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On the Theory of Cosmic-Ray Showers

II. Further Contributions to the Fluctuation Problem

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Further calculations are reported on the problem of the distribution in size of cosmic-ray showers. In §2 it is shown that this distribution is completely determined when the average energy distribution of the particles in a shower is known. In §3 previous calculations of the fluctuation in size of showers have been revised and extended. The main result is that for the simplified model chosen (the so-called Furry model) the fluctuations are small and roughly equal to twice the Poisson value for all values of the thickness (see Table II; Fig. 3). In §4 another simplified model is considered for which it is possible to take the ionization exactly into account. In §5 and §6 the calculations of the fluctuation are extended to the actual cosmic-ray problem. For one value of the initial energy and for one depth a numerical calculation has been made (see Table III). The result for the fluctuation is again a few times the Poisson value.

§1. INTRODUCTION

THE problem of the distribution in size of cosmic-ray showers has been treated by several authors but no satisfactory solution has yet been given. The question is to determine the probability $P(E_0, N, x)$ that N particles emerge from a layer of matter of thickness x, when an electron of energy E_0 falls normally upon it. Furry[1] succeeded in solving the problem for a special model in which the essential approximation consisted of the neglect of the ionization. He found:

$$P(N, x) = (1/\bar{N})[1-(1/\bar{N})]^{N-1}, \quad (1)$$

where $\bar{N}$ is the average number of particles, which in this case is equal to $\exp(Bx)$ and is independent of E_0. The quantity B is a material constant. On the other hand Bhabha and Heitler[2] have asserted that the function $P(N, x)$ will be essentially the Poisson distribution:

$$P(N, x) = e^{-\bar{N}}(\bar{N})^N/N! \quad (2)$$

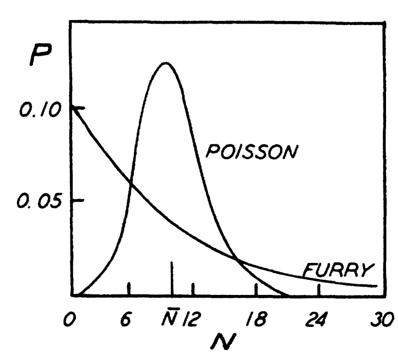

FIG. 1. The Furry and Poisson distributions (1) and (2) for $\bar{N} = 10$.

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[1] W. H. Furry, Phys. Rev. **52**, 569 (1937). See also B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

[2] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. **159**, 432 (1937). Compare also: H. Euler, Zeits. f. Physik **110**, 450 (1938), N. Arley, Proc. Roy. Soc. **168**, 519 (1938).

This may be obtained by assuming no correlation between the different particles in a shower. The striking difference between (1) and (2) can be seen from Fig. 1. In the actual cosmic-ray case one has to take into account both the correlation between different particles and the ionization. It seems likely that the result will lie somewhere between (1) and (2).

In order to estimate which one of the two results is more nearly correct, especially when one varies E_0 and x, an attempt was made in a previous paper[3] to calculate the fluctuation $\sigma=\langle N^2\rangle_{\text{Av}}-\langle N\rangle_{\text{Av}}^2$ more exactly. The same simplified model, as chosen by Furry, was considered, except that the ionization was now taken approximately into account by the so-called cut-off method (see I, p. 350). The results were found to lie between the values which follow from (1) and (2), namely:

$$\langle N^2\rangle_{\text{Av}}-\langle N\rangle_{\text{Av}}^2=\bar{N}(\bar{N}-1) \tag{1a}$$

for the Furry distribution[4] and:

$$\langle N^2\rangle_{\text{Av}}-\langle N\rangle_{\text{Av}}^2=\bar{N} \tag{2a}$$

for the Poisson distribution. It seemed to us necessary to refine and to extend these calculations, since no estimates of error were made in I and since the dependence of the fluctuation on the thickness x remained unclear. The results are given in §3; in contrast to the values found in I, p. 358, we now find that the fluctuation is much smaller than the Furry value (1a), and that it is roughly twice the Poisson value (2a) for the interesting range of values of x (see Fig. 3). The reason for this discrepancy was traced to the fact that in I the calculation of the integrals by the method of steepest descent was not made with sufficient accuracy. The actual behavior of the fluctuation as a function of x is quite curious and is difficult to explain in a qualitative way. In §4 we have therefore considered another simplified model, which has much less similarity with the cosmic-ray problem, but for which the influence of the ionization can be taken exactly into account. For this model one can show easily that *without* ionization the fluctuation is $\bar{N}-1$, or practically the Poisson value. However, here again the influence of the ionization on the fluctuation (which can be computed exactly) is strange, and difficult to explain qualitatively.

In §5 and §6 we have extended the calculations to the actual cosmic-ray equations. Again we have taken the ionization into account by means of the cut-off method. As was to be expected, the results for the fluctuation are quite similar to those for the Furry model.

Before going into the details of these calculations we shall show in §2 that the function $P(E_0, N, x)$ is completely determined if one knows the average energy distribution $F(E_0, E, x)$ of the particles in a shower. For the Furry model we shall write down a formal expression for the connection between these two functions. In the case of no ionization this leads again to (1) with $\bar{N}=\exp(Bx)$. We have, however, been unable to use this connection when the ionization is taken into account, so that the problem of the actual shape of the distribution function $P(N, x)$ remains unsolved.

