$\varphi_\alpha(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, quantum mechanics assigns the energy $E=\hbar\omega$ and the momentum $\mathbf{p}=\hbar\mathbf{k}$. Thus, corresponding to the ordinary waves, since in that case $\omega=ck$, the relation between the energy and momentum is

$$E^2=c^2p^2, \qquad (6.3)$$

which is the relativistic relation between the energy and momentum for a free particle of *zero* mass.

For the extraordinary waves, however, $\omega=c\tilde{k}$, which gives

$$\tilde{E}^2=(c\hbar\tilde{k})^2=c^2\tilde{p}^2+c^2\hbar^2/a^2. \qquad (6.4)$$

This is the relativistic relation between the energy and momentum of a free particle of a finite mass

$$m=\hbar/ac. \qquad (6.5)$$

Since we have assumed space to be free of electrified particles, we must suppose that the general electromagnetic field contains neutral particles, which I tentatively assume to be *neutrinos*. Then, m is the neutrino mass, and

$$a=\hbar/mc. \qquad (6.6)$$

Finally, we note that the second integral in Eq. (6.2) satisfies the differential equation

$$(1-a^2\Box)\psi=0, \qquad (6.7)$$

and is the de Broglie wave for a particle of mass given by Eq. (6.5). In the non-relativistic approximation Eq. (6.7) is the Schroedinger equation for a neutral particle.

I am grateful to Professor Samuel J. M. Allen for his friendly interest in this problem.

JULY 1 AND 15, 1942 PHYSICAL REVIEW VOLUME 62

The Vibration of Piezoelectric Plates

A. W. LAWSON
Randal Morgan Laboratory of Physics, University of Pennsylvania, Philadelphia, Pennsylvania

(Received April 24, 1942)

The characteristic frequencies of infinite piezoelectric plates vibrating between the grounded electrodes of a plane parallel condenser have been rigorously investigated. It is found that the frequencies depend on the piezoelectric constants as well as the elastic constants of the crystal. The effective elastic constants for a piezoelectric crystal do not in general satisfy the same symmetry relations as the true elastic constants. For odd harmonics, which are the only modes which can be excited by a uniform electric field, the strain does not vanish at the surface of the plate for a finite gap between the electrodes. Consequently, the frequencies of free vibration also depend on the separation of the electrodes, and the rigorous theory shows this dependence should not be linear as hitherto supposed. The effect of the gap decreases as the square of the harmonic number and hence the higher frequencies of vibration are not exactly harmonics of the fundamental.

THE object of this paper is to give a rigorous treatment of the free vibrations of an infinite piezoelectric plate vibrating between two grounded infinite electrodes. These frequencies are the resonant frequencies of the plate when driven by an alternating voltage on the electrodes. This theory is an excellent approximation for a finite plate whose thickness is small compared to its lateral dimensions. The general theory of non-piezoelectric plates has been given by Koga,[1] and special cases of piezoelectric plates have been discussed by Cady.[2] The general theory for piezoelectric plates is similar to that of Koga for ordinary plates but is complicated by the fact that it is necessary to solve Maxwell's equations simultaneously with the differential equations for the propagation of elastic waves. It will appear that Cady's particular solution is an excellent first approximation but is not self-consistent.

If, in an anisotropic substance, u is the displacement vector, θ the strain tensor, ϕ the

[1] I. Koga, Physics **3**, 70 (1932).

[2] W. G. Cady, Physics **7**, 237 (1936).

stress tensor, c the elastic tensor, E the electric field intensity, P the polarization, e the piezoelectric tensor, k the susceptibility tensor, and ρ the density, we have for our fundamental elastic wave equations

$$\sum_i \partial\varphi_{ij}/\partial x_i=\rho\ddot{u}_j, \quad j=1, 2, 3; \tag{1}$$

where

$$\varphi_{ij}=\sum_{\alpha\beta} c_{ij\alpha\beta}\theta_{\alpha\beta}-\sum_{\gamma} e_{ij\gamma}E_\gamma, \tag{2}$$

and

$$\begin{aligned}\theta_{\alpha\beta}&=\partial u_\alpha/\partial x_\beta, && \text{if} \quad \alpha=\beta;\\ \theta_{\alpha\beta}&=\partial u_\alpha/\partial x_\beta+\partial u_\beta/\partial x_\alpha, && \text{if} \quad \alpha\neq\beta.\end{aligned} \tag{3}$$

The use of this somewhat cumbersome notation is to ensure that the magnitudes of the elastic and piezoelectric constants will be the same as those employed by Voigt. Finally, we have

$$P_i=\sum_{\alpha\beta} e_{\alpha\beta i}\theta_{\alpha\beta}+\sum_{\gamma} k_{\gamma i}E_\gamma. \tag{4}$$

It is immediately evident from Eq. (4) that the polarization in the medium is not in general collinear with the electric field intensity.

