{3}Q_r^2+\frac{g}{\kappa\sqrt{2}}\sum_r Q_r\epsilon_r \qquad (\epsilon_1\epsilon_2\epsilon_3=-1). \quad (62)$$

The minimum corresponds to

$$\epsilon_1=\epsilon_2=\epsilon_3=-1, \quad Q_1=Q_2=Q_3=D=\frac{g}{\kappa\sqrt{2}}I,$$

$$\overset{0}{E}_{\text{min}}=-\frac{3}{4}\frac{g^2}{\kappa^2}I. \quad (63)$$

According to (53) the corresponding field is given by

$$\overset{0}{\varphi}_{\alpha k}=De_{\alpha k} \quad (64)$$

or, in vector notation

$$\boldsymbol{\phi}_\alpha{}^0 = D\mathbf{e}_\alpha. \tag{64a}$$

This field corresponds to the smallest eigenvalue of the interaction energy, determined by (36), and to the absence of free mesons [we neglected φ'][15] but still contains the q numbers $\mathbf{e}_\alpha$. Moreover, it corresponds to the state of the τ and σ where

$$\sum_{\alpha, k} \tau_\alpha \sigma_k e_{\alpha k} = -3.$$

By a suitable S transformation of the Hamiltonian (see Appendix) we can bring this to the form

$$S(\sum_{\alpha, k} \tau_\alpha \sigma_k e_{\alpha k})S^{-1} = \sum_{\alpha=1}^{3} \tau_\alpha \sigma_\alpha. \tag{65}$$

The S transformation is not uniquely determined, just as is true for the A and B matrices; it is, for instance, sufficient in this case to transform one system of the matrices τ_α or σ_k alone. In view of the application to the charged pseudoscalar theory, which is discussed in Section 6, we decided to leave the τ_α's unchanged and to transform the σ_k's,

$$S\tau_\alpha S^{-1} = \tau_\alpha; \quad S(\sum_k e_{\alpha k}\sigma_k)S^{-1} = \sigma_\alpha. \tag{65a}$$

This is equivalent to

$$\sigma_k' \equiv S\sigma_k S^{-1} = \sum_\alpha e_{\alpha k}\sigma_\alpha. \tag{65b}$$

The state in question can be described as the singlet state inasmuch as the composition of the τ_α and σ_α is analogous to the composition of two spins of equal magnitude. This state is not degenerate according to (36′), while the three other eigenvalues of the interaction energy are equal:

$$E_1 = -(g/\kappa\sqrt{2})3D, \quad E_2 = E_3 = E_4 = (g/\kappa\sqrt{2})D. \tag{66}$$

In the following we assume that it is legitimate to restrict our attention to the lowest, or singlet state. This implies a condition on the magnitude of such terms in the Hamiltonian as couple the lowest to the three upper states of the interaction energy.

For the eigenvalue E_1, the following expressions are diagonal:

$$\tau_1\sigma_1 = \tau_2\sigma_2 = \tau_3\sigma_3 = -1; \quad \tau_1\sigma_2 + \tau_2\sigma_1 = 0, \cdots \tag{67}$$

[15] It will be shown in Section 7 that the field φ' describes free mesons.

On the other hand, the matrices τ_r, σ_r, and $\tau_1\sigma_2 - \tau_2\sigma_1$, $\cdots$ are not diagonal. We shall simply neglect terms in the energy in which these matrices appear as a factor. In Section 6, we shall discuss the region of validity of this approximation.

As is shown in the Appendix, the S transformation changes the angular momentum $\mathbf{L}$ according to

$$\mathbf{L}' = S\mathbf{L}S^{-1} = \mathbf{L} - \tfrac{1}{2}\boldsymbol{\sigma}' = \mathbf{L} - \tfrac{1}{2}\sum_\alpha \sigma_\alpha \mathbf{e}_\alpha. \tag{68}$$

As a consequence of this and (54) we have the relation

$$SL_0{}^{\alpha\beta}S^{-1} = L_0{}^{\alpha\beta} - \tfrac{1}{2}\sigma_{\alpha\beta}. \tag{69}$$

One checks directly that $\mathbf{L}'$ commutes with $\boldsymbol{\sigma}'$ if $\mathbf{L}$ commutes with $\boldsymbol{\sigma}$ using (55). As the σ_α can be disregarded in our strong coupling approximation if they do not occur multiplied by the τ_α's, the $\mathbf{L}'$ have in this approximation the same eigenvalues as the $\mathbf{L}$, while the total sum L'^2 [see (42)] has eigenvalues $\frac{1}{4}$ larger than L^2. Moreover, the eigenfunction $\Psi' = S\Psi$ is double-valued in the angles which enter in S and in which Ψ is one-valued. Therefore $(L')^2$ has the eigenvalues $\frac{1}{4} + j(j+1)$ with half-integer j.

We now split the variables $\overset{0}{q}_{\alpha\beta}$ defined by (52) into two parts according to (64)

$$\overset{0}{q}_{\alpha\beta} = D\delta_{\alpha\beta} + \overset{(1)}{q}_{\alpha\beta}, \tag{70}$$

of which we consider the second part as small. With D given by (63)

$$E_{\text{pot}} = \frac{1}{2I}\sum_{\alpha, k} (\overset{0}{\varphi}_{\alpha k})^2 + \frac{g}{\kappa\sqrt{2}}\sum_{\alpha, k} \tau_\alpha \sigma_k \overset{0}{\varphi}_{\alpha k}.$$

This leads, by the S transformation (65) which does not change the $\overset{0}{\varphi}_{\alpha k}$ but does change the $\overset{0}{\pi}_{\alpha k}$, to

$$E'_{\text{pot}} = SE_{\text{pot}}S^{-1} = \frac{1}{2I}\sum_{\alpha, \beta} (\overset{0}{q}_{\alpha\beta})^2 + \frac{g}{\kappa\sqrt{2}}\sum_{\alpha, \beta} \overset{0}{q}_{\alpha\beta}\, \tfrac{1}{2}(\tau_\alpha\sigma_\beta + \sigma_\beta\tau_\alpha).$$

Using (67) we have

$$E'_{\text{pot}} = \frac{1}{2I}\sum_{\alpha, \beta} (\overset{0}{q}_{\alpha\beta})^2 - \frac{g}{\kappa\sqrt{2}}\sum_{\alpha, \beta} \overset{0}{q}_{\alpha\beta},$$

which simply reduces to

$$E'_{\text{pot}} = -\frac{3}{4}\frac{g^2}{\kappa^2}I + \frac{1}{2I}\sum_{\alpha, \beta} (\overset{(1)}{q}_{\alpha\beta})^2. \tag{71}$$

We now introduce (64) into (22_0), which means that we disregard $\overset{(1)}{q}_{\alpha\beta}$ and get

$$\overset{0}{L}_{ik}=D\sum_\alpha(e_{\alpha i}\overset{0}{\pi}_{\alpha k}-e_{\alpha k}\overset{0}{\pi}_{\alpha i}).$$

and from (54)

$$-L_0{}^{\alpha\beta}=D\sum_k(e_{\alpha k}\overset{0}{\pi}_{\beta k}-e_{\beta k}\overset{0}{\pi}_{\alpha k}).$$

It therefore follows from (59) that

$$\overset{0}{\pi}_{\alpha k}=\sum_\beta e_{\beta k}\left(\overset{0}{p}_{\alpha\beta}+\frac{1}{2D}L_0{}^{\alpha\beta}\right)$$

$$=\sum_\beta\left(\overset{0\dagger}{p}_{\alpha\beta}+\frac{1}{2D}L_0{}^{\alpha\beta}\right)e_{\beta k}. \quad (72)$$

Inserting $\overset{0}{\varphi}_{\alpha k}=De_{\alpha k}$ one can finally check directly

$$T_0{}^{\alpha\beta}=-L_0{}^{\alpha\beta}, \quad (73)$$

where the disregarded terms are of the relative order of magnitude D^{-2} in comparison with the main term. This is in agreement with the result of the last section. If we introduce $T_0{}^{rs}=-L_0{}^{rs}$ and $Q_r+Q_s=2D$ in the last term in (48) we get (72) directly. The relation (73) has the important consequence that the total square of the isotopic spin is equal to the total square of the angular momentum; after the S transformation, both correspond to half-integer quantum numbers. Therefore the quantum number n, which is equal to the charge number minus $\frac{1}{2}$ [see Eqs. (9), (10)], always has for the zero field (that means in absence of free mesons) an absolute value smaller than the total angular momentum quantum number j:

$$-j\leqq n\leqq j. \quad (74)$$

This is of course due to the fact that the τ_α and σ_α are in the "singlet state" defined by (67); this also enables us to disregard the terms linear in τ and σ as already mentioned.

If we introduce $\langle\overset{0}{p}_{\alpha\beta}\rangle_H=\frac{1}{2}(\overset{0}{p}_{\alpha\beta}+\overset{0\dagger}{p}_{\alpha\beta})$, the $\langle\ \rangle_H$ denoting Hermitization, we get

$$\overset{0}{\pi}_{\alpha\beta}=\sum_\beta e_{\beta k}\langle\overset{0}{p}_{\alpha\beta}\rangle_H+\frac{1}{2D}\langle L_0{}^{\alpha\beta}e_{\beta k}\rangle_H. \quad (72')$$

This leads to the following expression for the kinetic energy of the zero field (neglecting an additive constant in the term proportional to D^{-2}):

$$\frac{N}{2}\sum_{\alpha,k}(\overset{0}{\pi}_{\alpha k})^2=\frac{N}{2}\left\{\sum_{\alpha,\beta}(\overset{0}{p}_{\alpha\beta})_H{}^2+\frac{1}{2D^2}L_0{}^2\right\}, \quad (72a)$$

where the square of the total angular momentum is given by (42). A result identical with (72a) is obtained from (61) if we insert in the last term $Q_r+Q_s=2D$, $-T_0{}^{rs}=L_0{}^{rs}$. We note that the only result of the S transformation is that j gets half-integral values, the constant $\frac{1}{4}$ added to $L_0{}^2$ being negligible for our applications.

Inserting (71), (72), (72a) into the Hamiltonian (I), we get the "reduced" Hamiltonian of the strong coupling approximation

$$H=-\frac{3}{4}\frac{g^2}{\kappa^2}I+\frac{1}{2}N\left\{\sum_{\alpha,\beta}\langle\overset{0}{p}_{\alpha\beta}\rangle_H^2+\frac{1}{2D^2}L_0{}^2\right\}$$
$$+\frac{1}{2I}\sum_{\alpha,\beta}(\overset{(1)}{q}_{\alpha\beta})^2+(4\pi)^{\frac{1}{2}}\sum_{\alpha,\beta,k}e_{\beta k}\langle\overset{0}{p}_{\alpha\beta}\rangle_H$$
$$\times\int\frac{\partial K}{\partial x_k}\pi_\alpha'(x)dV+\frac{1}{2}\sum_\alpha\int\pi_\alpha'^2dV$$
$$+\frac{1}{2}\sum_\alpha\int\varphi_\alpha'(-\Delta+\kappa^2)\varphi_\alpha'dV$$
$$+(4\pi)^{\frac{1}{2}}\sum_{\alpha,\beta,k}\frac{1}{2D}\langle e_{\beta k}L_0{}^{\alpha\beta}\rangle_H\int\frac{\partial K}{\partial x_k}\pi_\alpha'(x)dV. \quad \text{(II)}$$

Inspection with respect to the approximation contained in (72) shows that the disregarded terms in H are of the following type: terms of the order D^{-2} independent of $\overset{0}{L}_{ik}$; terms of the order D^{-2} linear in the $\overset{0}{L}_{ik}$ and in the $\overset{(1)}{q}_{ik}$; and terms of higher order than D^{-2}. We shall see in the next two sections that such terms are of no importance for the questions which are treated in this paper. The last term in (II), however, will be essential.