§2. A FORMAL EXPRESSION FOR $P(N, x)$

As explained in I §3, all statistical questions regarding the formation of showers can be answered if one knows the probability of a given energy distribution of the particles which emerge after the thickness x. Let us assume first that the energy of any particle can only have the discrete values $\epsilon_1, \epsilon_2, \epsilon_3\cdots$. An energy distribution is then specified by giving the numbers $n_1, n_2, n_3\cdots$ which have these energy values. The function we seek is $W(n_1, n_2\cdots; x)$, the probability that at depth x we have the distribution n_i. We shall call it the "master function." Every other statistical function may be obtained from the master function by taking suitable average values. For instance, the average number of particles of energy ϵ_i is given by:

$$\bar{n}_i(x)=Sn_iW(n_1, n_2\cdots; x), \tag{3}$$

where the round summation signs will always mean a sum over all possible values that each n_i may have. Analogously one can form the quad-

[3] A. Nordsieck, W. E. Lamb, Jr., and G. E. Uhlenbeck, Physica **7**, 344 (1940). In the following this paper will be quoted as I and we shall use the same notations as much as possible.

[4] In the comparison the value of $\bar{N}$ is taken which follows from the multiplication curve *with* the ionization taken into account, instead of the value exp (Bx). This procedure was first proposed by C. G. and D. D. Montgomery [Phys. Rev. **53**, 955 (1938)]. It is hard to give a logical justification for this, but it is the best which one can do.

ratic averages:

$$\bar{n}_i n_j(x) = S n_i n_j W(n_1, n_2 \cdots; x), \quad (4)$$

and so on. For a continuous energy variation $\bar{n}_i(x)$ will become the average energy distribution $F(E, x)dE$; analogously $\langle n_i n_j \rangle_{Av}$ will become a function of two energy variables, which we denote by $F_2(E_1, E_2, x)dE_1 dE_2$, and so on.

The function $P(N, x)$ is obtained from W by the operation:

$$P(N, x) = S' W(n_1, n_2 \cdots; x), \quad (5)$$

where the primed summation sign S' means that one has to sum over all values of n_i with the restriction:

$$\sum_i n_i = N.$$

$P(N, x)$ is completely determined when one knows all the moments $\langle N^k(x) \rangle_{Av}$, and one may write:

$$P(N, x) = \sum_{k=1}^{\infty} a_{Nk} \langle N^k(x) \rangle_{Av}, \quad (6)$$

where the a_{Nk} are numerical coefficients, which clearly are independent of the distribution function $P(N, x)$.[5] Since on the other hand:

$$\bar{N}(x) = \sum_i \bar{n}_i(x), \quad \langle N^2(x) \rangle_{Av} = \sum_{ij} \langle n_i n_j(x) \rangle_{Av}, \quad (7)$$

and so on, it is clear that in this way $P(N, x)$ will be related to the energy distributions $F(E, x)$, $F_2(E_1, E_2, x)$, etc.

To discuss this connection further, we will consider first *the case of no correlation*. When the probabilities of finding a particle in the different energy intervals are completely independent of each other, the master function must have the form:

$$W(n_1, n_2, \cdots) = A \frac{(\bar{n}_1)^{n_1} (\bar{n}_2)^{n_2} \cdots}{n_1! n_2! \cdots}, \quad (8)$$

since $\bar{n}_i$ will then be proportional to the probability of finding a particle with the energy ϵ_i; the normalization constant A must be determined from:

$$SW = A \prod_i \sum_{n_i=0}^{\infty} \frac{(\bar{n}_i)^{n_i}}{n_i!} = A \exp\left(\sum \bar{n}_i\right) = 1,$$

so that $A = \exp(-\bar{N})$. From the definition (5) one then shows easily that $P(N, x)$ becomes the Poisson distribution (2). The quadratic averages (4) and also the higher order averages can in this case all be expressed in terms of the $\bar{n}_i$. One finds for instance:

$$\begin{aligned} \langle n_i n_j \rangle_{Av} &= \bar{n}_i \bar{n}_j + \delta_{ij} \bar{n}_i \\ \langle n_i n_j n_k \rangle_{Av} &= \bar{n}_i \bar{n}_j \bar{n}_k + \delta_{ij} \bar{n}_i \bar{n}_k + \delta_{jk} \bar{n}_j \bar{n}_i \\ &\quad + \delta_{ki} \bar{n}_k \bar{n}_j + \delta_{ij} \delta_{jk} \bar{n}_i \end{aligned} \quad (9)$$

and the generalization is clear. Consequently one can express $\langle N^k \rangle_{Av}$ as a polynomial of degree k in $\bar{N}$:

$$\langle N^k \rangle_{Av} = \sum_{l=1}^{k} b_{kl} (\bar{N})^l. \quad (10)$$

Of the coefficients b_{kl} we will only need the property:

$$\sum_{k=l}^{\infty} a_{Nk} b_{kl} = \frac{(-1)^{l-N}}{N!(l-N)!}, \quad (11)$$

which follows immediately by introducing (10) in (6) and remembering that $P(N, x)$ is now the Poisson distribution (2).

Consider now *the actual case*, in which the different energy intervals will be *correlated* because of the splitting processes. When a particle "splits" it adds simultaneously to the number in each of two energy intervals and hence the probabilities for particles to be in these ranges are not independent. A general method would now be to derive from the continuity equation, which the function W has to fulfill [the so-called "master" equation, see I, Eq. (22)], equations for all the average values $\bar{n}_i$, $\langle n_i n_j \rangle_{Av}$, etc. However, since these equations are almost self-evident, we shall omit the formal derivations and in addition pass immediately to the limit of a continuous energy variation. One finds that the quadratic averages $F_2(E_1, E_2, x)$ and the higher order averages $F_l(E_1, E_2 \cdots E_l, x)$ are singular whenever two or more of the energy variables coincide. Analogously to (9) one then can decompose the functions F_l into regular parts as follows:

$$\begin{aligned} F(E, x) &\equiv K_1(E, x), \\ F_2(E_1, E_2, x) &= K_2(E_1, E_2, x) \\ &\quad + \delta(E_1 - E_2) K_1(E_1, x), \\ F_3(E_1, E_2, E_3, x) &= K_3(E_1, E_2, E_3, x) \\ &\quad + \delta(E_1 - E_2) K_2(E_1, E_3, x) \\ &\quad + \delta(E_2 - E_3) K_2(E_2, E_1, x) \\ &\quad + \delta(E_3 - E_1) K_2(E_3, E_2, x) \\ &\quad + \delta(E_1 - E_2) \delta(E_2 - E_3) K_1(E_1, x), \end{aligned} \quad (12)$$