Now, if we restrict our attention to the case of a plate oriented perpendicular to x_1, then we have

$$\partial/\partial x_2=\partial/\partial x_3=0. \tag{5}$$

In virtue of Eqs. (2), (3), and (5), we may write Eq. (1) as

$$\sum_\alpha\left\{\frac{\partial^2}{\partial x_1^2}(c_{1j\alpha1}u_\alpha)-e_{1j\alpha}\frac{\partial E_\alpha}{\partial x_1}\right\}=\rho\ddot{u}_j. \tag{6}$$

Moreover, we have from electromagnetic theory

$$c\nabla\times\mathbf{E}=-\dot{\mathbf{B}}, \tag{7}$$

where $\dot{\mathbf{B}}$ is the magnetic field intensity, and

$$\nabla\cdot\mathbf{D}=0, \tag{8}$$

where $\mathbf{D}$ is the electric induction vector. From Eqs. (5) and (8), we obtain

$$\frac{\partial E_1}{\partial x_1}+4\pi\sum_\alpha\left\{k_{\alpha1}\frac{\partial E_\alpha}{\partial x_1}+e_{\alpha11}\frac{\partial u_\alpha}{\partial x_1}\right\}=0, \tag{9}$$

where the quantities E_α are the components of the electric field along axes chosen perpendicular and parallel to the surface of the plate. Thus E_1 is the electric field intensity along x_1. Neglecting $\dot{\mathbf{B}}$ in Eq. (7), we have

$$\partial E_\alpha/\partial x_\beta-\partial E_\beta/\partial x_\alpha=0,\ \beta=\alpha-1;\ \alpha=1, 2, 3, \tag{10}$$

whence

$$\partial E_2/\partial x_1=\partial E_3/\partial x_1=0. \tag{11}$$

So Eq. (9) reduces to

$$K_{11}\frac{\partial E_1}{\partial x_1}+4\pi\sum_\alpha e_{\alpha11}\frac{\partial u_\alpha}{\partial x_1}=0, \tag{12}$$

where

$$K_{11}\equiv1+4\pi k_{11}, \tag{13}$$

which when substituted in Eq. (6) yields

$$\sum_\alpha\frac{\partial^2}{\partial x_1^2}\{c^*_{1j\alpha1}u_\alpha\}=\rho\ddot{u}_j, \tag{14}$$

where

$$c^*_{1j\alpha1}\equiv c_{1j\alpha1}+\frac{4\pi e_{1j1}e_{\alpha11}}{K_{11}}. \tag{15}$$

Taking the origin at the center of the plate, we shall now assume that

$$u_j^r=A_j^r\sin\frac{n\pi x_1}{s_r}\exp i\omega_r t, \quad n \text{ odd}; \tag{16}$$

and

$$u_j^r=A_j^r\cos\frac{n\pi x_1}{s_r}\exp i\omega_r t, \quad n \text{ even}. \tag{17}$$

We then have the secular system

$$\sum_\alpha c^*_{1j\alpha1}A_\alpha^r-\kappa_r^2A_j^r=0, \quad j=1, 2, 3; \tag{18}$$

where

$$\kappa_r^2=4\rho s_r^2f_r^2/n^2, \quad f_r=\omega_r/2\pi. \tag{19}$$

We define

$$\xi_r=\sum_j A_j^ru_j^r. \tag{20}$$

Then our differential equations become

$$\frac{\partial^2\xi_r}{\partial x_1^2}=\frac{\rho}{\kappa_r^2}\ddot{\xi}_r, \quad r=1, 2, 3; \tag{21}$$

where κ_1^2, κ_2^2, κ_3^2 are the roots of the cubic equation

$$\begin{vmatrix} c^*_{1111}-\kappa^2 & c^*_{1121} & c^*_{1131}\\ c^*_{1211} & c^*_{1221}-\kappa^2 & c^*_{1231}\\ c^*_{1311} & c^*_{1321} & c^*_{1331}-\kappa^2\end{vmatrix}=0. \tag{22}$$