6. THE EXCITED STATES (ISOBARS) OF THE NUCLEON

We shall now treat two different kinds of problems. In the first kind we investigate the states of the nucleon in the absence of free mesons, while in the second kind we treat the scattering of free mesons by the nucleon and its proper field. It will turn out in the next section that the field which is described by $\overset{0}{p}_{\alpha\beta}$, $\pi_\alpha'(x)$; $\overset{(1)}{q}_{\alpha\beta}$, $\varphi_\alpha'(x)$ represents free mesons, which are partly undisturbed plane waves, partly scattered waves. On the other hand the problem described by $e_{\alpha k}$, $\overset{0}{L}_{\alpha\beta}$ is the problem of the possible states of the free nucleon in the absence of free mesons.

For the latter we have to retain the terms up to the order D^{-2} but we can omit terms of the order D^{-2} which are independent of $\overset{0}{L}_{ik}$ or terms of the order D^{-2} linear in $\overset{0}{L}_{ik}$ which are at the same time linear in $\overset{(1)}{q}_{ik}$ or bilinear in $\pi_\alpha'(x_1)$ and $\varphi_\beta'(x_2)$. The latter terms do not have an expectation value of the order D^{-2} in the state where the corresponding waves are unexcited, and the terms independent of $\overset{0}{L}_{ik}$ do not influence the energy difference between the excited states of the nucleon and the ground state (isobar separation). These more complicated terms, however, give rise to a correction of higher order to the scattering of free mesons, which question is not treated in this paper. In the following section we treat the meson scattering only as a problem of the order D^0 and disregard all terms of higher order, while in this section we treat the problem of the nucleon in the absence of free mesons up to the order D^{-2}.

In order to get rid of the last terms in (II) we have to shift the zero point of $\pi_\alpha'(x)$ but in such a way that the orthogonality relation (12′), namely $\int \pi_\alpha'(\partial\xi/\partial x_k)dV=0$ is not violated. The suitable assumption is

$$\pi_\alpha'(x)=\pi_\alpha''(x)-\frac{(4\pi)^{\frac{1}{2}}}{2D}\sum_{k,\beta}\langle e_{\beta k}L_0^{\alpha\beta}\rangle_H \times\left\{\frac{\partial K}{\partial x_k}-\frac{\partial\xi/\partial x_k}{\int(\partial\xi/\partial x)^2dV}\right\},^{16} \quad (75)$$

which agrees with the orthogonality condition (12′) because of (15). Indeed, we have, again using (12′),

$$\sum_\alpha \tfrac{1}{2}\int(\pi_\alpha'')^2dV=\sum_\alpha \tfrac{1}{2}\int(\pi_\alpha')^2dV +\frac{(4\pi)^{\frac{1}{2}}}{2D}\sum_{k,\alpha,\beta}\langle e_{\beta k}L_0^{\alpha\beta}\rangle_H\int\frac{\partial K}{\partial x_k}\pi_\alpha'(x)dV +\frac{4\pi}{2D^2}L_0^2\tfrac{1}{2}\int\left\{\frac{\partial K}{\partial x}-\frac{\partial\xi/\partial x}{\int(\partial\xi/\partial x)^2dV}\right\}^2dV.$$

Taking into account (15) and the definition (23) of N we find

$$4\pi\int\left\{\frac{\partial K}{\partial x}-\frac{\partial\xi/\partial x}{\int(\partial\xi/\partial x)^2dV}\right\}^2Vd=N-\frac{4\pi}{\int(\partial\xi/\partial x)^2dV}.$$

[16] We omit the index on the variable x if an integral has the same value for all three values of this index.

The term with N just gives the second term in (II); therefore we get finally, inserting $\xi(x)=X(x)/I$ and the value (63) of D

$$H=\frac{\kappa^2}{2g^2}\frac{4\pi}{\int(\partial X/\partial x)^2dV}L_0^2-\frac{3}{4}\frac{g^2}{\kappa^2}I+H_1, \quad (76)$$

$$H_1=\tfrac{1}{2}N\sum_{\alpha,\beta}\langle\overset{0}{p}_{\alpha\beta}\rangle_H^2+\frac{1}{2I}\sum_{\alpha,\beta}(\overset{(1)}{q}_{\alpha\beta})^2 +(4\pi)^{\frac{1}{2}}\sum_{\alpha,\beta,k}\langle\overset{0}{p}_{\alpha\beta}\rangle_H e_{\beta k}\int\frac{\partial K}{\partial x_k}\pi_\alpha''(x)dV +\tfrac{1}{2}\sum_\alpha\int(\pi_\alpha'')^2dV +\tfrac{1}{2}\sum_\alpha\int\varphi_\alpha'(-\Delta+\kappa^2)\varphi_\alpha'dV. \quad (77)$$

In the third term of the shift of $\pi_\alpha'(x)$ does not give any contribution, because

$$\sum_{\alpha,\beta}L_0^{\alpha\beta}p_{\alpha\beta}=0,$$

$L_0^{\alpha\beta}$ being antisymmetric, $p_{\alpha\beta}$ symmetric in α and β.

Before we proceed, it is necessary to complete the discussion of the shift (75) of $\pi_\alpha'(x)$. As this shift depends on $L_0^{\alpha\beta}$ and $e_{\beta k}$ it is a q number and necessitates corrections to $L_0^{\alpha\beta}$ and $e_{\beta k}$ themselves to insure that they commute with the new $\pi_\alpha''(x)$ and at the same time maintain their own commutation relations. This goal is reached by the canonical transformation

$$F=e^{iU}F'e^{-iU}=F'+i[U,F'] +\frac{i^2}{2!}[U,[U,F']]+\cdots. \quad (78)$$

We can substitute for F the old variables $\pi_\alpha'(x)$, $L_0^{\alpha\beta}$, $e_{\beta k}$, for F' the new variables $\pi_\alpha''(x)$, $L_0^{\alpha\beta\prime}$, $e_{\beta k}'$ if we choose for U

$$U=\frac{(4\pi)^{\frac{1}{2}}}{2D}\sum_{\alpha,\beta,k}\langle e_{\beta k}'L_0^{\alpha\beta\prime}\rangle_H\int\frac{\partial\xi}{\partial x_k}\varphi_\alpha'(x)dV \times\left[\int\left(\frac{\partial\xi}{\partial x}\right)^2dV\right]^{-1}, \quad (79)$$

which again gives the shift (75) for the first-order correction to $\pi_\alpha'(x)$ as a result of (21).

Fortunately it is not necessary to give here the explicit formulas for the corrections to $L_0^{\alpha\beta\prime}$,

$e'_{\alpha k}$ and the second-order corrections to $\pi_\alpha'(x)$ because the additional terms in the Hamiltonian of the order D^{-2} are either independent of $L_0^{\alpha\beta}$ or linear in $L_0^{\alpha\beta}$ and at the same time linear in $\varphi_\alpha'(x)$ or bilinear in $\pi_\alpha'(x)$, $\varphi_\beta'(x)$. The expectation value of the terms linear in $L_0^{\alpha\beta}$ is zero in the stationary states of the Hamiltonian (76), (77). It may be noted that $\overset{0}{L}_{ik}$ is only an approximate integral of the motion, but if we replace it in (76) by the exact integral (22′) of the total angular momentum, the additional terms in the Hamiltonian are again of the type which is disregarded here.

Because the eigenvalues of L_0^2 are $j(j+1)$ with half-integer j after the transformation (54), we get directly from (76) for the energy of excitation of the levels of the nucleon above the ground level

$$\Delta E=\frac{3}{2}\frac{\kappa^2}{g^2}\frac{4\pi}{\int(\nabla X)^2dV}\{j(j+1)-\tfrac{3}{4}\}. \quad (80)$$

The charge quantum number n does not enter but is restricted by (74) to values less than or equal to j. It is characteristic of the *symmetrical* pseudoscalar theory that the system is degenerate and the energy of all states with the same j and different charges has the same value. For the ground state, $j=\frac{1}{2}$, the number n has the two possible values $n=-\frac{1}{2}$ and $n=\frac{1}{2}$, which correspond to neutron and proton.

The integral

$$\frac{1}{4\pi}\int(\nabla X)^2dV=-\frac{1}{4\pi}\int X\Delta X dV$$
$$=\int XKdV-\frac{\kappa^2}{4\pi}\int X^2dV,$$

[see (13)] and can be evaluated in the two limiting cases of small sources $\kappa a\ll 1$ and large sources $\kappa a\gg 1$ as follows. Consider the expression

$$\int XKdV=\int\int K(x')\frac{e^{-\kappa r}}{r}K(x)dVdV'.$$

In the first limit we may substitute 1 for $e^{-\kappa r}$, the error being of the relative order κa. This means, according to (4), that we put $\int XKdV=a^{-1}$. In this limit the second integral $(\kappa^2/4\pi)\int X^2dV$ can be disregarded because it becomes independent of a if κa is small. Therefore we get

$$\frac{1}{4\pi}\int(\nabla X)^2dV=\frac{1}{a},$$
$$\frac{\Delta E}{\mu c^2}=\frac{3}{2}\frac{\kappa a}{g^2}\{j(j+1)-\tfrac{3}{4}\}\text{ for }\kappa a\ll 1. \quad (80a)$$

The expression $(\Delta E/\mu c^2)$, μ being the rest mass of the meson, refers to ordinary units of the energy and it is equal to $\Delta E/\kappa$ in our units.