[5] This follows by considering (6) as the solution of the equations

$$\langle N^k(x) \rangle_{Av} = \sum_{N=1}^{\infty} N^k P(N, x); \quad k = 1, 2, 3 \cdots.$$

and so on. The functions $K_l(E_1\cdots E_l, x)$ are regular and symmetric in all the energy variables. Since the decomposition rules (12) are exactly the same as (9), except for replacing the products of average values $\bar{n}_i$ by the functions K_l, one obtains:

$$\langle N^k(x)\rangle_{\text{Av}}=\sum_{l=1}^{k} b_{kl}M_l(x), \tag{13}$$

where

$$M_l(x)=\int\cdots\int dE_1\cdots dE_l K_l(E_1\cdots E_l, x) \tag{14}$$

and the coefficients b_{kl} are the *same* as in (10). Introducing (13) in (6) and using the relation (11) one gets:

$$P(N, x)=\sum_{l=1}^{\infty}\frac{(-1)^{l-N}}{N!(l-N)!}M_l(x). \tag{15}$$

The average energy distribution $F(E, x) \equiv K_1(E, x)$ fulfills a continuity equation of the form:

$$\partial F/\partial x=L_E F(E, x), \tag{16}$$

where L_E is a linear operator acting on the variable E. Equation (16) is an abbreviation for:

$$\frac{\partial F}{\partial x}=-F(E, x)\int_0^E q(E, u)du +2\int_E^{\infty} duq(u, E)F(u, x)+\beta\frac{\partial F}{\partial E}, \tag{16a}$$

where $q(E, u)du$ is the probability per unit thickness that a particle of energy E splits into two particles of which one has the energy between u and $u+du$, while the other has the remaining energy $E-u$; $q(E, u)=q(E, E-u)$ since one does not distinguish between the two particles. Finally β is the average energy loss per unit thickness due to ionization.[6] For the higher order averages $K_l(E_1\cdots E_l, x)$ one then finds inhomogeneous equations of the form:

$$\partial K_l/\partial x=(L_{E1}+L_{E2}+\cdots L_{El})K_l +I_l(E_1\cdots E_l, x). \tag{17}$$

The inhomogeneous part I_l consists of a sum of $l(l-1)/2$ terms of which a typical one is:

$$2q(E_1+E_2, E_1)K_{l-1}(E_1+E_2, E_3\cdots E_l, x). \tag{18}$$

[6] See I, §2; the following considerations would remain valid for an arbitrary ionization probability $p(E, u)$.

The other terms correspond to the other ways in which one can get the l energy variables occurring in K_l by *one* splitting process from $(l-1)$ energy variables occurring in K_{l-1}. The interpretation of (17) is clear: for $I_l=0$ Eq. (17) would be separable and the solution would be $F(E_1, x)F(E_2, x)\cdots \times F(E_l, x)$; Eqs. (12) would therefore become identical with (9), and one would get the case of no correlation. The correlation is therefore due to the inhomogeneous part I_l, and one may say that I_l is the probability per unit thickness that l particles with energies $E_1, E_2\cdots E_l$ are produced from $(l-1)$ particles by one splitting process. Since *one* particle of energy E_0 is falling in, one has to solve Eqs. (17) with the initial condition

$$F_l(E_1\cdots E_l, 0)=\delta(E_1-E_0)\cdots\delta(E_l-E_0).$$

From (12) one sees that this means that $F(E, 0)=\delta(E-E_0)$ while all K_l for $l>1$ are zero for $x=0$. Writing for the solution of (16) with this initial condition $F(E_0, E, x)$ it is clear that the product $F(\xi_1, E_1, x)\cdots F(\xi_l, E_l, x)$ is not only a solution of the homogeneous part of (17) but that it is also the so-called fundamental solution. One then verifies easily that with $K_l(E_1\cdots E_l, 0)=0$ the solution of the inhomogeneous Eq. (17) is given by:

$$K_l=\int_0^x dt\int_0^{\infty}\cdots\int_0^{\infty} d\xi_1\cdots d\xi_l F(\xi_1, E_1, x-t)\cdots \times F(\xi_l, E_l, x-t)I_l(\xi_1\cdots\xi_l, t). \tag{19}$$

This is a recurrence relation for K_l; it becomes physically obvious when one considers the physical meaning of the K_l. From (19), (14), and (15) one sees therefore that $P(N, x)$ is completely determined when the average energy distribution $F(E_0, E, x)$ is known. Unfortunately, these equations seem to be of little use for the practical determination of $P(N, x)$. One can still write:

$$M_l(x)=\int_0^{\infty}\cdots\int_0^{\infty} d\xi_1\cdots d\xi_l\int_0^x dt I_l(\xi_1\cdots\xi_l, t) \times\bar{N}(\xi_1, x-t)\cdots\bar{N}(\xi_l, x-t), \tag{20}$$

but only for the case of no ionization ($\beta=0$) and "homogeneous" splitting probability $[q(E, u)=\chi(u/E)/E$; see I, §2] can one obtain from this a recurrence relation for the M_l. In this case:

$$\bar{N}(E_0, x) = e^{Bx}$$

with

$$B = \int_0^1 d\xi \chi(\xi).$$

Substituting in (20) one finds:

$$M_l(x) = l(l+1)Be^{lBx}\int_0^x dte^{-lBt}M_{l-1}(t).$$

Since $M_1(x) = \bar{N}(x)$ one then finds by induction:

$$M_l(x) = l!e^{lBx}(1-e^{-Bx})^{l-1}.$$

Substitution in (15) leads to the Furry formula (1).