Corresponding to Eqs. (16) and (17), we assume that

$$\xi_r=\xi_r^0\sin\frac{n\pi x_1}{s_r}\exp i\omega_r t, \quad n \text{ odd}; \tag{23}$$

and

$$\xi_r=\xi_r^0\cos\frac{n\pi x_1}{s_r}\exp i\omega_r t, \quad n \text{ even}. \tag{24}$$

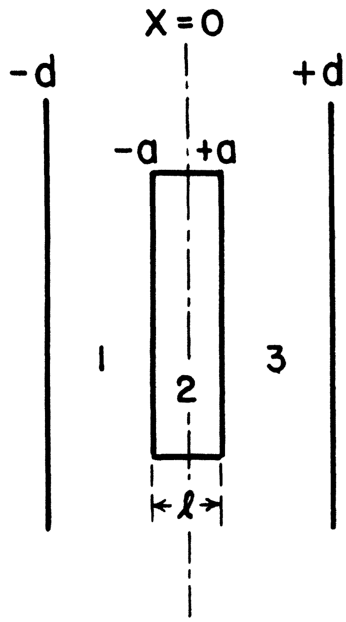

FIG. 1. Arrangement of crystal air gaps and electrodes.

Here we regard s_r as an unknown to be determined by the boundary conditions, which are at $x_1 = \pm a$.

$$X_x = X_z = X_y = 0, \tag{25}$$

$$\mathbf{n}\times(\mathbf{E}-\mathbf{E}^0)=0, \tag{26}$$

$$\mathbf{n}\cdot(\mathbf{D}-\mathbf{D}^0)=0, \tag{27}$$

when the origin is chosen at the center of the plate and $2a \equiv l$, the thickness of the plate.

The exact formulation of these conditions depends on the physical situation. In practice, the crystalline plate is inserted between two electrodes which are parallel to the surface of the plate. An alternating difference of potential between the two plates is used to excite resonant vibrations in the plate. We shall, however, assume that the plates are at zero potential and inquire into the characteristic frequencies of free vibration in this case. These are also the frequencies at which a driving voltage on the electrodes will produce resonance, but the expressions for the displacement are then considerably more complicated.[3] Since only the resonant frequencies are of practical interest, we shall not attempt to formulate the expressions for the displacements in the case of forced vibrations. Confining our attention to the case when the electrodes are grounded, we conclude by symmetry that the external electric field E^0 is entirely along the x_1 direction, that is, perpendicular to the surface of the plate, or

$$E^0{}_1 = E^0, \quad E^0{}_2 = E^0{}_3 = 0. \tag{28}$$

Consequently our boundary conditions at $x_1 = \pm a$ reduce to

$$K_{11}E_1 + 4\pi \textstyle\sum_\alpha e_{\alpha 11}(\partial u_\alpha/\partial x_1) = E^0, \tag{29}$$

$$\textstyle\sum_\alpha c_{1j\alpha 1}(\partial u_\alpha/\partial x_1) - e_{1j1}E_1 = 0, \quad j = 1, 2, 3; \tag{30}$$

and, solving these equations simultaneously, we obtain

$$\sum_\alpha c^*{}_{1j\alpha 1}\frac{\partial u_\alpha}{\partial x_1} = \frac{e_{1j1}}{K_{11}}E^0, \quad j = 1, 2, 3. \tag{31}$$

It remains to determine E^0 in terms of the quantities u_j. This requires the solution of Poisson's equation, which is most conveniently carried out in a manner suggested by Cady.

We treat the cases n odd and n even separately. We assume that in both cases, however, the configuration is that shown in Fig. 1, where $2d$ is the total gap between the electrodes in which the specimen is symmetrically inserted. Denoting by 1, 2, and 3 the regions occupied by the first air gap, the specimen, and the second air gap, respectively, we have, since

$$\nabla^2 V_q = -4\pi\rho_q(x_1), \quad q = 1, 2, 3, \tag{32}$$

the relation

$$V_q = -4\pi\left\{\int\left[\int \rho_q(x_1)dx_1\right]dx_1 + C_q x_1 + C_q'\right\}, \tag{33}$$

where V_q is the electrostatic potential and $\rho_q(x_1)$ is the charge density in the qth region. These latter quantities have the values:

$$\rho_1(x_1) = \rho_3(x_1) = 0, \tag{34}$$

$$\rho_2(x_1) = -(d/dx_1)P_2'(x_1), \tag{35}$$

where $P_2'(x_1)$ is that part of the polarization arising from the strain. Confusion may arise at this juncture because some authors consider the strain to produce free charge, while others consider the strain to produce polarization. It

[3] See, for example, P. M. Morse, *Vibration and Sound* (McGraw-Hill, 1936).

is clear that we have assumed the strain to produce polarization equivalent to that arising from a free charge distribution $\rho_2(x_1)$. While both points of view lead to the same result, confusion may arise due to the difference in definition of **P** and **D**. In order to proceed further and evaluate the constants C_q and C_q', it is necessary to know the space dependence of $P_2'(x_1)$. Here, and until further notice, we omit the index r, since it is understood that these equations obtain for each value of r.