For large sources $\kappa a\gg 1$ it is possible to integrate (13) by the following development

$$X(x)=\frac{4\pi}{\kappa^2}\left(1-\frac{1}{\kappa^2}\Delta\right)^{-1}K(x)$$
$$=\frac{4\pi}{\kappa^2}\left(1+\frac{1}{\kappa^2}\Delta+\frac{1}{\kappa^4}\Delta\Delta+\cdots\right)K(x),$$

which corresponds to a development in increasing powers of $(\kappa a)^{-1}$ in the resulting integrals. The leading term gives [see (23)]

$$\frac{1}{4\pi}\int(\nabla X)^2dV=\frac{4\pi}{\kappa^4}\int(\nabla K)^2dV=\frac{3}{\kappa^4}N.$$

Hence we get, using $N=\alpha a^{-5}$ (with a numerical factor α depending on the shape of the source function)

$$\frac{1}{4\pi}\int(\nabla X)^2dV=3\alpha(\kappa a)^{-4}a^{-1},$$
$$\frac{\Delta E}{\mu c^2}=\frac{1}{\alpha}\frac{(\kappa a)^5}{g^2}\{j(j+1)-\tfrac{3}{4}\}\text{ for }\kappa a\gg 1. \quad (80b)$$

We have now to consider the various terms neglected and the conditions which are implied by such procedures. In the first place, we neglected terms in the Hamiltonian which had as factors τ_r, σ_r, since these had no diagonal matrix elements for the system of states chosen, but introduced coupling between them. In order of magnitude, such terms are given by

$$h\sim\frac{\kappa^2}{g^2}\left[\int(\nabla X)^2dV\right]^{-1}.$$

For small sources $(\kappa a\ll 1)$ we get from (80a)

$$h\sim\kappa^2a/g^2.$$

The energy of separation between the states of (62) is of the order g^2I/κ^2. An evaluation of the integral I, defined by (14), shows that in the limit $\kappa a \ll 1$, $I \sim 1/a^3$. Consequently, if we demand that h be small compared with the energy of separation we get

$$\kappa a/g \ll 1. \qquad (81)$$

The second approximation was introduced in (70) and (72) when $\overset{(1)}{q}_{\alpha\beta}$ was treated as small compared to D; we restricted ourselves at this point to small oscillations of the field about its position of equilibrium. This restriction is not necessarily essential to the strong coupling hypothesis, but was introduced for mathematical simplicity. If it failed, the isobar energy (80) would have had a more complicated dependence on the angular momentum quantum number j. We estimate the validity of the approximation by computing $\langle(\overset{(1)}{q}_{\alpha\beta})^2\rangle_H/D^2$; we do this for the case where no free mesons are present, so that the only contribution to $\langle(\overset{(1)}{q}_{\alpha\beta})^2\rangle_H$ comes from the zero-point oscillations of the field. We find for small sources

$$\langle(\overset{(1)}{q}_{\alpha\beta})^2\rangle_H \sim a^{-4} \text{ for } \kappa a \ll 1.$$

In the same limit $D = gI/\kappa \sim g/\kappa a^3$. Once again, the criterion proves to be

$$\kappa a/g \ll 1.$$

Finally, we consider the unitary transformation (78). To the first order, its effect was simply to shift $\pi_\alpha'(x)$ according to (75) in such a way as to separate the Hamiltonian completely into a bound part and a part (77) associated with unbound mesons. However, higher order terms in (78) spoil this separation; they give a dependence of the isobar energy on the distribution of free mesotrons, or conversely a dependence of the scattering of mesotrons on the isobaric state. They make possible such processes as the transition from one isobaric state to another with the inelastic scattering of mesons. If we demand that the ratio of these terms to the isobar separation be small (i.e., that the half-width of the isobaric states be small compared to their separation), we arrive at the condition

$$\int \frac{\partial \xi}{\partial x_k}\varphi_\alpha'(x)dV \Big/ \int \left(\frac{\partial \xi}{\partial x}\right)^2 dV \ll 1. \qquad (82)$$

Again we substitute for φ_α' values corresponding to the zero-point oscillation of the unbound meson field. Evaluating the expressions in the small source limit, we find[17]

$$\frac{(\kappa a)^2 \ln \kappa a}{g^2} \ll 1. \qquad (83)$$

For $\kappa a < 1$, this condition is certainly fulfilled for values of g large enough to satisfy (81). For comparison we give the preceding expressions in the large source limit ($\kappa a \gg 1$):

$$h \sim \kappa^6 a^5/g^2, \quad I \sim 1/\kappa^2 a^5, \quad D \sim g/\kappa^3 a^5,$$
$$\langle(\overset{(1)}{q}_{\alpha\beta})^2\rangle_H \sim 1/\kappa a^5.$$

Condition (81) for the first and second neglects is replaced by

$$(\kappa a)^5/g^2 \ll 1. \qquad (81a)$$

Inequality (81a) also expresses the validity of (82) for this limit.

7. THE SCATTERING OF FREE MESONS

As mentioned already, in this paper we treat the scattering with disregard of all terms of order D^{-2} which means that we do not distinguish between elastic and inelastic scattering. Our treatment of the problem therefore will give us the total scattering cross section. Omitting the irrelevant self-energy constant, we can use here the Hamiltonian (77) directly, where the difference between $\pi_\alpha'(x)$ and $\pi_\alpha''(x)$ can be disregarded. The quantities $\mathbf{e}_\alpha$ or $e_{\alpha k}$ can be treated here as c numbers for they commute with all other observables in the Hamiltonian.[18] It is convenient to go back from the variables in (77) with the help of (16), (16′), (70), (72), to the functions

$$p_\alpha(x) = (4\pi)^{\frac{1}{2}} \sum_{i,\beta} \overset{0}{p}_{\alpha\beta} e_{\beta i}\frac{\partial K}{\partial x_i} + \pi_\alpha'(x)$$
$$= \pi_\alpha(x) - \frac{1}{2D}\sum_{\beta,k} e_{\beta k} L_0{}^{\alpha\beta}\frac{\partial K}{\partial x_k}, \qquad (84)$$
$$q_\alpha(x) = 1/(4\pi)^{\frac{1}{2}} \sum_{i,\beta} \overset{(1)}{q}_{\alpha\beta}\, e_{\beta i}(\partial\xi/\partial x_i) + \varphi_\alpha'(x)$$
$$= \varphi_\alpha(x) - 1/(4\pi)^{\frac{1}{2}} \textstyle\sum_k e_{\alpha k}(\partial\xi/\partial x_k). \qquad (85)$$

[17] The condition from which (82) was derived also restricts the values of j for which the approximation holds. We find $j \ll g/\kappa a$.

[18] If, on the other hand, one uses a matrix representation where $\overset{0}{T}_{12}$, $\overset{0}{L}_{12}$, and $L_0{}^2$ are diagonal, the $e_{\alpha k}$ are matrices which are a generalization of the rotator matrices already used by Heitler and Ma (reference 8).

We see that $p_\alpha(x)$, $q_\alpha(x)$ represents a field of free mesons, the meson eigenfield of the nucleon being subtracted. It fulfills the commutation relations [see (21), (55)]

$$i[p_\alpha(\mathbf{x}_1), q_\beta(\mathbf{x}_2)] = \delta_{\alpha\beta}\delta(\mathbf{x}_1-\mathbf{x}_2) - \tfrac{1}{2}\sum_k \frac{\partial K}{\partial x_{1k}}\frac{\partial \xi}{\partial x_{2k}}\delta_{\alpha\beta} + \tfrac{1}{2}\sum_{i,k} e_{\alpha k}e_{\beta i}\frac{\partial K}{\partial x_{1i}}\frac{\partial \xi}{\partial x_{2k}}. \quad (86)$$

These relations correspond to the fact that the symmetry of the $\overset{(1)}{q}_{\alpha\beta}$ and $\overset{0}{p}_{\alpha\beta}$ makes the field $p_\alpha(x)$, $q_\alpha(x)$ fulfill the subsidiary conditions [compare (12), (12′)]

$$\sum_k \left\{ e_{\alpha k}\int \frac{\partial \xi}{\partial x_k} p_\beta dV - e_{\beta k}\int \frac{\partial \xi}{\partial x_k} p_\alpha dV \right\} = 0, \quad (87a)$$

$$\sum_k \left\{ e_{\alpha k}\int \frac{\partial K}{\partial x_k} q_\beta dV - e_{\beta k}\int \frac{\partial K}{\partial x_k} q_\alpha dV \right\} = 0. \quad (87b)$$

On the other hand, the Hamiltonian (77) becomes formally a Hamiltonian for free plane waves

$$H = \tfrac{1}{2}\sum_\alpha \int p_\alpha^2 dV + \tfrac{1}{2}\sum_\alpha \int q_\alpha(-\Delta+k^2)q_\alpha dV. \quad (88)$$

The fact that the equations of motion differ from the wave equation of free particles is entirely due to the symmetry conditions or to the difference of the commutation relations (86) from the usual canonical ones.[19]

From (87a, b) and (88) we get, by the method of Lagrangian multipliers which have here the form of antisymmetric constants $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$ and $\lambda'_{\alpha\beta} = -\lambda'_{\beta\alpha}$, the equations of motion

$$\dot{q}_\alpha = p_\alpha - \sum_{\beta,k} \lambda_{\alpha\beta} e_{\beta k}(\partial \xi/\partial x_k),$$

$$-\dot{p}_\alpha = (-\Delta+k^2)q_\alpha - \sum_{\beta,k} \lambda'_{\alpha\beta} e_{\beta k}(\partial K/\partial x_k).$$

With the help of the time derivatives of (87a), (87b) in which one has to insert the expressions for $\dot{q}_\alpha$, $\dot{p}_\alpha$ one gets

$$\lambda_{\alpha\beta} = \tfrac{1}{2}\sum_k \int (p_\alpha e_{\beta k} - p_\beta e_{\alpha k})\frac{\partial K}{\partial x_k} dV, \quad \lambda'_{\alpha\beta} = 0.$$

Therefore we get the equations of motion in the form

$$\dot{q}_\alpha = p_\alpha - \tfrac{1}{2}\sum_{i,k,\beta} e_{\beta k}\frac{\partial \xi}{\partial x_k} \times \int (p_\alpha e_{\beta i} - p_\beta e_{\alpha i})\frac{\partial K}{\partial x_i} dV, \quad (89a)$$

$$-\dot{p}_\alpha = (-\Delta+k^2)q_\alpha. \quad (89b)$$

The same equations follow from (86) with the help of the relation $\dot{f} = i[H, f]$ for $f = p_\alpha$ and $f = q_\alpha$.

If we first consider q_α, p_α as classical quantities, we can integrate (89a, b) by putting

$$p_\alpha = \dot{u}_\alpha, \quad q_\alpha = u_\alpha - \tfrac{1}{2}\sum_{i,k,\beta} e_{\beta k}\frac{\partial \xi}{\partial x_k} \times \int (u_\alpha e_{\beta i} - u_\beta e_{\alpha i})\frac{\partial K}{\partial x_i} dV, \quad (90)$$

where u_α has, according to (89b), to fulfill the wave equation

$$\left(-\Delta+k^2+\frac{\partial^2}{\partial t^2}\right)u_\alpha = \frac{4\pi}{2I}\sum_{i,k,\beta} e_{\beta k}\frac{\partial K}{\partial x_k} \times \int (u_\alpha e_{\beta i} - u_\beta e_{\alpha i})\frac{\partial K}{\partial x_i} dV \quad (91)$$

and the subsidiary condition

$$\sum_k \left\{ e_{\beta k}\int \frac{\partial \xi}{\partial x_k} u_\alpha dV - e_{\alpha k}\int \frac{\partial \xi}{\partial x_k} u_\beta dV \right\} = 0. \quad (91a)$$

This condition is, however, not independent of the wave Eq. (91), for the second time derivative of (91a) vanishes as a consequence of (91). The subsidiary condition essentially excludes only the static solution

$$u_\alpha^0 = \sum_k e_{\alpha k}\frac{\partial \xi}{\partial x_k} \text{ of (91).}$$

The quantum-theoretical formalism leads to an analogous result. Let $u_\alpha^{(n)}$ be a normalized orthogonal set of functions (c numbers), which fulfill (with the usual boundary conditions in a hole)

$$(-\Delta+k^2-\omega_n^2)u_\alpha^{(n)} = \frac{4\pi}{2I}\sum_{i,k,\beta} e_{\beta k}\frac{\partial K}{\partial x_k} \times \int (u_\alpha^{(n)} e_{\beta i} - u_\beta^{(n)} e_{\alpha i})\frac{\partial K}{\partial x_i} dV, \quad (91b)$$

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[19] The analogous result in the charged scalar theory was first obtained by J. Schwinger.