§3. CALCULATION OF THE FLUCTUATION FOR THE FURRY MODEL

As in I we shall take $q(E, u) = 1/E$ (which makes $B = 1$) and we shall take the ionization into account by means of the cut-off method. One finds for $\bar{N}$ and $\langle N^2\rangle_{\text{Av}}$ the expressions (see I, Eqs. (18) and (38)):

$$\bar{N}(z, x) = \frac{e^{-x}}{2\pi i}\int \frac{ds}{s}\exp\left[sz + 2x/(s+1)\right], \quad (21)$$

$$\langle N^2(z, x)\rangle_{\text{Av}} = \bar{N}(z, x) + 2\int_0^x d\xi I(z, x, \xi),$$

$$I(z, x, \xi) = \frac{e^{\xi-2x}}{(2\pi i)^2}\int\int dsdt\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+2)} \times\exp\left[(s+t)z + \frac{2\xi}{s+t+1} + 2(x-\xi)\left(\frac{1}{s+1}+\frac{1}{t+1}\right)\right]. \quad (22)$$

Here $z = \log(E_0/\epsilon)$, where ϵ is the cut-off energy;[7] all the integrals must be taken along paths parallel to the imaginary axis and to the right of all singularities of the integrand. They can be computed approximately with the saddle point method. The following improvements were made over the analogous calculations reported in I:

1. All variable factors in the integrand were written into the exponent and included in the saddle point expansion. Or in other words: no part of the integrand was considered as "slowly varying."
2. The equations which determine the position of the saddle points for given values of x [in (21)] or of x and ξ [in (22)] now become quite involved. To circumvent this difficulty we have turned the question around; convenient values of the saddle points were chosen and the corresponding values of x or of ξ were then determined from the equations.
3. In order to estimate the errors involved in the saddle point method we calculated for a few values of x a next approximation by extending the Taylor expansion around the saddle points up to the fourth-order terms.

For $\bar{N}$ one obtains:

$$\bar{N} = \frac{\exp(-x+\varphi(s_0))}{[2\pi\varphi''(s_0)]^{\frac{1}{2}}} \times\left[1 + \frac{\varphi^{\text{IV}}(s_0)}{8\varphi''(s_0)^2} - \frac{5\varphi'''(s_0)^2}{24\varphi''(s_0)^3}\cdots\right], \quad (23)$$

where

$$\varphi(s) = sz + \frac{2x}{s+1} - \log s \quad (23\text{a})$$

and s_0 is the saddle point determined by $\varphi'(s_0) = 0$.

TABLE I. $\bar{N}(x)$ for the Furry model with $z = 4.75$.

s_0	x	$\bar{N}$ main term	% correction	$\bar{N}$ corrected
$1/z = 0.21$	0.00	1.084	−8.3	0.996
0.3	1.20	2.89		
0.4	2.20	5.57		
0.5	3.09	8.68		
0.6	3.95	11.93	−1.1	11.80
0.7	4.80	15.1		
0.8	5.67	17.9	−1.7	17.6
0.9	6.57	20.0		
1.0	7.50	21.15	−1.8	20.8
1.1	8.48	21.25		
1.2	9.47	20.4		
1.3	10.54	18.5	−1.6	18.2
1.4	11.6	16.1		
1.5	12.8	13.3	−1.5	13.1
1.6	13.9	10.4		
1.7	15.2	7.96	−1.5	7.84
1.8	16.4	5.84		
1.9	17.8	3.97		
2.0	19.1	2.65	−1.3	2.62

[7] The connection between ϵ and β can be fixed by means of the relation:

$$\int_0^\infty \bar{N}(x)dx = E_0/\beta,$$

which follows strictly from (16a) and which is also physically obvious. Introducing (21) and carrying out the integrals one finds: $2e^z - 1 = E_0/\beta$ so that $\epsilon \cong 2\beta$.

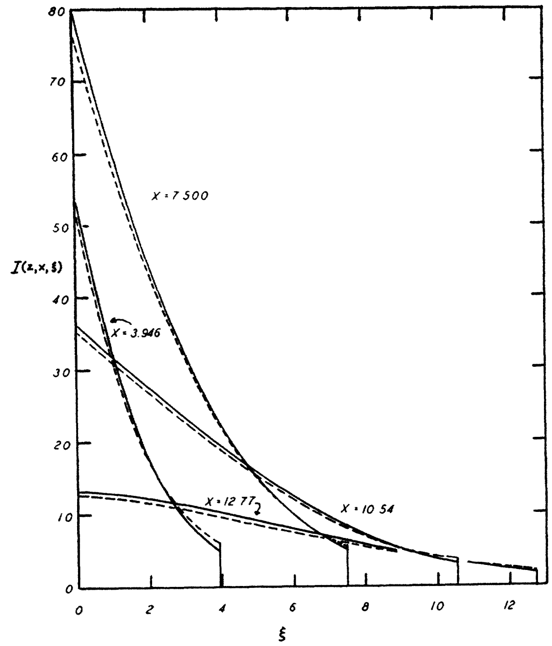

FIG. 2. The integrand $I(z, x, \xi)$ of Eq. (22) as function of ξ for $z=4.75$ and for the four values of x indicated. For the calculation Eq. (24) is used; the full lines are the result of taking only the first term, while for the dotted lines the other terms given are also used.

In Table I one finds the results for $z=4.75$. The correction given in the fourth column is computed from the terms with the third and fourth derivatives in (23). One sees that by using the main term only the error is already less than 10 percent over the whole range of x, while with the correction it is even less than 0.5 percent.