For n odd, we have according to Eqs. (4) and (16)

$$P_2'(x_1)\equiv P_0\cos\frac{n\pi x_1}{s}=\frac{n}{s}\sum_\alpha e_{\alpha 11}A_\alpha\cos\frac{n\pi x_1}{s}. \quad (36)$$

Consequently, we obtain for the electrostatic potentials

$$V_1=-4\pi[C_1x_1+C_1'], \quad (37)$$

$$V_2=-\frac{4\pi}{K_{11}}\left[-\frac{P_0s}{n\pi}\sin\frac{n\pi x_1}{s}+C_2x_1+C_2'\right], \quad (38)$$

$$V_3=-4\pi[C_3x_1+C_3']. \quad (39)$$

We seek to evaluate the undetermined constants. By symmetry,

$$C_1=C_3. \quad (40)$$

Moreover, at $x=-d$:

$$V_1=-4\pi[-C_1d+C_1']=0, \quad (41)$$

so

$$C_1'=Cd, \quad (42)$$

and, at $x=d$:

$$V_3=-4\pi[C_3d+C_3']=0, \quad (43)$$

so

$$C_3'=-C_3d. \quad (44)$$

Consequently, we obtain

$$V_1=-4\pi C_1(x+d), \quad (45)$$

and

$$V_3=-4\pi C_1(x-d). \quad (46)$$

At $x=-a$, $V_1=V_2$, so

$$-4\pi[-a+d]C_1=-\frac{4\pi}{K_{11}}\left[-\frac{P_0s}{n\pi}\sin\frac{-n\pi a}{s}-C_2a+C_2'\right] \quad (47)$$

and $D_1=D_2$, so

$$-4\pi C_1+4\pi\left[-P_0\cos\frac{-n\pi a}{s}+C_2\right]=-4\pi P_0\cos\frac{-n\pi a}{s}, \quad (48)$$

whence immediately

$$C_1=C_2. \quad (49)$$

Similarly, at $x=a$, $V_2=V_3$, so

$$-4\pi C_1[a-d]=-\frac{4\pi}{K_{11}}\left[-\frac{P_0s}{n\pi}\sin\frac{n\pi a}{s}+C_2a+C_2'\right]. \quad (50)$$

Adding Eqs. (47) and (50) yields

$$C_2'=0. \quad (51)$$

Hence from Eq. (50) we obtain

$$-4\pi C_1[a-d]=-\frac{4\pi}{K_{11}}\left[-\frac{P_0s}{n\pi}\sin\frac{n\pi a}{s}+C_1a\right], \quad (52)$$

so

$$E^0=4\pi C_1=(8\pi P_0s/n\pi w)\sin(n\pi a/s), \quad (53)$$

where

$$w\equiv 2[a+K_{11}(d-a)]. \quad (54)$$

This differs from Cady's solution of the problem by a factor $\sin n\pi a/s$.

Inserting the value for the external electric field in the boundary conditions, we have at $x=\pm a$,

$$\sum_\alpha c^*_{1j\alpha 1}\frac{\partial u_\alpha{}^r}{\partial x_1}=\frac{8e_{1j1}}{wK_{11}}\sin\frac{n\pi a}{s_r}\sum_\alpha e_{\alpha 11}A_\alpha{}^r, \quad (55)$$

in which we have again inserted the index r. Multiplying each equation by $A_j{}^r$, adding and taking cognizance of Eq. (20), we have for the corresponding boundary conditions on ξ_r at $x=\pm a$:

$$\frac{n\pi\kappa_r{}^2}{s_r}\xi_r{}^0\cos\frac{n\pi a}{s_r}=\frac{8\pi}{wK_{11}}[\textstyle\sum_\alpha e_{\alpha 11}A_\alpha{}^r]^2\sin\frac{n\pi a}{s_r}, \quad (56)$$

or

$$\cos\frac{n\pi a}{s_r}=\frac{8s_re_r'^2}{nw\kappa_r{}^0K_{11}}\sin\frac{n\pi a}{s_r}, \quad (57)$$

where

$$e_r'^2\equiv[\textstyle\sum_\alpha e_{\alpha 11}A_\alpha{}^r]^2/[\sum_\alpha A_\alpha{}^r]^2. \quad (58)$$

In Eq. (58) it is not necessary to know the absolute value of the A_α^r to determine $e_r'^2$ but only the ratio $A_1^r : A_2^r : A_3^r$ which is determined by Eq. (18) for a given κ_r^2.