and the subsidiary condition (91a). We put

$$p_\alpha=\sum_n A_n u_\alpha^{(n)}\left(\frac{\omega_n}{2}\right)^{\frac{1}{2}}, \quad (92)$$

$$q_\alpha=\sum_n A_n(2\omega_n)^{-\frac{1}{2}}\left\{u_\alpha^{(n)}-\frac{1}{2}\sum_{i,k,\beta} e_{\beta k}\frac{\partial \xi}{\partial x_k}\times\int(u_\alpha^{(n)}e_{\beta i}-u_\beta^{(n)}e_{\alpha i})\frac{\partial K}{\partial x_i}dV\right\},$$

$$A_{-n}=A_n^+,\quad u_\alpha^{(-n)}=u_\alpha^{(n)*},\quad \omega_{-n}=\omega_n>0. \quad (92a)$$

If we assume

$$[A_n, A_m^+]=[A_n, A_{-m}]=\delta_{nm}, \quad (93)$$

we get just the commutation relation (86), the system $u_\alpha^{(n)}$ being incomplete because, as a result of (91a), the static solution $u_\alpha^0=\sum_k e_{\alpha k}(\partial\xi/\partial x_k)$ is missing. Moreover, the Hamiltonian (77) assumes the normal form

$$H=\sum_n \omega_n(A_n^+A_n+\tfrac{1}{2}). \quad (94)$$

Our scattering problem is therefore reduced to the determination of the solution of (91) [or, aside from a factor $\exp(-i\omega t)$, of (91b)] which describes a plane wave incident in a given direction and an outgoing scattered wave. Corresponding to the three pseudoscalar fields u_α ($\alpha=1, 2, 3$) we shall get three linear independent solutions $u_\alpha^{(\rho)}$ or $u_\alpha^{(1)}$, $u_\alpha^{(2)}$, $u_\alpha^{(3)}$ of this problem, which correspond to an incident particle with $\rho=1$, $\rho=2$, or $\rho=3$.

Instead of this we can also use the solutions

$$u_\alpha^+=(1/\sqrt{2})(u_\alpha^{(1)}+iu_\alpha^{(2)}),$$

$$u_\alpha^-=(1/\sqrt{2})(u_\alpha^{(1)}-iu_\alpha^{(2)}),\ u_\alpha^{(3)},$$

which correspond to an incident positive or negative charged or neutral particle.

We now solve (91b) by the assumption

$$u_\alpha^{(\rho)}(x)=\delta_{\alpha\rho}\exp[ik(\mathbf{n}\cdot\mathbf{x})]+A\sum_{i,k}(\delta_{\alpha\rho}n_k-e_{\alpha i}e_{\rho k}n_i)\frac{\partial}{\partial x_k}\times\int K(x')\frac{e^{ikr}}{r}dV'. \quad (95)$$

Here $\rho=1, 2, 3$ enumerates the three different solutions, $\mathbf{n}$ is a unit vector in the direction of the incident wave, $k=(\omega^2-\kappa^2)^{\frac{1}{2}}$, and the constant A has to be determined. The first part, the plane wave, fulfills the wave equation for free particles without the right side of (91) while the scattered wave and therefore the whole expression (95) fulfills the equation

$$-(\Delta+k^2)u_\alpha^{(\rho)}=4\pi A\sum_{i,k}(\delta_{\alpha\rho}n_k-e_{\alpha i}e_{\rho k}n_i)(\partial K/\partial x_k). \quad (95')$$

In order to check (91b) we introduce the abbreviations

$$\int K(x)\exp[ik(\mathbf{n}\cdot\mathbf{x})]dV=\int K(x)\frac{\sin kR}{kR}dV=F, \quad (96)$$

(where $R=|\mathbf{x}|$, $r=|\mathbf{x}-\mathbf{x}'|$)

$$\int\int\frac{\partial K(x)}{\partial x_i}\frac{\cos kr}{r}\frac{\partial K(x')}{\partial x_k'}dVdV'=\delta_{ik}J. \quad (97)$$

Further we have to evaluate the integral

$$\int\int\frac{\partial K(x)}{\partial x_i}\frac{\sin kr}{r}\frac{\partial K(x')}{\partial x_k'}dVdV'=\frac{1}{3}\int\int K(x)\left(-\Delta\frac{\sin kr}{r}\right)K(x')dVdV'=\frac{k^2}{3}\int\int K(x)\frac{\sin kr}{r}K(x')dVdV'.$$

But as a consequence of the fact that $\sin kr/kr$ has no singularity for $x=0$, $\int(\sin kr/r)K(x')dV'$ is a solution of the wave equation $(\Delta+k^2)u=0$, and therefore

$$\int(\sin kr/r)K(x')dV'=(\sin kR/R)C.$$

Moreover, $C=F$ because for $R=0$ the left side of the equation is equal to kF, the right side equal to kC. Therefore

$$\int\int\frac{\partial K(x)}{\partial x_i}\frac{\sin kr}{r}\frac{\partial K(x')}{\partial x_k'}dVdV'=\delta_{ik}\frac{k^3}{3}F^2, \quad (98)$$

and finally

$$\int\int\frac{\partial K(x)}{\partial x_i}\frac{e^{ikr}}{r}\frac{\partial K(x')}{\partial x_k}dVdV'=\delta_{ik}\left(J+i\frac{k^3}{3}F^2\right). \quad (99)$$

With the assumption (95) we get

$$\sum_k \int (e_{\beta k}u_\alpha^{(\rho)}-e_{\alpha k}u_\beta^{(\rho)})\frac{\partial K}{\partial x_k}dV$$
$$=\sum_k (\delta_{\rho\alpha}e_{\beta k}n_k-\delta_{\rho\beta}e_{\alpha k}n_k)\times\left\{-ikF+2A\left(J+\frac{ik^3}{3}F^2\right)\right\}.$$

The differential Eq. (89b) therefore gives

$$-(\Delta+k^2)u_\alpha^{(\rho)}=\frac{4\pi}{2I}\left\{\sum_{i,k}(n_k\delta_{\alpha\rho}-e_{\rho k}e_{\alpha i}n_i)\frac{\partial K}{\partial x_k}\right\}\times\left\{-ikF+2A\left(J+\frac{ik^3}{3}F^2\right)\right\},$$

and the comparison with (95′) gives

$$IA=-\frac{ikF}{2}+A\left(J+\frac{ik^3}{3}F^2\right),\qquad A=\frac{ikF/2}{J-I+\frac{ik^3}{3}F^2}.\tag{100}$$

Introducing

$$\tan\delta=\frac{k^3}{3}F^2\Big/(J-I),\qquad \sin\delta=\frac{k^3}{3}F^2\Big/\left[(J-I)^2+\left(\frac{k^3}{3}F^2\right)^2\right]^{\frac12},\tag{101}$$

we get

$$A=(3i/2k^2F)\sin\delta e^{i\delta}.\tag{100′}$$

We get the asymptotic behavior of the scattered wave for very large r in the direction $\mathbf{n}_s$ of the scattered wave by putting $r=R-ik(\mathbf{n}_s\cdot\mathbf{x})$, $\nabla=-ik\mathbf{n}_s$,

$$[u_\alpha^{(\rho)}(x)]_{\text{scatt}}\sim ikFA[\delta_{\rho\alpha}(\mathbf{n}\cdot\mathbf{n}_s)-(\mathbf{e}_\alpha\cdot\mathbf{n})(\mathbf{e}_\rho\cdot\mathbf{n}_s)](e^{ikR}/R).\tag{102}$$

We now compute the square of this amplitude for a given $\mathbf{n}_s$ and sum over α,

$$\sum_\alpha [u_\alpha^{(\rho)}(x)]^2_{\text{scatt}}=k^2F^2A^2[(\mathbf{n}\cdot\mathbf{n}_s)^2-2(\mathbf{n}\cdot\mathbf{n}_s)(\mathbf{e}_\rho\cdot\mathbf{n})(\mathbf{e}_\rho\cdot\mathbf{n}_s)+(\mathbf{e}_\rho\cdot\mathbf{n}_s)^2].$$

Further, we take the average over-all directions of the axis system $\mathbf{e}_\alpha$ and get for the right side,[20] with $\cos\theta=(\mathbf{n}\cdot\mathbf{n}_s)$, θ being the scattering angle

$$(kFA)^2\tfrac13(1+\cos^2\theta).$$

[20] This is in the approximation where the energy loss of the meson in the scattering is disregarded, the same as averaging over-all directions of the angular momentum $\overset{0}{L}_{ik}$ in the initial and in the final states.

This is the cross section dq for the solid angle $d\Omega=2\pi\sin\theta d\theta$; inserting (100′) we get

$$dq=\frac{\sin^2\delta}{k^2}\frac{3}{4}(1+\cos^2\theta)d\Omega,\tag{103}$$

and the total cross section

$$q=(\sin^2\delta/k^2)4\pi.\tag{104}$$

On the other hand, if we integrate the square of the amplitude over all angles θ, which means averaging over $\mathbf{n}_s$ without summation over α, we get

$$(kFA)^2\tfrac13[\delta_{\alpha\rho}-2\delta_{\alpha\rho}(\mathbf{e}_\rho\cdot\mathbf{n})+(\mathbf{e}_\alpha\cdot\mathbf{n})^2].$$

Averaging over the orientations of the axis system gives

$$(kFA)^2\tfrac19(\delta_{\alpha\rho}+1)$$

which means

$$q_\alpha=4\pi(\sin^2\delta/k^2)\tfrac14(1+\delta_{\alpha\rho}).\tag{105}$$

In other words, if the incident particle is a "1 particle" the number of scattered "1 particles," "2-particles," and "3-particles" is $\frac12$, $\frac14$, $\frac14$. This also holds if one takes for "1," "2," "3" *in any order, neutral, positive, and negative* as can be shown by using the above-mentioned eigenfunctions

$$u_\alpha^+=(1/\sqrt2)(u_\alpha^{(1)}+iu_\alpha^{(2)}),$$
$$u_\alpha^-=(1/\sqrt2)(u_\alpha^{(1)}-iu_\alpha^{(2)})$$

as well as $u_\alpha^{(3)}$.

Finally we evaluate the integrals J, I, and F for the physically important case $ka\ll1$, $\kappa a\ll1$. In this case we can first put $F=1$. Further we have, according to (97),

$$3J=\int\int K(x)[(-\Delta)(\cos kr/r)]K(x')dVdV'=4\pi\int K^2(x)dV+k^2\int\int K(x)(\cos kr/r)K(x')dVdV'.$$

In the same way

$$3I=\int\int K(x)\left[(-\Delta)\frac{e^{-\kappa r}}{r}\right]K(x')dVdV'=4\pi\int K^2(x)dV-\kappa^2\int\int K(x)\frac{e^{-\kappa r}}{r}K(x')dVdV'.$$

Therefore

$$3(J-I)=\int\int K(x)(k^2\cos kr+\kappa^2 e^{-\kappa r})\times\frac{1}{r}K(x')dVdV'. \quad (106)$$

So far this result is exact, but for $ka\ll 1$, $\kappa a\ll 1$ we can substitute 1 for $\cos kr$ and $e^{-\kappa r}$ and get, with $\omega^2=k^2+\kappa^2$

$$3(J-I)=\omega^2/a \quad \text{for} \quad ka\ll 1, \quad \kappa a\ll 1, \quad (106')$$

hence

$$\sin\delta\sim\tan\delta\sim(k^3a/\omega^2)$$

and for (103), (104) we get finally on writing $v/c=p/E$ for k/ω

$$\left.\begin{aligned} dq&=\left(\frac{p}{E}\right)^4 a^2\tfrac{3}{4}(1+\cos^2\theta)d\Omega \\ q&=\left(\frac{p}{E}\right)^4 a^2 4\pi \end{aligned}\right\} ka\ll 1,\ \kappa a\ll 1, \quad (107)$$

which was quoted in the second section.