For $I(z, x, \xi)$ one obtains:

$$I(z,x,\xi)=\frac{\exp\left[\xi-2x+\psi(s_0,s_0)\right]}{2\pi(\psi_{ss}^2-\psi_{st}^2)^{\frac{1}{2}}}\times\left[1+\frac{\psi_{ssss}+4\psi_{ssst}+3\psi_{sstt}}{16(\psi_{ss}+\psi_{st})^2}+\frac{\psi_{ssss}-\psi_{sstt}}{8(\psi_{ss}^2-\psi_{st}^2)}+\frac{\psi_{ssss}-4\psi_{ssst}+3\psi_{sstt}}{16(\psi_{ss}-\psi_{st})^2}-\frac{5(\psi_{sss}+3\psi_{sst})^2}{48(\psi_{ss}+\psi_{st})^3}-\frac{\psi_{sss}^2+2\psi_{sss}\psi_{sst}-3\psi_{sst}^2}{8(\psi_{ss}+\psi_{st})(\psi_{ss}^2-\psi_{st}^2)}-\frac{3(\psi_{sss}-\psi_{sst})^2}{16(\psi_{ss}^2-\psi_{st}^2)(\psi_{ss}-\psi_{st})}+\cdots\right], \quad (24)$$

where

$$\psi(s,t)=(s+t)z+\frac{2\xi}{s+t+1}+2(x-\xi)\left(\frac{1}{s+1}+\frac{1}{t+1}\right)+\log\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+2)} \quad (24a)$$

and the subscripts denote differentiations after s and t. Since $\psi(s, t)$ is symmetric in s and t, the saddle points s_0 and t_0 for the s and t integrations will be equal to each other; they are determined by $\psi_s(s_0, t_0)=\psi_t(s_0, t_0)=0$. The terms with the third- and fourth-order differential quotients are again correction terms and they have been simplified by making use of the symmetry of $\psi(s, t)$. For a given value of x, one now chooses a suitable set of values for the saddle points s_0 and one then computes the corresponding values of ξ from $\psi_s(s_0, s_0)=0$ and of $I(z, x, \xi)$ from (24). Figure 2 shows the integrand $I(\xi)$ for a few values of x and for $z=4.75$; the full lines are the result of taking only the main term of (24) into account, while for the dotted lines the complete expression (24) is used. The integration over ξ must be done graphically, and in Table II one finds the results. One sees that the influence of the correction terms in (24) amounts to roughly 3 percent for all values of x. Table II also gives the values of the fluctuation $\sigma=\langle N^2\rangle_{Av}-\langle N\rangle_{Av}^2$ divided by $\bar{N}$, which are plotted in Fig. 3. The accuracy of these results depends mainly on the accuracy of the values for $\langle N^2\rangle_{Av}$. We believe that *with* the correction terms in (24), the error in $\langle N^2\rangle_{Av}$ is at most 1.5 percent. Since the error in $\langle N\rangle_{Av}^2$ is certainly much smaller, the error in σ will be at most of the order of 25 percent.

TABLE II. $\langle N^2\rangle_{Av}$ for the Furry model with $x=4.75$.

x	$M=2\int_0^x I d\xi$ main term	$\langle N^2\rangle_{Av}=\bar{N}+M$ main term	$\frac{\sigma}{\bar{N}}=\frac{\langle N^2\rangle_{Av}-\langle N\rangle^2_{Av}}{\bar{N}}$	% corr. to M	$\langle N^2\rangle_{Av}$ corrected	$\sigma/\bar{N}$ corrected
0	0	1	0			
2.20	38.40	43.97	2.33			
3.95	169.2	181.1	3.25	−3.4	175.4	3.07
5.67	349.0	366.9	2.60	−3.3	355.6	2.56
7.50	459.2	480.4	1.57	−3.4	464.8	1.59
8.48	455.0	476.3	1.16			
9.47	418.6	439.0	1.14			
10.54	348.6	367.1	1.34	−3.6	354.8	1.24
12.8	191.1	204.4	2.00	−3.2	198.1	1.96
15.2	79.38	87.34	3.01			
17.8	24.74	28.71	3.26			
19.1	12.83	15 48	3.19	−2.2	15.18	3.18

We believe therefore that for the Furry model with ionization *the fluctuation of N is small* and roughly equal to *twice the Poisson value* for all values of x. This result is quite different from the one reported in I (see table on p. 358), where for instance for $z=4.75$ and near the maximum of the multiplication curve $\sigma/\bar{N}$ was found to be equal to 9.6. The origin of this discrepancy can be traced to the fact that in I the factor $\Gamma(s)\Gamma(t)/\Gamma(s+t+2)$ was considered to be "slowly varying." This is *not* correct, because this factor has a pole for $s=0$ and $t=0$, and the saddle point $s_0=t_0$ lies near zero when x is small. We have verified that as a consequence the correction terms [analogous to those in (24)] become now quite appreciable and since the values of σ are very sensitive with regard to errors in $\langle N^2\rangle_{\text{Av}}$, large errors in σ can be expected.[8] We have also verified the same fact by only taking the factor $1/st$ into the exponent, considering $\Gamma(s+1)\Gamma(t+1)/\Gamma(s+t+2)$ as "slowly varying." The correction terms then again become quite small, and the results for σ are in good agreement with the values of Table II. This is of importance for the actual cosmic-ray problem, since there it is quite complicated to take all variable factors into the exponent.

The shape of the curve $\sigma/\bar{N}$ as a function of x is quite curious and we have been unable to find a qualitative physical explanation for it. We believe that the shape is real and *not* due to possible errors in computation. It probably arises from the mathematical form of σ. One can show that for very large x, $\sigma/\bar{N}$ will approach one, but whether there will be further maxima in the curve is difficult to decide without numerical computations.

§4. THE MODEL WITH $q(E,u)=1$

One may look upon the effect of the ionization on the fluctuation as due to the fact that because of the ionization the particles lose their potency for producing pairs. Arley has considered a model in which the particles "degenerate" after a certain number of "generations" and he found that as a consequence the fluctuation soon reaches

[8] The pole at $s=t=0$ also explains the difference between Fig. 2 and the analogous Fig. 3 in I. By taking all variable factors into the exponent no special consideration of the region $\xi \gtreqqless x$ is necessary.

