A Generalized Electrodynamics

Part I—Non-Quantum

BORIS PODOLSKY
Department of Physics, University of Cincinnati, Cincinnati, Ohio
(Received March 23, 1942)

If one wishes to derive generalized field equations from a Lagrangian, at the same time preserving the linear character of the equations, one must admit terms involving derivatives of the field quantities. It turns out that the only non-trivial generalization of this kind, leading to differential equations of order below eighth, is obtained by taking $L_f=(1/8\pi)\{\frac{1}{2}F_{\alpha\beta}^2+a^2(\partial F_{\alpha\beta}/\partial x_\beta)^2\}$. This leads to a theory that contains the Landé-Thomas theory and accounts for the choice of sign required when one wishes to consider the total field as consisting of the Maxwell-Lorentz and the Yukawa fields.

1. INTRODUCTION

If one assumes that the equations of electrodynamics are derivable from some Lagrangian L, and wishes to preserve the linear character of the field equations (The Principle of Superposition) in order to make the quantization easy, then, unless one is prepared to introduce new kinds of field quantities, the only way of generalizing the Maxwell-Lorentz theory appears to be by permitting the Lagrangian of the field to contain terms involving derivatives of the field quantities $\mathbf{E}$ and $\mathbf{H}$.

One then obtains, as the field equations, partial differential equations of an order higher than the usual second. Far from being objectionable, this appears to be what is needed. For, the various proposed methods of "cutting off" effects of higher frequencies seems to indicate clearly that the higher derivatives, which become important for higher frequencies, are not properly taken care of by the usual second-order equations. Further, the extra freedom of choice of a solution to be used in any particular problem, provided by equations of higher order, permits of an imposition of finiteness conditions, analogous to Schroedinger's procedure, which serves also to remove infinities inherent in the usual treatment of point charges.

2. NON-RELATIVISTIC CASE

The usual Lagrangian of the field in this case, in electrostatic units, is:

$$\bar{L}_f \equiv \int L_f dV = \frac{1}{8\pi}\int \mathbf{E}^2 dV. \qquad (2.1)$$

To preserve the linearity of the field equations the additional terms have to be quadratic in $\mathbf{E}$ and its derivatives. If we limit ourselves to field equations of an order not higher than *sixth*, the highest derivative of $\mathbf{E}$ that may occur is second. Investigating all possible combinations of the operator ∇ and $\mathbf{E}$ satisfying these requirements, one finds that all such combinations either vanish identically, by virtue of the condition $\mathbf{E}=-\nabla\varphi$ (the result of preserving unchanged the term in the total Lagrangian representing the interaction of the field and particles), or differ by a divergence from $(\nabla\cdot\mathbf{E})^2$. Since addition of a divergence to L_f does not alter the field equations, we may take, as the only generalization giving anything new,

$$L_f=(1/8\pi)[\mathbf{E}^2\pm a^2(\nabla\cdot\mathbf{E})^2]. \qquad (2.2)$$

The constant a thus introduced has the dimension of length, but otherwise remains arbitrary, as does also the sign of the whole additional term.

The field equations are now:

$$(1\mp a^2\nabla^2)\nabla\cdot\mathbf{E}=4\pi\rho \quad \text{and} \quad \nabla\times\mathbf{E}=\mathbf{0}. \qquad (2.3)$$

Although both choices of the sign admit of solutions without infinities, I am inclined to the belief, based on the study of the types of waves occurring in the corresponding relativistic generalization, that eventually only the upper sign will turn out to give physically significant results. The following investigation is therefore based on the tentative assumption that the upper sign is to be used in Eqs. (2.2) and (2.3).

The generalized Poisson equation is now:

$$(1-a^2\nabla^2)\nabla^2\varphi=-4\pi\rho; \qquad (2.4)$$

and if we put

$$\rho = e\delta(\mathbf{r}), \tag{2.5}$$

which corresponds to a point charge e located at the origin, the only solution of Eq. (2.4), finite everywhere and vanishing at infinity, is:

$$\varphi = (e/r)(1-e^{-r/a}). \tag{2.6}$$

This result is of the same form as the electrostatic potential obtained by Landé and Thomas.[1] However, we are here not limited to values of a consistent with their special assumption of meson involvement in the electronic interaction. In fact, later considerations seem to suggest that

$$a = \hbar/mc, \tag{2.7}$$

which, on the Landé-Thomas theory, would correspond to the meson mass being equal to m (instead of 2×137 m). This would mean that the meson here is a neutrino. However, no such interpretation is here necessary, nor apparently desirable. In Section 4 we shall consider more fully the relation of the present theory to that of Landé and Thomas.