8. THE CHARGED PSEUDOSCALAR THEORY

Although the symmetrical theory is, from a mathematical standpoint, more elegant than the "charged pseudoscalar theory" which uses only positive and negative charged, but no neutral, mesons, it is also of interest to develop the corresponding results for the latter theory, because neutral mesons have never been observed and because in the case of strong coupling the charged theory also leads to forces between two like particles.

With a complex pseudoscalar field φ the Hamiltonian of this theory can be written

$$H=\int dV\left\{\pi^*\pi+(\nabla\varphi^*)\cdot(\nabla\varphi)+\kappa^2\varphi^*\varphi -\frac{g}{\kappa}(4\pi)^{\frac{1}{2}}K(x)(\nabla\varphi\cdot\tau_+\boldsymbol{\sigma}+\nabla\varphi^*\cdot\tau_-\boldsymbol{\sigma})\right\}, \quad (108)$$

where

$$\tau_+=\tfrac{1}{2}(\tau_1+i\tau_2), \quad \tau_-=\tfrac{1}{2}(\tau_1-i\tau_2). \quad (109)$$

In order to facilitate the comparison with the symmetrical theory we split π and φ into the real and imaginary parts which are normalized according to

$$\begin{aligned} \pi&=\frac{1}{\sqrt{2}}(\pi_1+i\pi_2), \quad & \varphi&=\frac{1}{\sqrt{2}}(\varphi_1-i\varphi_2), \\ \pi^*&=\frac{1}{\sqrt{2}}(\pi_1-i\pi_2), \quad & \varphi^*&=\frac{1}{\sqrt{2}}(\varphi_1+i\varphi_2), \end{aligned} \quad (110)$$

and get

$$H=\frac{1}{2}\sum_{\alpha=1}^{2}\left\{\int[\pi_\alpha^2+(\nabla\varphi_\alpha)^2+\kappa^2\varphi_\alpha^2]dV -\frac{g}{\kappa\sqrt{2}}(4\pi)^{\frac{1}{2}}\int K(x)\tau_\alpha\boldsymbol{\sigma}\cdot\nabla\varphi_\alpha dV\right\}. \quad (111)$$

We can therefore take over the whole development of Section 3 if we simply drop the field components π_3, φ_3 and let the summation index α take only the values $\alpha=1, 2$. Of course there exists as an integral of motion only the T_{12} component of the isotopic spin in this theory, which gives the electric charge minus $\frac{1}{2}$.

The determination of the eigenvalue of the interaction energy

$$H_{\text{int}}=\frac{g}{\kappa\sqrt{2}}\sum_{k=1}^{3}(\tau_1\sigma_k\overset{0}{\varphi}_{1k}+\tau_2\sigma_k\overset{0}{\varphi}_{2k})$$

here is somewhat simpler than in the case of the symmetrical theory. Computing the square of the interaction energy, we get,

$$[\tau_1(\boldsymbol{\sigma}\cdot\boldsymbol{\phi}_1^0)+\tau_2(\boldsymbol{\sigma}\cdot\boldsymbol{\phi}_2^0)]^2 =(\boldsymbol{\phi}_1^0)^2+(\boldsymbol{\phi}_2^0)^2-2\tau_3(\boldsymbol{\sigma}\cdot[\boldsymbol{\phi}_1^0,\boldsymbol{\phi}_2^0]).$$

The four eigenvalues of the interaction energy are therefore

$$\begin{aligned} E_{\text{int}}&=\pm\frac{g}{\kappa\sqrt{2}}\{(\boldsymbol{\phi}_1^0)^2+(\boldsymbol{\phi}_2^0)^2\pm|[\boldsymbol{\phi}_1^0,\boldsymbol{\phi}_2^0]|\}^{\frac{1}{2}} \\ &=\pm\frac{g}{\kappa\sqrt{2}}\{(\boldsymbol{\phi}_1^0)^2+(\boldsymbol{\phi}_2^0)^2\pm 2[(\boldsymbol{\phi}_1^0)^2(\boldsymbol{\phi}_2^0)^2-(\boldsymbol{\phi}_1^0\cdot\boldsymbol{\phi}_2^0)^2]^{\frac{1}{2}}\}^{\frac{1}{2}}, \end{aligned} \quad (112)$$

with all possible combinations of the + and − signs.

The application of the formalism of Section 4 leads here to the case where one of the eigenvalues Q of (C) [Eq. (30)] or (C') [Eq. (30')] is zero because $\overset{0}{\varphi}_{3i}\equiv 0$. The matrix C' has therefore only two rows and two columns and is given by

$$C'=\begin{pmatrix}(\boldsymbol{\phi}_1{}^0)^2 & (\boldsymbol{\phi}_1{}^0\cdot\boldsymbol{\phi}_2{}^0)\\ (\boldsymbol{\phi}_1{}^0\cdot\boldsymbol{\phi}_2{}^0) & (\boldsymbol{\phi}_2{}^0)^2\end{pmatrix}$$

and $Q_1{}^2$, $Q_2{}^2$ are the roots of the equation

$$\{(\boldsymbol{\phi}_1{}^0)^2-x\}\{(\boldsymbol{\phi}_2{}^0)^2-x\}-(\boldsymbol{\phi}_1{}^0\cdot\boldsymbol{\phi}_2{}^0)^2=0,$$

which gives

$$Q_{1,2}=\frac{1}{\sqrt{2}}\{(\boldsymbol{\phi}_1{}^0)^2+(\boldsymbol{\phi}_2{}^0)^2\pm[(\boldsymbol{\phi}_1{}^0)^2+(\boldsymbol{\phi}_2{}^0)^2 -4((\boldsymbol{\phi}_1{}^0)^2(\boldsymbol{\phi}_2{}^0)^2-(\boldsymbol{\phi}_1{}^0\cdot\boldsymbol{\phi}_2{}^0)^2)]^{\frac{1}{2}}\}^{\frac{1}{2}}. \quad (113)$$

The expression (36) for the interaction energy

$$E_{\text{int}}=\frac{g}{\kappa\sqrt{2}}(\pm Q_1\pm Q_2) \quad (112')$$

is in agreement with (112). The orthogonal matrix $B_{r\alpha}$ has here only two rows and two columns and can be assumed as

$$B_{r\alpha}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \quad (114)$$

If we write $\mathbf{n}^{(r)}$ for A_{kr} the normal form (29) for $\overset{0}{\varphi}_{\alpha k}$ gives us

$$\begin{aligned}\boldsymbol{\phi}_1{}^0&=Q_1\cos\theta\mathbf{n}^{(1)}-Q_2\sin\theta\mathbf{n}^{(2)},\\ \boldsymbol{\phi}_2{}^0&=Q_1\sin\theta\hat{\mathbf{n}}^{(1)}+Q_2\cos\theta\mathbf{n}^{(2)}.\end{aligned} \quad (115)$$

The significance of $\mathbf{n}^{(1)}$, $\mathbf{n}^{(2)}$ is therefore that $\mathbf{n}^{(3)}$ is orthogonal to the plane through $\boldsymbol{\phi}_1{}^0$ and $\boldsymbol{\phi}_2{}^0$ and the orientation of the $\mathbf{n}^{(1)}$, $\mathbf{n}^{(2)}$ axes which we may call for the moment the ξ and η axes is defined by the condition

$$\overset{0}{\varphi}_{1\xi}\overset{0}{\varphi}_{1\eta}+\overset{0}{\varphi}_{2\xi}\overset{0}{\varphi}_{2\eta}=0. \quad (116)$$

This can be immediately derived from (115) and means that the matrix C defined by (30) is diagonalized by going to the (ξ,η,ζ) axes, the diagonal matrix elements being Q_1, Q_2, 0.

In the Eqs. (45), (45′) we have to put

$\overset{0}{\pi}_{3k}\equiv 0$, $B_{3\alpha}\equiv 0$, $B_{r3}\equiv 0$ for $\alpha, r=1, 2$; $P_3\equiv 0$;

hence

$$\sum_{\alpha=1,2}(\boldsymbol{\pi}_\alpha{}^0)^2=\sum_{r=1,2}P_r{}^+P_r+\frac{(L_0{}^{12}+T_0{}^{12})^2}{2(Q_1-Q_2)^2} +\frac{(L_0{}^{12}-T_0{}^{12})}{2(Q_1+Q_2)^2}+\sum_{r=1,2}\frac{(L_0{}^{r3})^2}{Q_r{}^2}. \quad (50\text{c})^{21}$$

[21] We denote corresponding equations in the symmetric and charged theories with the same number and add in the latter a c.

The symmetrical quantities $\overset{0}{q}_{\alpha\beta}$, $\overset{0}{p}_{\alpha\beta}$ have to be defined only for α, $\beta=1, 2$ and according to (51) we get

$$\begin{aligned}\mathbf{e}^{(1)}&=\mathbf{n}^{(1)}\cos\theta-\mathbf{n}^{(2)}\sin\theta,\\ \mathbf{e}^{(2)}&=\mathbf{n}^{(1)}\sin\theta+\mathbf{n}^{(2)}\cos\theta,\\ \mathbf{e}^{(3)}&=\mathbf{n}^{(3)},\end{aligned} \quad (117)$$

and from (52)

$$\begin{aligned}\overset{0}{q}_{11}&=Q_1\cos^2\theta+Q_2\sin^2\theta,\\ \overset{0}{q}_{12}=\overset{0}{q}_{21}&=(Q_1-Q_2)\sin\theta\cos\theta,\\ \overset{0}{q}_{22}&=Q_1\sin^2\theta+Q_2\cos^2\theta.\end{aligned} \quad (118)$$

For the description of $\mathbf{n}^{(r)}$ and $L_0{}^{rs}$ by Euler angles and their conjugate momenta, see Appendix.

For the minimum of the potential energy we get from (62) the old value (63) for $Q_1=Q_2=D$; but because $Q_3=0$ we now have

$$\overset{0}{E}_{\min}=-\frac{D^2}{I}=-\frac{1}{2}\frac{g^2}{\kappa^2}I. \quad (63\text{c})$$

In the approximation where $\overset{(1)}{q}_{\alpha\beta}$ is disregarded, we now get

$$\begin{aligned}L_0{}^{12}&=D\{(\mathbf{e}_\alpha\cdot\boldsymbol{\pi}_\beta{}^0)-(\mathbf{e}_\beta\cdot\boldsymbol{\pi}_\alpha{}^0)\},\\ L_0{}^{3\alpha}&=-L_0{}^{\alpha 3}=D(\mathbf{e}_3\cdot\boldsymbol{\pi}_\alpha{}^0)\quad(\alpha=1, 2).\end{aligned}$$

Therefore

$$\boldsymbol{\pi}_\alpha{}^0=\sum_{\beta=1,2}\left(\mathbf{e}_\beta\overset{0}{p}_{\alpha\beta}+\frac{1}{2D}\langle\mathbf{e}_\beta L_0{}^{\alpha\beta}\rangle_H\right)+\frac{1}{D}\mathbf{e}_3L_0{}^{\alpha 3},$$

hence

$$\sum_\alpha(\boldsymbol{\pi}_\alpha{}^0)^2=\sum_{\alpha,\beta}(\overset{0}{p}_{\alpha\beta})^2+\frac{1}{2D^2}(L_0{}^{12})^2+\frac{1}{D^2}\sum_{\alpha=1,2}(L_0{}^{\alpha 3})^2 =\sum_{\alpha,\beta}(\overset{0}{p}_{\alpha\beta})^2+\frac{1}{2D^2}[2L_0{}^2-(L_0{}^{12})^2].$$

This is in accordance with (50c) if we put in the last two terms $Q_1=Q_2=D$, $T_0{}^{12}=-L_0{}^{12}$. In (II) we have therefore to substitute $2L_0{}^2-(T_0{}^{12})^2$ for $L_0{}^2$.