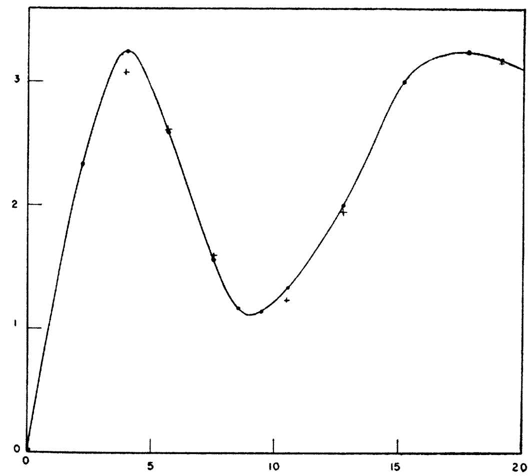

FIG. 3. The relative fluctuation $\sigma/\bar{N}$ as a function of x for the Furry model with $z=4.75$. ● = one term; + = two terms.

the Poisson value (2a). It is therefore perhaps of interest to study a Furry-like model for which the splitting probability $q(E,u)$ is a constant. This makes the total splitting probability proportional to the energy E of the particle, so that it will really diminish by the successive splitting processes. Furthermore, this model has the advantage that the influence of an ionization term $\beta\partial F/\partial E$ can be studied exactly, although of course the model has very little similarity with the actual cosmic-ray problem.

Without ionization one easily sees that $P(N;x)$ will fulfill the equation:

$$\partial P(N,x)/\partial x = -E_0P(N,x)+E_0P(N-1,x). \quad (25)$$

This is because the total energy is conserved and since the splitting probability of each particle is proportional to its energy, the sum of the splitting probabilities for all particles will be constant, and equal to the initial energy E_0, when the unit of x is so chosen that $q(E,u)=1$. The solution of (25), with the initial condition $P(N,0)=\delta_{1N}$ is given by:

$$P(N,x)=\exp(-E_0x)\frac{(E_0x)^{N-1}}{(N-1)!}, \quad (26)$$

so that $\bar{N}=1+E_0x$ and $\langle N^2\rangle_{\text{Av}}-\langle N\rangle_{\text{Av}}^2=E_0x=\bar{N}-1$. As soon as $E_0x\gg1$ one gets therefore the Poisson distribution.

With the ionization taken into account, we were unable to determine $P(N,x)$, but from the general formula of §2 one can of course again compute the fluctuation. Equation (16a) becomes for our case:

$$\frac{\partial F}{\partial x}=-EF+2\int_E^\infty duF(u,x)+\beta\frac{\partial F}{\partial E}. \quad (27)$$

The solution, with the initial condition $F(E,0)=\delta(E_0-E)$ can be found exactly and is given by:

$$F(E_0,E,x)=\exp\left(-E_0x+\tfrac{1}{2}\beta x^2\right)\delta(E_0-E-\beta x)+x\exp\left(-Ex-\tfrac{1}{2}\beta x^2\right)\times[2+x(E_0-E-\beta x)] \quad (28)$$
$$=0 \quad (E+\beta x>E_0),$$

from which follows:

$$\bar{N}(E_0,x)=\int_0^{E_0-\beta x}dEF(E_0,E,x)=[1+x(E_0-\beta x)]\exp\left(-\tfrac{1}{2}\beta x^2\right) \quad (29)$$
$$=0 \quad (x>E_0/\beta).$$

One then finds from (20), after a lengthy computation:

$$\langle N^2(E_0,x)\rangle_{\text{Av}}=2\beta x(2+E_0x-2\beta x^2)\exp(-\beta x^2)\int_0^x\exp\left(\tfrac{1}{2}\beta t^2\right)dt+(3\beta x^2-E_0x-1)\exp\left(-\tfrac{1}{2}\beta x^2\right)+[4\beta^2x^4-4\beta E_0x^3+(E_0^2-10\beta)x^2+4E_0x+2]\times\exp(-\beta x^2); \quad (x<E_0/2\beta)$$
$$=2\beta x(2+E_0x-2\beta x^2)\exp(-\beta x^2)\int_0^x\exp\left(\tfrac{1}{2}\beta t^2\right)dt+(3\beta x^2-E_0x-1)\exp\left(-\tfrac{1}{2}\beta x^2\right)+2(1-\beta x^2)\exp\left(\beta x^2-2E_0x+E_0^2/2\beta\right); \quad \left(\frac{E_0}{2\beta}<x<\frac{E_0}{\beta}\right)$$
$$=0; \quad \left(x>\frac{E_0}{\beta}\right). \quad (30)$$

In Fig. 4 the results are plotted for $\bar{N}$ and for $\sigma/\bar{N}$; here $t=E_0x$ and β/E_0^2 is taken equal to 0.01; the dotted lines are the results when the ionization is neglected. One sees the same peculiar behavior as in the Furry model case, but the deviation from the Poisson value unity is considerably less. $\sigma/\bar{N}$ presumably begins to approach one for t somewhat greater than 30.

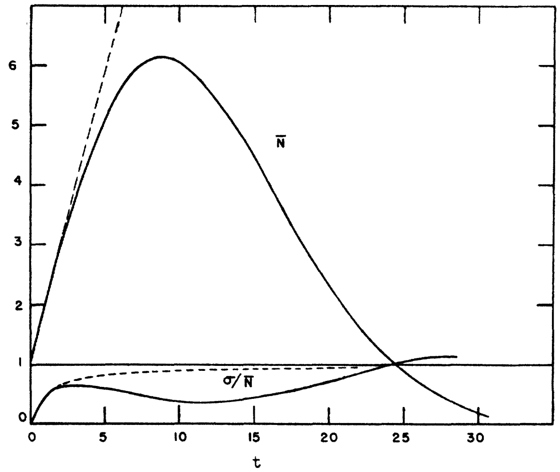

FIG. 4. The average number of particles $\bar{N}$ and the relative fluctuation $\sigma/\bar{N}$ as a function of $t=E_0x$ for the model with $q(E,u)=1$; β/E_0^2 is taken equal to 0.01. The dotted lines show the corresponding quantities when the ionization is neglected.