3. RELATIVISTIC EQUATIONS

Using $x_4 = ict$ and $-ds^2 = dx_\alpha{}^2$, with the usual summation convention, we need not distinguish between covariant and contravariant tensors. Letting

$$F_{\alpha\beta} = -F_{\beta\alpha} = (\partial\varphi_\beta/\partial x_\alpha) - (\partial\varphi_\alpha/\partial x_\beta) \tag{3.1}$$

with

$$\varphi_\alpha = (\mathbf{A}, i\varphi), \tag{3.2}$$

where $\mathbf{A}$ is the vector potential and φ the scalar potential, we have: $F_{12} = H_3$, $F_{23} = H_1$, $\cdots F_{14} = -iE_1$, $\cdots$ etc.; and, as usual,

$$\mathbf{E} = -\nabla\varphi - (1/c)\dot{\mathbf{A}} \quad\text{and}\quad \mathbf{H} = \nabla\times\mathbf{A}. \tag{3.3}$$

One set of the field equations is then the usual

$$\partial F_{\alpha\beta}/\partial x_\gamma + \partial F_{\beta\gamma}/\partial x_\alpha + \partial F_{\gamma\alpha}/\partial x_\beta = 0, \tag{3.4}$$

or

$$\nabla\times\mathbf{E} + (1/c)\dot{\mathbf{H}} = \mathbf{0} \quad\text{and}\quad \nabla\cdot\mathbf{H} = 0. \tag{3.5}$$

Equations of motion of a particle are, in the usual way,

$$(d^2x_\alpha/ds^2) = (e/mc^2)F_{\alpha\beta}(dx_\beta/ds), \tag{3.6}$$

or

$$\frac{d}{dt}\left[\frac{m\mathbf{v}}{(1-v^2/c^2)^{\frac{1}{2}}}\right] = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right). \tag{3.7}$$

[1] A. Landé and L. H. Thomas, Phys. Rev. **60**, 514 (1941).

The only independent relativistically invariant generalization of Eq. (2.2) is found to be[2]

$$L_f = \frac{1}{8\pi}\left\{\frac{1}{2}F_{\alpha\beta}{}^2 + a^2\left(\frac{\partial F_{\alpha\beta}}{\partial x_\beta}\right)^2\right\} \tag{3.8}$$

$$= \frac{1}{8\pi}\left\{\mathbf{E}^2 - \mathbf{H}^2 + a^2\left[(\nabla\cdot\mathbf{E})^2 - \left(\nabla\times\mathbf{H} - \frac{1}{c}\dot{\mathbf{E}}\right)^2\right]\right\}. \tag{3.9}$$

The resulting field equations are:

$$\left(1 - a^2\frac{\partial^2}{\partial x_\alpha{}^2}\right)\frac{\partial F_{\beta\gamma}}{\partial x_\gamma} = 4\pi j_\beta, \tag{3.10}$$

or

$$(1-a^2\Box)\nabla\cdot\mathbf{E} = 4\pi\rho$$

and

$$(1-a^2\Box)\left(\nabla\times\mathbf{H} - \frac{1}{c}\dot{\mathbf{E}}\right) = 4\pi\rho\mathbf{v}/c, \tag{3.11}$$

where the four-vector j_α is defined by

$$j_\alpha = (\rho\mathbf{v}/c, i\rho), \tag{3.12}$$

with

$$\rho(\mathbf{r}) = \textstyle\sum_s e_s\delta(\mathbf{r} - \mathbf{r}_s). \tag{3.13}$$

The summation here is over all the particles; the sth particle having the charge e_s and the position $\mathbf{r}_s = \mathbf{r}_s(t)$.

With the usual restriction on the potentials,

$$(\partial\varphi_\alpha/\partial x_\alpha) \equiv \nabla\cdot\mathbf{A} + (1/c)\dot{\varphi} = 0, \tag{3.14}$$

we obtain:

$$(1-a^2\Box)\Box\varphi_\alpha = -4\pi j_\alpha, \tag{3.15}$$

a set of fourth-order partial differential equations.

4. RELATION TO THE LANDÉ-THOMAS THEORY

Although the present generalization consists merely of an addition of a comparatively simple term to the Lagrangian of the field, see Eq. (3.8), it contains the Landé-Thomas theory as a special case, and shows in a way why their particular way of combining Maxwell-Lorentz and Yukawa fields should work.