In the same way we have to substitute in (75)

$$\frac{1}{2D}\{\sum_\beta e_{\beta i}L_0{}^{\alpha\beta}+2e_{3i}L_0{}^{\alpha 3}\}\quad\text{for}\quad\frac{1}{2D}\sum_\beta e_{\beta i}L_0{}^{\alpha\beta}$$

and get instead of (76):

$$H=\frac{\kappa^2}{2g^2}\frac{4\pi}{\int(\partial X/\partial x)^2dV}[2L_0{}^2-(T_0{}^{12})^2] -\frac{1}{2}\frac{g^2}{\kappa^2}I+H_1 \quad (76\text{c})$$

where H_1 is again given by (73), and instead of (80), (80a), (80b)

$$\Delta E=\frac{3}{2}\frac{\kappa^2}{g^2}\frac{4\pi}{\int(\nabla X)^2dV}[2j(j+1)-n^2-5/4], \quad (80c)$$

$$\frac{\Delta E}{\mu c^2}=\frac{3}{2}\frac{\kappa a}{g^2}[2j(j+1)-n^2-5/4] \text{ for } \kappa a\ll 1, \quad (80ac)$$

$$\frac{\Delta E}{\mu c^2}=\frac{1}{\alpha}\frac{(\kappa a)^5}{g^2}[2j(j+1)-n^2-5/4] \quad \text{for } \kappa a\gg 1. \quad (80bc)$$

The restriction (74) $-j\leqq n\leqq j$ is of course unchanged.

In the computation of the scattering the following modifications of Section 7 take place. The functions $p_\alpha(x)$, $q_\beta(x)$ (α, $\beta=1$, 2 only in the following) fulfill, besides (87a, b), the condition

$$\mathbf{e}_3\cdot\int\nabla Kq_\alpha dV=0, \quad \mathbf{e}_3\cdot\int\nabla\xi p_\alpha dV=0, \quad (87c)$$

and the commutation relations (86) get a corresponding additional term:

$$i[p_\alpha(\mathbf{x}_1), q_\beta(\mathbf{x}_2)]=\delta_{\alpha\beta}\delta(\mathbf{x}_1-\mathbf{x}_2) -\frac{1}{2}\sum_k\frac{\partial K}{\partial x_{1k}}\frac{\partial\xi}{\partial x_{2k}}\delta_{\alpha\beta}+\frac{1}{2}\sum_{i,k}e_{\alpha k}e_{\beta i}\frac{\partial K}{\partial x_{1i}}\frac{\partial\xi}{\partial x_{2k}} -\delta_{\alpha\beta}\frac{1}{2}\sum_{i,k}e_{3i}e_{3k}\frac{\partial K}{\partial x_{1i}}\frac{\partial\xi}{\partial x_{2k}}. \quad (86c)$$

With

$$p_\alpha=\dot{u}_\alpha,$$

$$q_\alpha=u_\alpha-\frac{1}{2}\sum_{i,k}\frac{\partial\xi}{\partial x_k}\int(u_\alpha\delta_{ik}+u_\alpha e_{3i}e_{3k} -\textstyle\sum_\beta u_\beta e_{\beta k}e_{\alpha i})\frac{\partial K}{\partial x_i}dV, \quad (90c)$$

we now get

$$\mathbf{e}_\beta\cdot\int\nabla\xi u_\alpha dV-\mathbf{e}_\alpha\cdot\int\nabla\xi u_\beta dV=0,$$

$$\mathbf{e}_3\cdot\int\nabla\xi u_\alpha dV=0, \quad (91c)$$

$$(-\Delta+\kappa^2-\omega^2)u_\alpha=\frac{4\pi}{2I}\sum_{i,k,\beta}\frac{\partial K}{\partial x_k} \times\int\{u_\alpha(\delta_{ik}+e_{3i}e_{3k})-u_\beta e_{\beta k}e_{\alpha i}\}\frac{\partial K}{\partial x_i}dV. \quad (91c)$$

The solution is

$$u_\alpha^{(\rho)}(x)=\delta_{\rho\alpha}\exp[ik(\mathbf{n}\cdot\mathbf{x})] +A\{\delta_{\rho\alpha}[(\mathbf{n}\cdot\nabla)+(\mathbf{n}\cdot\mathbf{e}_3)(\mathbf{e}_3\cdot\nabla)] -(\mathbf{n}\cdot\mathbf{e}_\alpha)(\mathbf{e}_\rho\cdot\nabla)\}\int K(x')\frac{e^{ikr}}{r}dV', \quad (95c)$$

where A is again given by (88), (89). Asymptotically for large R we have

$$[u_\alpha^{(\rho)}(x)]_{\text{Scatt.}}\sim ikFA[\delta_{\rho\alpha}(\mathbf{n}\cdot\mathbf{n}_s)+\delta_{\rho\alpha}(\mathbf{n}\cdot\mathbf{e}_3)(\mathbf{n}_s\cdot\mathbf{e}_3) -(\mathbf{n}\cdot\mathbf{e}_\alpha)(\mathbf{n}_s\cdot\mathbf{e}_\rho)]\frac{e^{ikR}}{R}.$$

$$\sum_{\alpha=1,2}|u_\alpha^{(\rho)}(x)|^2{}_{\text{Scatt.}}=(kFA)^2\{[(\mathbf{n}\cdot\mathbf{n}_s) +(\mathbf{n}\cdot\mathbf{e}_3)(\mathbf{n}_s\cdot\mathbf{e}_3)]^2-2(\mathbf{n}\cdot\mathbf{n}_s)(\mathbf{n}\cdot\mathbf{e}_\rho)(\mathbf{n}_s\cdot\mathbf{e}_\rho) -2(\mathbf{n}\cdot\mathbf{e}_3)(\mathbf{n}\cdot\mathbf{e}_\rho)(\mathbf{n}_s\cdot\mathbf{e}_3)(\mathbf{n}_s\cdot\mathbf{e}_\rho) +(\mathbf{n}_s\cdot\mathbf{e}_\rho)^2-(\mathbf{n}\cdot\mathbf{e}_3)^2(\mathbf{n}_s\cdot\mathbf{e}_\rho)^2\}.$$

The process of averaging over the directions of the axis system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ gives (for any value of ρ, and for the solutions which correspond to an incident charged particle)

$$dq=(\sin^2\delta/k^2)\tfrac{3}{4}(1+3\cos^2\theta)d\Omega, \quad (103c)$$

$$q=(\sin^2\delta/k^2)\tfrac{3}{2}4\pi, \quad (104c)$$

where for $\sin^2\delta/k^2$ we can again substitute

$$\left(\frac{v}{c}\right)^4a^2=\left(\frac{p}{E}\right)^4a^2$$

if $ka\ll 1$, $\kappa a\ll 1$.

If the incident particle has a + charge, we find for the total cross section (integrating over all scattering angles) for scattering with the same (+) and the opposite (−) charge, respectively,

$$q_+=(\sin^2\delta/k^2)4\pi(5/4),\qquad q_-=(\sin^2\delta/k^2)4\pi(1/4), \quad (105c)$$

the sum of both agreeing with (104c).

9. THE MAGNETIC MOMENT OF THE NUCLEON

The meson field surrounding the heavy particle gives rise to a magnetic moment which will prove to be of equal magnitude, but of opposite sign for the proton and neutron, just as in the case of the weak coupling theory.[22] The results of this section are equally valid for the symmetrical and charged pseudoscalar theories, since the neutral mesons give no contribution to the magnetic moment.

The magnetic moment is related to the current density S by

$$M=\tfrac{1}{2}e\int[\mathbf{x}\cdot\mathbf{S}]dV \qquad (119)$$

or, in tensor notation,

$$M_{ik}=-M_{ki}=\tfrac{1}{2}e\int[x_iS_k-x_kS_i]dV, \qquad (119a)$$

e being the absolute value of the elementary electric charge. The definition of the charge and current densities S_0 and $\mathbf{S}$ becomes essentially ambiguous in a theory which uses finite distance operators, such as the source function $K(x)$. Indeed, in such a theory it is not possible to fulfill the continuity equation

$$\partial S_0/\partial t+\text{div }\mathbf{S}=0 \qquad (120)$$

everywhere inside the source. It is fulfilled outside the source, as well as (on the average) over a volume V of dimensions of order a in which the source is included.

$$\int_V[\partial S_0/\partial t+\text{div }\mathbf{S}]dV=0. \qquad (120a)$$

Expressions for S_0 and $\mathbf{S}$ which fulfill this weaker condition and which agree with the definition (9) of the total charge are given by

$$S_0=\varphi_1\pi_2-\varphi_2\pi_1+\tfrac{1}{2}(1+\tau_3)K(x), \qquad (121)$$

$$S=(\varphi_2\,\mathbf{grad}\;\varphi_1-\varphi_1\,\mathbf{grad}\;\varphi_2)+\frac{g}{\kappa}(2\pi)^{\frac{1}{2}}(\phi_1\tau_2-\phi_2\tau_1)\boldsymbol{\sigma}K(x).$$

To prove that these satisfy the conservation conditions, we make use of the relations

$$\left(-\Delta+\kappa^2+\frac{\partial^2}{\partial t^2}\right)\varphi_\alpha+\frac{g}{\kappa}(2\pi)^{\frac{1}{2}}\tau_\alpha(\boldsymbol{\sigma}\cdot\mathbf{grad}\;K)=0,$$

$$\frac{1}{2}\frac{\partial\tau_3}{\partial t}=-\frac{g}{\kappa}(2\pi)^{\frac{1}{2}}\boldsymbol{\sigma}\cdot\int(\tau_2\,\mathbf{grad}\;\varphi_1-\tau_1\,\mathbf{grad}\;\varphi_2)K(x)dV.$$

These follow from the general rule $\partial f/\partial t=i[H,f]$ and the commutation relations (2), (6) applied to the Hamiltonian given by (1), (8) with $f=\pi_\alpha(X)$, $\phi_\alpha(X)$, τ_3. We obtain

$$\frac{\partial S_0}{\partial t}+\text{div }\mathbf{S}=-\boldsymbol{\sigma}\cdot[\tau_2\,\mathbf{grad}\;\varphi_1-\tau_1\,\mathbf{grad}\;\varphi_2]K(x)+K(X)\boldsymbol{\sigma}\cdot\int dV(\tau_2\,\mathbf{grad}\;\varphi_1-\tau_1\,\mathbf{grad}\;\varphi_2)K(x).$$

Therefore the continuity condition is fulfilled both for points outside the source, where $K(x)=0$, and on the average over the source as indicated by (120a). As a result of the lack of validity of the continuity equation inside the source, we may apply the assumed expression for the current density to the computation of the magnetic moment only if this moment is generated in a volume large compared with the dimensions of the source; in other words, we are restricted to small source size, $\kappa a\ll 1$.