§5. THE COSMIC-RAY PROBLEM

Before considering the fluctuations we shall collect here briefly the results for the average energy distributions of the electrons $F(E,x)$ and of the photons $\Phi(E,x)$. We shall mainly follow the notations and method of Landau and Rumer.[9] F and Φ fulfill the equations:

$$\frac{\partial F}{\partial x}=-F(E,x)\int_0^E\pi(E,u)du+\int_E^\infty duF(u,x)\pi(u,u-E)+2\int_E^\infty du\Phi(u,x)\gamma(u,E)\equiv L_E^{(1)}F(E,x)+L_E^{(2)}\Phi(E,x), \quad (31a)$$

$$\frac{\partial \Phi}{\partial x}=-\Phi(E,x)\int_0^E\gamma(E,u)du+\int_E^\infty duF(u,x)\pi(u,E)\equiv L_E^{(3)}\Phi(E,x)+L_E^{(4)}F(E,x). \quad (31b)$$

[9] L. Landau and G. Rumer, Proc. Roy. Soc. **166**, 213 (1938).

These are the analogues of Eq. (16) for the Furry problem. They must be solved with the initial condition $F(E, 0)=\delta(E_0-E)$; $\Phi(E, 0)=0$, corresponding to the fact that one electron is falling in. The term $\beta\partial F/\partial E$ is omitted since we will take the ionization into account by the cut-off method. The thickness x is measured in radiation units; $\pi(E, u)du$ is the probability per unit thickness that an electron or positron of energy E produces a photon of energy between u and $u+du$; $\gamma(E, u)du$ is the probability per unit thickness that a photon of energy E produces a positron-electron pair of energies u to $u+du$ and $E-u$ to $E-u-du$; $\gamma(E, u)=\gamma(E, E-u)$ since no distinction is made between electrons and positrons. The functions $\pi(E, u)$ and $\gamma(E, u)$ have again the homogeneity property:

$$\pi(E, u)=\frac{1}{E}\bar{\pi}\left(\frac{u}{E}\right), \quad \gamma(E, u)=\frac{1}{E}\bar{\gamma}\left(\frac{u}{E}\right) \tag{32}$$

and from the calculations of Bethe-Heitler follows that:

$$\bar{\pi}(\xi)=\frac{4}{3\xi}-\frac{4}{3}+\xi, \quad \bar{\gamma}(\xi)=\frac{4}{3}\xi^2-\frac{4}{3}\xi+1. \tag{32a}$$

Because of (32) Eqs. (31) can be solved again by the momentum method (see I, §2), and one finds:

$$\left.\begin{aligned} F(E_0, E, x)&=\frac{1}{2\pi i}\int dsE^{-s-1}g(s, x),\\ \Phi(E_0, E, x)&=\frac{1}{2\pi i}\int dsE^{-s-1}\theta(s, x)\end{aligned}\right\} \tag{33}$$

where

$$\left.\begin{aligned} g(s, x)&=E_0{}^s\frac{(D-\lambda)e^{-\lambda x}-(D-\mu)e^{-\mu x}}{\mu-\lambda},\\ \theta(s, x)&=E_0{}^s\frac{C}{\mu-\lambda}(e^{-\lambda x}-e^{-\mu x}),\end{aligned}\right\} \tag{33a}$$

$$\left.\begin{matrix}\lambda\\ \mu\end{matrix}\right\}=\tfrac{1}{2}(A+D)\mp\tfrac{1}{2}[(A-D)^2+4BC]^{\frac{1}{2}} \tag{33b}$$

and A, B, C, and D are functions of s, given by:

$$A(s)=\int_0^1 d\xi(1-\xi^s)\bar{\pi}(1-\xi)=\frac{4}{3}[\Psi(s)+C_1]-\frac{5}{6}+\frac{1}{3(s+1)}+\frac{1}{s+2},$$

$$\left.\begin{aligned} B(s)&=2\int_0^1 d\xi\xi^s\bar{\gamma}(\xi)=\frac{2}{s+1}-\frac{8}{3(s+2)}+\frac{8}{3(s+3)},\\ C(s)&=\int_0^1 d\xi\xi^s\bar{\pi}(\xi)=\frac{4}{3s}-\frac{4}{3(s+1)}+\frac{1}{s+2},\\ D&=\int_0^1 d\xi\bar{\gamma}(\xi)=\frac{7}{9}.\end{aligned}\right. \tag{34}$$

Here C_1 is Euler's constant and $\Psi(s)=\Gamma'(s+1)/\Gamma(s+1)$. The functions A, B, C, λ, and μ, together with some of their derivatives, have recently been given by Rossi and Greisen.[10]

For the average number of particles $\bar{N}(E_0, x)$ one then finds:

$$\begin{aligned}\bar{N}(E_0, x)&=\int_\epsilon^{E_0} dEF(E_0, E, x)\\ &=\frac{1}{2\pi i}\int ds\frac{D-\lambda(s)}{s[\mu(s)-\lambda(s)]}e^{sz-x\lambda(s)},\end{aligned} \tag{35}$$

where $z=\log(E_0/\epsilon)$ and ϵ is the cut-off energy. In F we have neglected the term with $\exp(-\mu x)$ since for the values of x in which we are interested it contributes at the most 0.7 percent to $\bar{N}$, and generally considerably less. The value of ϵ can be related to β, the ionization loss per unit thickness, by means of the relation:

$$\int_0^\infty \bar{N}(E_0, x)dx=E_0/\beta, \tag{36}$$

which is a strict consequence of the Eqs. (31) when the ionization term $\beta\partial F/\partial E$ is added to (31a). Introducing (35) (and using the complete expression (33) for F) one obtains:

$$\frac{E_0}{\beta}=\frac{D}{2\pi i}\int ds\frac{e^{sz}}{s(AD-BC)}.$$

From (34) it follows that $s(AD-BC)$ has only one zero point, namely for $s=1$. By computing the residue there, one finds:

$$E_0/\beta=[63/(14\pi^2+5)]e^z$$

or $\epsilon=0.44\beta$.