Since $\cos n\pi a/s_r = \cos -n\pi a/s_r$ it is apparent that our solution $\xi_r = \xi_r^0 \sin n\pi x_1/s_r \exp i\omega_r t$ will satisfy both boundary conditions provided s_r is the solution of the transcendental Eq. (57). This completes the formal solution of the problem for n odd.

It should be noted, however, that in general s_r is not equal to the thickness of the plate. Physically, this means that in virtue of the electric field arising from the charge distribution caused by the strain, neither the stress nor the strain vanish at the surface of the plate. This is an important qualitative difference between this solution and that of Cady, who assumed that there was a node of strain at the boundary of the plate and then calculated the external electric field and found it to be finite. Thus, his result is fundamentally inconsistent with his initial assumption, which, however, is an excellent approximation owing to the small difference between l and s_r. Indeed, the difference is so small that it is worthy of mention only for the sake of clarity of interpretation.

The foregoing state of affairs is somewhat altered when n is even. In this case we must write

$$V_1 = -4\pi(C_1x + C_1'), \tag{59}$$

$$V_2 = -(4\pi/K_{11})[-(P_0s/n\pi)\cos(n\pi x_1/s) + C_2x_1 + C_2'], \tag{60}$$

$$V_3 = -4\pi[C_3x + C_3']. \tag{61}$$

From $V=0$ at $x=\pm d$, we obtain

$$V_1 = -4\pi C_1(x_1+d), \tag{62}$$

and

$$V_3 = -4\pi C_3(x_1-d). \tag{63}$$

From continuity of potential we have, at $x=+a$,

$$C_3[-4\pi(a-d)] = -(4\pi/K_{11})[-(P_0s/n\pi)\cos(n\pi a/s) + C_2a + C_2'], \tag{64}$$

and, at $x=-a$

$$C_1[-4\pi(-a+d)] = -(4\pi/K_{11})[-(P_0s/n\pi)\cos(-n\pi a/s) - C_2a + C_2']. \tag{65}$$

From continuity of electric displacement, we have, at $x=+a$,

$$-4\pi C_3 + 4\pi[P_0\sin(n\pi a/s) + C_2] = 4\pi P_0\sin(n\pi a/s), \tag{66}$$

and, at $x=-a$

$$-4\pi C_1 + 4\pi[P_0\sin(-n\pi a/s) + C_2] = 4\pi P_0\sin(-n\pi a/s), \tag{67}$$

whence $C_1=C_2=C_3$. Adding Eqs. (64) and (65) we find

$$C_2' = (P_0s/n\pi)\cos(n\pi a/s). \tag{68}$$

Subtracting the same equations we find

$$C_1 = C_2 = C_3 = 0. \tag{69}$$

So, for even harmonics, we conclude that the external field vanishes. Consequently,

$$\xi_r = \xi_r^0 \cos n\pi x_1/l,$$

where l is the thickness of the plate. The faces of the plate are nodes of displacement for the even harmonics and consequently cannot be excited by a uniform electric field applied perpendicular to the plate.

Now by returning to Eq. (57) and realizing from a physical standpoint that $s_r \cong l$, we are in a position to obtain a sufficiently accurate, explicit approximation for the value of s_r and hence of f_r, the frequency of free vibration when n is odd.

First, we demonstrate that $s_r > l$ for n odd. In Eq. (57), if $s_r \cong l$, we must have $\sin n\pi a/s_r$ and $\cos n\pi a/s_r$ with the same sign, since $8e_r'^2 s_r/\kappa_r^2 nwK_{11}$ is essentially positive. This is only possible if $s_r > 2a = l$.