Introducing the strong coupling approximation, we apply the S transformation (65a, b) to the second part of the current (121b); then we insert the values (68) for $\tau_r\sigma_s$, namely $\tau_r\sigma_s=-\delta_{rs}$. This gives

$$\mathbf{S}=(\varphi_2\,\mathbf{grad}\;\varphi_1-\varphi_1\,\mathbf{grad}\;\varphi_2)-\frac{g}{\kappa}(2\pi)^{\frac{1}{2}}(\varphi_1\mathbf{e}^{(2)}-\varphi_2\mathbf{e}^{(1)})K(x). \qquad (122)$$

We insert for the field its value corresponding to the absence of free mesons and the lowest eigenvalue of the interaction energy, (16), (64).

$$\varphi_\alpha(x)=\frac{1}{(4\pi)^{\frac{1}{2}}}D\mathbf{e}^{(\alpha)}\cdot\text{grad }\xi \quad (\alpha=1,2).$$

If now we use

$$D=\frac{g}{\kappa\sqrt{2}}I\ [\text{Eq. (63)}],\quad \xi=\frac{1}{I}\mathrm{X}\ \text{Eq. (15)},$$

$$\varphi_\alpha(x)=\frac{1}{(4\pi)^{\frac{1}{2}}}\frac{g}{\kappa\sqrt{2}}\mathbf{e}^{(\alpha)}\cdot\text{grad X}.$$

The current becomes

$$S_k=\frac{g^2}{2\kappa^2}\sum_{l,m=1}^{3}\left\{\frac{1}{4\pi}(e_l^{(1)}e_m^{(2)}-e_l^{(2)}e_m^{(1)})\frac{\partial^2\mathrm{X}}{\partial x_k\partial x_l}\frac{\partial\mathrm{X}}{\partial x_m}-(e_l^{(1)}e_k^{(2)}-e_l^{(2)}e_k^{(1)})\frac{\partial\mathrm{X}}{\partial x_l}K(x)\right\}$$

[22] H. Fröhlich, W. Heitler, and N. Kemmer, Proc. Roy. Soc. **A166**, 154 (1938). Their result, given in Eq. (63) of their paper, agrees formally (apart from a numerical factor) with our result, Eq. (124) below.

and the magnetic moment (119a) becomes

$$M_{ik}=\frac{e}{2}\frac{g^2}{2\kappa^2}\sum_{l,m=1}^{3}\left\{\int dV\frac{1}{4\pi}(e_l^{(1)}e_m^{(2)}-e_l^{(2)}e_m^{(1)})\times\left(x_i\frac{\partial^2 X}{\partial x_k\partial x_l}-x_k\frac{\partial^2 X}{\partial x_i\partial x_l}\right)\frac{\partial X}{\partial x_m}-\int dV[e_l^{(1)}e_k^{(2)}-e_l^{(2)}e_k^{(1)})x_i-(e_l^{(1)}e_i^{(2)}-e_l^{(2)}e_i^{(1)})x_k]\frac{\partial X}{\partial x_l}K(x)\right\}. \quad (123)$$

The first volume integral is transformed by partial integration with respect to $\partial/\partial x_l$, half the terms vanishing in the sum over l and m. In the second integral one substitutes for $K(x)$ its expression $(4\pi)^{-1}(-\Delta+\kappa^2)X$ taken from (13) and gets

$$4\pi\int x_i\frac{\partial X}{\partial x_l}K(x)dV=\int x_i\frac{\partial X}{\partial x_l}(-\Delta+\kappa^2)XdV$$
$$=\tfrac{1}{2}\int dVx_i\frac{\partial}{\partial x_l}X(-\Delta+\kappa^2)X+\tfrac{1}{2}\sum_r\int dVx_i\frac{\partial}{\partial x_r}\left(X\frac{\partial^2 X}{\partial x_l\partial x_r}-\frac{\partial X}{\partial x_l}\frac{\partial X}{\partial x_r}\right)$$
$$=\tfrac{1}{2}\delta_{il}\int X\Delta XdV-\frac{\kappa^2}{2}\delta_{il}\int X^2dV+\tfrac{1}{2}\int\left(\frac{\partial X}{\partial x_l}\frac{\partial X}{\partial x_i}-X\frac{\partial^2 X}{\partial x_l\partial x_i}\right)dV$$
$$=-\tfrac{1}{2}\delta_{il}\int(\mathbf{grad}\ X)^2dV-\frac{\kappa^2}{2}\delta_{il}\int X^2dV+\int\frac{\partial X}{\partial x_l}\frac{\partial X}{\partial x_i}dV.$$

Using the spherical symmetry of X we have

$$\int\frac{\partial X}{\partial x_i}\frac{\partial X}{\partial x_l}dV=\tfrac{1}{3}\delta_{il}\int(\mathbf{grad}\ X)^2dV.$$

We get

$$M_{ik}=+\frac{e}{8\pi}\frac{g^2}{2\kappa^2}(e_i^{(1)}e_k^{(2)}-e_i^{(2)}e_k^{(1)})\times\left[-\tfrac{1}{3}\int(\mathbf{grad}\ X)^2dV+\kappa^2\int X^2dV\right].$$

As was shown in Section 4 the second integral can be neglected for $\kappa a\ll 1$ while

$$\frac{1}{4\pi}\int(\mathbf{grad}\ X)^2dV=1/a.$$

Changing to vector notation and using $[\mathbf{e}^{(1)}\times\mathbf{e}^{(2)}]=\mathbf{e}^{(3)}$ we get the final result

$$\mathbf{M}=-\frac{e}{\kappa}\frac{g^2}{\kappa a}\frac{1}{12}\mathbf{e}^{(3)}. \quad (124)$$

We now compute the component of $\mathbf{M}$ in a fixed direction, say along the x_3 axis. We introduce a matrix representation for the $e_k^{(\alpha)}$, where $L_{12}^0\equiv L_3^0$, $(L_0)^2$, and $T_0^{12}=-L_0^{(3)}$ [cf. Eq. (73)] are diagonal with eigenvalues m, $j(j+1)$, and n, respectively, with $-j\leq m\leq j$ and $-j\leq n\leq j$. From the commutation relations (39), (55), (56) it follows that $L^{(3)}\equiv L^{12}$ commutes with all L_k and is therefore diagonal with respect to j, m, and L_k.

Moreover, $e^{(3)}$ commutes with $L^{(3)}$ and is therefore diagonal with respect to n. However, $e_3^{(3)}$ is not diagonal with respect to j and we need the diagonal element $(j,n,m|e_3^{(3)}|j,n,m)$.

We have the following relations:

$$(m|L_1+iL_2|m-1)=[(j+m)(j-m+1)]^{\frac{1}{2}},$$
$$(m|L_1-iL_2|m+1)=[(j+m-1)(j-m)]^{\frac{1}{2}}.$$

To satisfy the commutation relations [analogous to (38)] of $e_l^{(3)}$ with $\overset{0}{L}_{ik}$ we set

$$(j,m,n|e_3^{(3)}|j,m,n)=(j,n|C|j,n)m,$$
$$(j,m,n|e_1^{(3)}+ie_2^{(3)}|j,m-1,n)=(j,n|C|j,n)[(j+m)(j-m+1)]^{\frac{1}{2}},$$
$$(j,m,n|e_1^{(3)}-ie_2^{(3)}|j,m+1,n)=(j,n|C|j,n)[(j+m+1)(j-m)]^{\frac{1}{2}}.$$

Inserting this in the equation

$$-(\mathbf{L}\cdot\mathbf{e}^{(3)})=-\mathbf{L}^{(3)}=\mathbf{T}^{(3)}$$

we get

$$-(j,n|C|j,n)j(j+1)=n$$

or

$$-(j,m,n|e_3^{(3)}|j,m,n)=\frac{nm}{j(j+1)}.$$

Therefore, for a given isobar, we find that to the leading order

$$M_3=-\frac{e}{\kappa}\frac{g^2}{\kappa a}\frac{1}{12}\frac{nm}{j(j+1)}. \quad (125)$$

For the proton $n=m=j=\frac{1}{2}$, hence

$$M_3=\frac{e}{\kappa}\frac{g^2}{\kappa a}\frac{1}{36}. \tag{126}$$

For the neutron $n=-\frac{1}{2}$, hence M_3 simply changes sign, as was mentioned above.

If, for a first orientation, we identify this magnetic moment with the total empirical magnetic moment of the neutron—1.93 proton magnetons—and assume a meson mass equal to 1/10 proton mass ($e/\kappa=10$ proton magnetons), we find $g^2/\kappa a\sim 6.95$. Inserting the value of $\kappa a\sim 0.1$ determined from consideration of the scattering, we find $g^2\sim 0.695$, or $g/\kappa a\sim 8.3$, well within the range of the strong coupling approximation.

However, the fact that the proton and neutron magnetic moments are not of equal magnitude cannot be explained in any simple way in this theory. Of course, (126) does not give the total magnetic moment, but only the part due to the meson cloud. To it must be added the moment due to the nucleon itself.[23] If we assume that a "bare" proton or neutron would have a magnetic moment or one or zero proton magnetons, respectively, then we can compute the nucleon moment in the two lowest states by calculating the expectation value of

$$\mathbf{M}'=\frac{e\hbar}{m_pc}\frac{1}{2}(1+\tau_3)\boldsymbol{\sigma}. \tag{127}$$

In the weak coupling theory τ_3 is diagonal with the value $+\frac{1}{2}$ for the proton and $-\frac{1}{2}$ for the neutron, giving rise to the same values for the moment which were assumed for the bare core. However, in the strong coupling theory we have, after performing the S transformation,

$$\mathbf{M}'=-\frac{e}{m_pc}\frac{1}{2}\mathbf{e}^{(3)},$$

the expectation value of $\boldsymbol{\sigma}$ being zero. We compute the component of M' along the x_3 axis as before and get

$$M_3'=\frac{1}{2}\frac{nm}{j(j+1)}\frac{e}{m_pc}. \tag{128}$$

[23] We are indebted to Dr. J. Schwinger for valuable discussion of this question.