For, let

$$(1-a^2\Box)\varphi_\alpha \equiv \varphi_\alpha{}''; \tag{4.1}$$

[2] A possible addition of a term $b\mathbf{E}\cdot\mathbf{H}$ is trivial, since, by virtue of Eq. (3.4), $\mathbf{E}\cdot\mathbf{H} = \partial J_\alpha/\partial x_\alpha$, a four-dimensional divergence of the four-vector $J_\alpha = (i/4)\epsilon_{\alpha\beta\gamma\delta}F_{\beta\gamma}\varphi_\delta$, and does not affect the resulting field equations.

then Eq. (3.15) becomes

$$\Box\varphi_\alpha''=-4\pi j_\alpha, \quad (4.2)$$

so that φ_α'' is the usual Maxwell-Lorentz potential. If we now put

$$\varphi_\alpha\equiv\varphi_\alpha''-\varphi_\alpha', \quad (4.3)$$

substitution into Eq. (4.1), with the use of Eq. (4.2), gives

$$(1-a^2\Box)\varphi_\alpha'=4\pi a^2 j_\alpha. \quad (4.4)$$

Equations (4.2) and (4.4), with a suitable choice of a, are just the Landé-Thomas equations for the Maxwell-Lorentz and the Yukawa parts of the field, respectively. The minus sign in Eq. (4.3), which Landé and Thomas found necessary to introduce *ad hoc*, is here required to give Eq. (4.4). In other words, *if* we wish to consider φ_α as the sum of the Maxwell-Lorentz and Yukawa fields, these *must* be combined by subtracting the latter from the former. The same kind of analysis applies also to the energies. The choice of the opposite sign in Eq. (2.2) would lead to solutions of entirely different form, and has nothing to do with the present question.

5. ENERGY-MOMENTUM TENSOR

The derivation of the energy-momentum tensor is closely related to the problem of finding the Hamiltonian corresponding to the Lagrangian (3.8), which is necessary for the quantization of the field. I therefore reserve it for Part II of this report, giving here merely the result:

$$\begin{aligned}4\pi icT_{\mu\nu}=&F_{\mu\alpha}F_{\nu\alpha}-\tfrac{1}{4}F_{\alpha\beta}F_{\alpha\beta}\delta_{\mu\nu}\\&+(a^2/2)[F_{\alpha\beta}\Box F_{\alpha\beta}+(\partial F_{\alpha\beta}/\partial x_\beta)(\partial F_{\alpha\gamma}/\partial x_\gamma)]\delta_{\mu\nu}\\&-a^2[F_{\mu\alpha}\Box F_{\nu\alpha}+F_{\nu\alpha}\Box F_{\mu\alpha}\\&\qquad+(\partial F_{\mu\alpha}/\partial x_\alpha)(\partial F_{\nu\beta}/\partial x_\beta)]. \quad (5.1)\end{aligned}$$

This leads to the expression for the energy

$$\begin{aligned}E=&\frac{c}{i}\int T_{44}dV=\frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{H}^2)dV\\&-\frac{a^2}{8\pi}\int\left\{(\nabla\cdot\mathbf{E})^2+\left(\nabla\times\mathbf{H}-\frac{1}{c}\dot{\mathbf{E}}\right)^2\right.\\&\left.+2(\mathbf{H}\cdot\Box\mathbf{H}+\mathbf{E}\cdot\Box\mathbf{E})\right\}dV. \quad (5.2)\end{aligned}$$

In electrostatics this reduces to

$$E=\frac{1}{8\pi}\int\{\mathbf{E}^2-a^2[(\nabla\cdot\mathbf{E})^2+2\mathbf{E}\cdot\nabla^2\mathbf{E}]\}dV. \quad (5.3)$$

Making use of $\nabla\times\mathbf{E}=\mathbf{0}$, and assuming that $\mathbf{E}\nabla\cdot\mathbf{E}$ vanishes at infinity faster than $1/r^2$, one finds that Eq. (5.3) can easily be put in the form

$$E=\frac{1}{8\pi}\int\{\mathbf{E}^2+a^2(\nabla\cdot\mathbf{E})^2\}dV, \quad (5.4)$$

which is obviously positive. For the field of a point charge given by Eq. (2.2) this turns out to be $e^2/2a$.

6. WAVES

In the absence of *electrified matter* Eq. (3.15) becomes

$$(1-a^2\Box)\Box\varphi_\alpha(\mathbf{r},t)=0. \quad (6.1)$$

When we assume $\varphi_\alpha(\mathbf{r},t)$ to be real, the general solution of this equation is:

$$\begin{aligned}\varphi_\alpha(\mathbf{r},t)=&\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\{\varphi_\alpha(\mathbf{k})\exp i(\mathbf{k}\cdot\mathbf{r}-ckt)\\&+\varphi_\alpha^*(\mathbf{k})\exp -i(\mathbf{k}\cdot\mathbf{r}-ckt)\}dk\\&+\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\{\tilde{\varphi}_\alpha(\mathbf{k})\exp i(\mathbf{k}\cdot\mathbf{r}-c\tilde{k}t)\\&+\tilde{\varphi}_\alpha^*(\mathbf{k})\exp -i(\mathbf{k}\cdot\mathbf{r}-c\tilde{k}t)\}dk; \quad (6.2)\end{aligned}$$