Next we show that Cady's result for an X-cut crystal vibrating in its fundamental thickness mode is the quantitatively correct first approximation. Now Cady defined a quantity

$$c_r' = 4\rho f_r^2 l^2, \tag{70}$$

and found

$$c_{11}' = c_{11}^* - (32e_{11}^2 l/\pi wK_{11}), \quad n=1. \tag{71}$$

To achieve this result we return to Eq. (57) and shift the origin from $x_1=0$ to $x_1=-\frac{1}{2}s$ and thus obtain

$$\tan \Delta l_r n\pi/s_r = 8s_r e_r'^2/c_r^* nwK_{11}, \tag{72}$$

where

$$2\Delta l_r \equiv s_r - l, \tag{73}$$

and

$$c_r^* \equiv \kappa_r^2. \qquad (74)$$

From Eq. (72), we obtain as the first approximation for Δl_r:

$$\Delta l_r = 8e_r'^2 l^2 / n^2 \pi w K_{11} c_r^*. \qquad (75)$$

From Eqs. (19) and (73) we have

$$4\rho^2 (f_r^2/n^2)(l^2 + 4\Delta l_r l) = c_r^*, \qquad (76)$$

which with Eq. (75) yields

$$4\rho \frac{f_r^2}{n^2} l^2 \left(1 + \frac{32 e_r'^2 l}{\pi w K_{11} c_r^* n^2}\right) = c_r^*, \qquad (77)$$

or

$$c_r' = c_r^* - 32 e_r'^2 l / \pi n w K_{11}. \qquad (78)$$

For an X-cut crystal this expression reduces to that of Cady. It is further evident that the effect of the gap on the frequency vanishes when the gap is infinite, and decreases as the square of the harmonic number for a finite gap.

Finally, this theory should be capable of explaining the observed dependence of frequency on gap. According to Cady, the frequency should be related to the reciprocal of the equivalent electrical gap w by a linear relation. Actually, Dye[4] and Cady observed a slight curvature to the relation. The second approximation for Δl_r should yield the sign of this curvature. We may write

$$f_r/n = v_r/2s_r = y_r \tfrac{1}{2} v_r, \qquad (79)$$

where

$$v_r \equiv (\kappa_r^2/\rho)^{\frac{1}{2}}, \qquad (80)$$

and

$$y_r \equiv 1/s_r. \qquad (81)$$

Then the second approximation for s_r yields

$$l y_r^2 - y_r + D_r = 0, \qquad (82)$$

where

$$D_r \equiv 16 e_r'^2 / \kappa_r^2 n^2 w K_{11} \pi, \qquad (83)$$

so to the second approximation

$$y_r = \frac{1}{l} - D_r - l D_r^2. \qquad (84)$$

If Δf_r is the increment in the frequency as the gap is decreased from an infinite value, for which the frequency is f_r^0, we have

$$\Delta f_r / f_r^0 = -(l D_r + l^2 D_r^2). \qquad (85)$$

The frequency therefore drops more rapidly than linearly. This is qualitatively in agreement with experiment, although experiment would indicate that this drop is somewhat more rapid than the theory predicts. This quantitative discrepancy is possibly due to the lack of a uniform field and the fact that the experiments were conducted on the fundamental mode of vibration where the edge effects are more important than for higher modes. This is because the frequencies of vibration of a finite plate will only be equal to those of an infinite plate as the wave-length of the sound waves approaches zero.

In summary, we state the main result of our considerations. The frequencies of free vibration of a piezoelectric crystal in the form of an infinite plate inserted between two grounded electrodes are given by

$$f_r = \frac{n}{2s_r}\left(\frac{c_r^*}{\rho}\right)^{\frac{1}{2}},$$

where c_r^* are the roots of the secular determinant (22) and s_r is the root of the transcendental equation $\cot n\pi a/s_r = 8 s_r e_r'^2 / n w c_r^* K_{11}$, for n odd; and $s_r = l$, for n even where the various quantities have the connotations ascribed in the text.

It has been pointed out elsewhere[5] that this theory is capable of explaining the discrepancies in the elastic constants of quartz calculated by Atanasoff and Hart[6] on the basis of a somewhat less rigorous theory. The agreement between this theory and the careful results of Atanasoff and Hart is very satisfactory. It is worthy of emphasis that the effective elastic constants of a piezoelectric plate, the so-called c^*, do not obey the same symmetry relations as those obeyed by the ordinary elastic constants.

The author is indebted to Professor W. G. Cady and Professor J. V. Atanasoff for stimulating conversations and correspondence, and considerable constructive criticism.

[4] D. W. Dye, Proc. Phys. Soc. London **38**, 399 (1926).

[5] A. W. Lawson, Phys. Rev. **59**, 838 (1941).

[6] J. V. Atanasoff and P. J. Hart, Phys. Rev. **59**, 85 (1941).