This moment, while small compared to the empirical moments of proton and neutron, is again equal in magnitude and opposite in sign for the two. For the two states $n=\pm\frac{1}{2}$, $m=j=\frac{1}{2}$

$$M_3'=\pm\tfrac{1}{6} \text{ proton magneton.}$$

APPENDIX

1. Introduction of Polar Angles

It is possible to express by means of Euler angles the orthogonal matrices (or, in other words, the three orthogonal-unit-vectors) introduced in Section 2. For instance, we can put

$$\begin{aligned}
A_{1k}&=\cos\theta\cos\varphi\cos\psi-\sin\varphi\sin\psi,\\
&\quad\cos\theta\sin\varphi\cos\psi+\cos\varphi\sin\psi,\quad -\sin\theta\cos\psi,\\
A_{2k}&=-\cos\theta\cos\varphi\sin\psi-\sin\varphi\cos\psi,\\
&\quad-\cos\theta\sin\varphi\sin\psi+\cos\varphi\cos\psi,\quad \sin\theta\sin\psi,\\
A_{3k}&=\sin\theta\cos\varphi,\quad \sin\theta\sin\varphi,\quad \cos\theta,
\end{aligned} \tag{1}$$

which means that the right side is the transposed matrix **A**. In the same way we put for the matrix

$$\begin{aligned}
B_{1\alpha}&=\cos\Theta\cos\Phi\cos\Psi-\sin\Phi\sin\Psi,\\
&\quad\cos\Theta\sin\Phi\cos\Psi+\cos\Phi\sin\Psi,\quad -\sin\Theta\cos\Psi,\\
B_{2\alpha}&=-\cos\Theta\cos\Phi\sin\Psi-\sin\Phi\cos\Psi,\\
&\quad-\cos\Theta\sin\Phi\sin\Psi+\cos\Phi\cos\Psi,\quad \sin\Theta\sin\Psi,\\
B_{3\alpha}&=\sin\Theta\cos\Phi,\quad \sin\Theta\sin\Phi,\quad \cos\Theta
\end{aligned} \tag{2}$$

and for the $e_{\alpha k}$ defined by (51)

$$\begin{aligned}
e_{1k}&=\cos a\cos b\cos c-\sin b\sin c,\\
&\quad\cos a\sin b\cos c+\cos b\sin c,\quad -\sin a\cos c,\\
e_{2k}&=-\cos a\cos b\sin c-\sin b\cos c,\\
&\quad-\cos a\sin b\sin c+\cos b\cos c,\quad \sin a\sin c,\\
e_{3k}&=\sin a\cos b,\quad \sin a\sin b,\quad \cos a.
\end{aligned} \tag{3}$$

Defining the operators p_θ, p_φ, p_ψ simply by $p_\theta=-i\partial/\partial\theta$, $p_\varphi=-i\partial/\partial\varphi$, $p_\psi=-i\partial/\partial\psi$, we find from the volume element $\sin\theta d\theta d\varphi d\psi$ that p_φ, p_ψ are Hermitian, but

$$p_\theta{}^\dagger=(\sin\theta)^{-1}p_\theta\sin\theta;$$

similar expressions are found for $p_\Theta{}^\dagger$, $p_a{}^\dagger$.

A direct evaluation of the angular momentum operators gives

$$\begin{aligned}
L_1\equiv L_{23}&=-\sin\varphi p_\theta+(\cos\varphi/\sin\theta)(p_\psi-\cos\theta p_\varphi)\\
&=-p_\theta{}^\dagger\sin\varphi+(p_\psi-p_\varphi\cos\theta)(\cos\varphi/\sin\theta),\\
L_2\equiv L_{31}&=\cos\varphi p_\theta+(\sin\varphi/\sin\theta)(p_\psi-\cos\theta p_\varphi)\\
&=p_\theta{}^\dagger\cos\varphi+(p_\psi-p_\varphi\cos\theta)(\sin\varphi/\sin\theta),\\
L_3\equiv L_{12}&=p_\varphi,
\end{aligned} \tag{4}$$

and also

$$\begin{aligned}
L_1\equiv L_{23}&=-\sin b p_a+(\cos b/\sin a)(p_c-\cos a p_b)\\
&=-p_a{}^\dagger\sin b+(p_c-p_b\cos a)(\cos b/\sin a),\\
L_2=L_{31}&=\cos b p_a+(\sin b/\sin a)(p_c-\cos a p_b)\\
&=p_a{}^\dagger\cos b+(p_c-p_b\cos a)(\sin b/\sin a),\\
L_3=L_{12}&=p_b.
\end{aligned} \tag{5}$$

Analogously for the isotopic spin

$$\begin{aligned}
T_1\equiv T_{23}&=-\sin\Phi p_\Theta+(\cos\Phi/\sin\Theta)(p_\Psi-\cos\Theta p_\Phi)\\
&=-p_\Theta{}^\dagger\sin\Phi+(p_\Psi-p_\Phi\cos\Theta)(\cos\Phi/\sin\Theta),
\end{aligned}$$

$$T_2 \equiv T_{31} = \cos\Phi p_\Theta + (\sin\Phi/\sin\Theta)(p_\Psi - \cos\Theta p_\Phi)$$
$$= p_\Theta^\dagger \cos\Phi + (p_\Psi - p_\Phi\cos\Theta)(\sin\Phi/\sin\Theta), \quad (6)$$
$$T_3 \equiv T_{12} = p_\Phi.$$

For the components L^{rs} and T^{rs} we find

$$L^1 \equiv L^{23} = \sin\psi p_\theta + (\cos\psi/\sin\theta)(\cos\theta p_\psi - p_\varphi)$$
$$= p_\theta^\dagger \sin\psi + (p_\psi\cos\theta - p_\varphi)(\cos\psi/\sin\theta),$$
$$L^2 \equiv L^{31} = \cos\psi p_\theta - (\sin\psi/\sin\theta)(\cos\theta p_\psi - p_\varphi) \quad (7)$$
$$= p_\theta^\dagger \cos\psi - (p_\psi\cos\theta - p_\varphi)(\sin\psi/\sin\theta),$$
$$L^3 \equiv L^{12} = p_\psi,$$

$$T^1 \equiv T^{23} = \sin\Psi p_\Theta + (\cos\Psi/\sin\Theta)(\cos\Theta p_\Psi - p_\Phi)$$
$$= p_\Theta^\dagger \sin\Psi + (p_\Psi\cos\Theta - p_\Phi)(\cos\Psi/\sin\Theta),$$
$$T^2 \equiv T^{31} = \cos\Psi p_\Theta - (\sin\Psi/\sin\Theta)(\cos\Theta p_\Psi - p_\Phi) \quad (8)$$
$$= p_\Theta^\dagger \cos\Psi - (p_\Psi\cos\Theta - p_\Phi)(\sin\Psi/\sin\Theta),$$
$$T^3 \equiv T^{12} = p_\Psi.$$

And, analogously for the components $L^{\alpha\beta}$ which we write $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ or $L^{(23)}$, $L^{(31)}$, $L^{(12)}$ to distinguish them from L^{rs}

$$L^{(1)} \equiv L^{(23)} = \sin c p_a + (\cos c/\sin a)(\cos a p_c - p_b)$$
$$= p_a^\dagger \sin c + (p_c\cos a - p_b)(\cos c/\sin a),$$
$$L^{(2)} \equiv L^{(31)} = \cos c p_a - (\sin c/\sin a)(\cos a p_c - p_b) \quad (9)$$
$$= p_a^\dagger \cos c - (p_c\cos a - p_b)(\sin c/\sin a),$$
$$L^{(3)} \equiv L^{(12)} = p_c.$$

We note that all components L^{rs}, T^{rs}, $L^{\alpha\beta}$ turn out to be Hermitian. For the total square of L and T we get

$$L^2 = p_\theta^\dagger p_\theta + \frac{(p_\psi - \cos\theta p_\varphi)^2}{\sin^2\theta} + p_\varphi^2$$
$$= p_\theta^\dagger p_\theta + \frac{(\cos\theta p_\psi - p_\varphi)^2}{\sin^2\theta} + p_\psi^2 \quad (10)$$

$$T^2 = p_\Theta^\dagger p_\Theta + \frac{(p_\Psi - \cos\Theta p_\Phi)^2}{\sin^2\Theta} + p_\Phi^2$$
$$= p_\Theta^\dagger p_\Theta + \frac{(\cos\Theta p_\Psi - p_\Phi)^2}{\sin^2\Theta} + p_\Psi^2 \quad (11)$$

and also

$$L^2 = p_a^\dagger p_a + \frac{(p_c - \cos a p_b)^2}{\sin^2 a} + p_b^2$$
$$= p_a^\dagger p_a + \frac{(\cos a p_c - p_b)^2}{\sin^2 a} + p_c^2. \quad (12)$$

The expressions (7), (8) can be substituted in the expression (50) for

$$\sum_{\alpha,k}^{0} (\pi_{\alpha k})^2$$

which, however, gets rather complicated even in the approximation where one puts $Q_r + Q_s = 2D$ because of the terms containing $Q_r - Q_s$. The analogous expression (50c) for the charged scalar theory is simpler, because here we have simply $T^{12} = T_{12} = p_\Phi$. In the approximation where we can put in the last two terms $Q_1 + Q_2 = 2D$, $Q_r = D$ we get

$$(\pi_1)^2 + (\pi_2)^2 = P_1^\dagger P_1 + P_2^\dagger P_2 + \frac{(p_c + p_\varphi)^2}{2(Q_1 - Q_2)^2} + \frac{(p_c - p_\varphi)^2}{8D^2}$$
$$+ \left\{\frac{(\cos a p_c - p_b)^2}{\sin^2 a} + p_a^\dagger p_a\right\} D^{-2}. \quad (13c)$$

2. The S Transformation

By using the expression (3) for the $e_{\alpha k}$ the S transformation defined by Eq. (65a) is simply obtained by putting[24]

$$S = \exp\left(i\sigma_3\frac{c}{2}\right)\exp\left(i\sigma_2\frac{a}{2}\right)\exp\left(i\sigma_3\frac{b}{2}\right);$$
$$S^{-1} = \exp\left(-i\sigma_3\frac{b}{2}\right)\exp\left(-i\sigma_2\frac{a}{2}\right)\exp\left(-i\sigma_3\frac{c}{2}\right) \quad (14)$$

which can also be written out as

$$S = \cos\frac{a}{2}\left(\cos\frac{b}{2}\cos\frac{c}{2} - \sin\frac{b}{2}\sin\frac{c}{2}\right)I$$
$$+ i\sin\frac{a}{2}\left(-\sin\frac{b}{2}\cos\frac{c}{2} + \cos\frac{b}{2}\sin\frac{c}{2}\right)\sigma_1$$
$$+ i\sin\frac{a}{2}\left(\cos\frac{b}{2}\cos\frac{c}{2} + \sin\frac{b}{2}\sin\frac{c}{2}\right)\sigma_2$$
$$+ i\cos\frac{a}{2}\left(\sin\frac{b}{2}\cos\frac{c}{2} + \cos\frac{b}{2}\sin\frac{c}{2}\right)\sigma_3. \quad (15)$$

The expression for S^{-1} is obtained from it by changing i into $-i$. One easily finds

$$Sp_aS^{-1} = p_a - \tfrac{1}{2}(\sigma_1\sin c + \sigma_2\cos c),$$
$$Sp_bS^{-1} = p_b - \tfrac{1}{2}(-\sigma_1\sin a\cos c + \sigma_2\sin a\sin c + \sigma_3\cos a), \quad (16)$$
$$Sp_cS^{-1} = p_c - \tfrac{1}{2}\sigma_3.$$

Inserting this in the expressions (5) and (9) of the last section, one easily checks the Eqs. (69), (69[1]) of the text, namely

$$S\mathbf{L}S^{-1} = \mathbf{L} - \tfrac{1}{2}S\boldsymbol{\sigma}S^{-1} = \mathbf{L} - \tfrac{1}{2}\textstyle\sum_\alpha \sigma_\alpha \mathbf{e}_\alpha,$$
$$SL^{\alpha\beta}S^{-1} = L^{\alpha\beta} - \tfrac{1}{2}\sigma_{\alpha\beta}.$$

[24] See W. Pauli, Helv. Phys. Acta 12, 147 (1939), Section 3, where the special case $c = 0$ is treated.