where $\mathbf{k}$ is the wave vector, the direction of which is the direction of propagation of the component plane wave and the magnitude of which is $k=2\pi/\lambda$, λ being the wave-length; $dk=dk_xdk_ydk_z$; $\tilde{k}=(1+a^2k^2)^{\frac{1}{2}}/a$; $\varphi_\alpha(\mathbf{k})$ and $\tilde{\varphi}_\alpha(\mathbf{k})$ are two arbitrary independent functions of $\mathbf{k}$; and the asterisk denotes the complex conjugate of the quantity to which it is attached. If the reality assumption is omitted, the asterisk then denotes merely another independent function of $\mathbf{k}$.

The first integral in Eq. (6.2) is the general solution of the ordinary wave equation, which was previously used and found convenient.[3] I shall refer to waves thus represented as the *ordinary waves*. The second integral gives the *extraordinary waves*, and are distinguished by a *tilde* (˜).

If a wave is considered as being associated with a particle, then, corresponding to a term

[3] V. Fock and B. Podolsky, Physik. Zeits. der Sowjetunion 1, 801 (1932).

$\varphi_\alpha(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, quantum mechanics assigns the energy $E=\hbar\omega$ and the momentum $\mathbf{p}=\hbar\mathbf{k}$. Thus, corresponding to the ordinary waves, since in that case $\omega=ck$, the relation between the energy and momentum is

$$E^2=c^2p^2, \tag{6.3}$$

which is the relativistic relation between the energy and momentum for a free particle of *zero* mass.

For the extraordinary waves, however, $\omega=c\tilde{k}$, which gives

$$\tilde{E}^2=(c\hbar\tilde{k})^2=c^2\tilde{p}^2+c^2\hbar^2/a^2. \tag{6.4}$$

This is the relativistic relation between the energy and momentum of a free particle of a finite mass

$$m=\hbar/ac. \tag{6.5}$$

Since we have assumed space to be free of electrified particles, we must suppose that the general electromagnetic field contains neutral particles, which I tentatively assume to be *neutrinos*. Then, m is the neutrino mass, and

$$a=\hbar/mc. \tag{6.6}$$

Finally, we note that the second integral in Eq. (6.2) satisfies the differential equation

$$(1-a^2\Box)\psi=0, \tag{6.7}$$

and is the de Broglie wave for a particle of mass given by Eq. (6.5). In the non-relativistic approximation Eq. (6.7) is the Schroedinger equation for a neutral particle.

I am grateful to Professor Samuel J. M. Allen for his friendly interest in this problem.

JULY 1 AND 15, 1942 PHYSICAL REVIEW VOLUME 62

The Vibration of Piezoelectric Plates

A. W. Lawson
Randal Morgan Laboratory of Physics, University of Pennsylvania, Philadelphia, Pennsylvania
(Received April 24, 1942)

The characteristic frequencies of infinite piezoelectric plates vibrating between the grounded electrodes of a plane parallel condenser have been rigorously investigated. It is found that the frequencies depend on the piezoelectric constants as well as the elastic constants of the crystal. The effective elastic constants for a piezoelectric crystal do not in general satisfy the same symmetry relations as the true elastic constants. For odd harmonics, which are the only modes which can be excited by a uniform electric field, the strain does not vanish at the surface of the plate for a finite gap between the electrodes. Consequently, the frequencies of free vibration also depend on the separation of the electrodes, and the rigorous theory shows this dependence should not be linear as hitherto supposed. The effect of the gap decreases as the square of the harmonic number and hence the higher frequencies of vibration are not exactly harmonics of the fundamental.

THE object of this paper is to give a rigorous treatment of the free vibrations of an infinite piezoelectric plate vibrating between two grounded infinite electrodes. These frequencies are the resonant frequencies of the plate when driven by an alternating voltage on the electrodes. This theory is an excellent approximation for a finite plate whose thickness is small compared to its lateral dimensions. The general theory of non-piezoelectric plates has been given by Koga,[1] and special cases of piezoelectric plates have been discussed by Cady.[2] The general theory for piezoelectric plates is similar to that of Koga for ordinary plates but is complicated by the fact that it is necessary to solve Maxwell's equations simultaneously with the differential equations for the propagation of elastic waves. It will appear that Cady's particular solution is an excellent first approximation but is not self-consistent.

If, in an anisotropic substance, u is the displacement vector, θ the strain tensor, ϕ the

[1] I. Koga, Physics **3**, 70 (1932).

[2] W. G. Cady, Physics **7**, 237 (1936).