[10] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941). These authors have used slightly more accurate forms of the probability functions $\bar{\pi}$ and $\bar{\gamma}$, (32a), in which the factor 4/3 is increased by 0.027. With the exception of the second derivative, the resulting values of the functions differ from ours by 2 percent at the most.

The integral over s in (35) must again be computed by the saddle point method. The factor $(D-\lambda)/(\mu-\lambda)$ can be considered to be "slowly varying," but the $1/s$ must be taken into the exponent. One obtains:

$$\bar{N}(E_0, x) = \frac{D-\lambda(s_0)}{s_0[\mu(s_0)-\lambda(s_0)]} \times\left[2\pi\left\{-\lambda''(s_0)x+\frac{1}{s_0^2}\right\}\right]^{-\frac{1}{2}} e^{s_0 z - x\lambda(s_0)}, \quad (37)$$

where x and s_0 are related by:

$$z - x\lambda'(s_0) - (1/s_0) = 0. \quad (38)$$

§6. CALCULATION OF THE FLUCTUATION FOR THE COSMIC-RAY PROBLEM

To calculate $\langle N^2\rangle_{\text{Av}}$ one must first generalize the results of §2. We have to introduce three new distribution functions. $F_2(E_1, E_2, x)dE_1dE_2$ denotes as before the average product of the numbers of electrons in the energy ranges dE_1 and dE_2; $\Phi_2(E_1, E_2, x)dE_1dE_2$ denotes the corresponding quantity for the photons, while $H_2(E_1, E_2, x) \times dE_1dE_2$ denotes the average product of the number of electrons in dE_1 and the number of photons in dE_2. The functions F_2 and Φ_2 are symmetric in E_1 and E_2, and are singular for $E_1=E_2$; the function H_2 is not symmetric but is regular for all values of E_1 and E_2. As in (12) one can separate off the singular parts of F_2 and Φ_2 by introducing two new functions $K_2(E_1, E_2, x)$ and $J_2(E_1, E_2, x)$, such that:

$$\begin{aligned} F_2(E_1, E_2, x) &= \delta(E_1-E_2)F(E_1, x) + K_2(E_1, E_2, x) \\ \Phi_2(E_1, E_2, x) &= \delta(E_1-E_2)\Phi(E_1, x) + J_2(E_1, E_2, x). \end{aligned} \quad (39)$$

Using the operator notation of Eqs. (31) we can now write the equations for the regular functions K_2, H_2, and J_2 in the form:

$$\begin{aligned}
\frac{\partial K_2}{\partial x} &= (L_{E_1}^{(1)}+L_{E_2}^{(1)})K_2(E_1, E_2, x) + L_{E_2}^{(2)}H_2(E_1, E_2, x) + L_{E_1}^{(2)}H_2(E_2, E_1, x) + 2\gamma(E_1+E_2, E_1)\Phi(E_1+E_2, x), \\
\frac{\partial H_2}{\partial x} &= (L_{E_1}^{(1)}+L_{E_2}^{(3)})H_2(E_1, E_2, x) + L_{E_2}^{(4)}K_2(E_1, E_2, x) + L_{E_1}^{(2)}J_2(E_1, E_2, x) + \pi(E_1+E_2, E_2)F(E_1+E_2, x), \quad (40) \\
\frac{\partial J_2}{\partial x} &= (L_{E_1}^{(3)}+L_{E_2}^{(3)})J_2(E_1, E_2, x) + L_{E_1}^{(4)}H_2(E_1, E_2, x) + L_{E_2}^{(4)}H_2(E_2, E_1, x).
\end{aligned}$$

These equations may be derived from a "master equation," but they are almost self-evident from the physical meaning of the operators $L^{(i)}$; the inhomogeneous terms again give the correlation between the different energy intervals, because of the splitting processes.

The solution of (40) is straightforward. One introduces the momenta

$$k(s, t, x) = \iint_0^\infty dE_1 dE_2 E_1^s E_2^t K_2(E_1, E_2, x)$$

and the corresponding functions $h(s, t, x)$ and $j(s, t, x)$ formed with H_2 and J_2. These fulfill then the differential equations:

$$\begin{aligned}
\partial k/\partial x &= -[A(s)+A(t)]k(s, t, x) + B(t)h(s, t, x) + B(s)h(t, s, x) + S(s, t)\theta(s+t, x), \\
\partial h(s, t, x)/\partial x &= -[A(s)+D]h(s, t, x) + C(t)k(s, t, x) + B(s)j(s, t, x) + T(s, t)g(s+t, x), \\
\partial h(t, s, x)/\partial x &= -[A(t)+D]h(t, s, x) + C(s)k(s, t, x) + B(t)j(s, t, x) + T(t, s)g(s+t, x), \\
\partial j/\partial x &= -2Dj(s, t, x) + C(s)h(s, t, x) + C(t)h(t, s, x),
\end{aligned} \quad (41)$$

TABLE III. $\langle N^2\rangle_{\text{Av}}$ for the cosmic-ray case; $z=5.67$ and $x=z$.

	$\bar{N}$	$\langle N^2\rangle_{\text{Av}}$	$\sigma=\langle N^2\rangle_{\text{Av}}-\langle N\rangle_{\text{Av}}^2$	$\sigma/\bar{N}$	σ_{Furry}	σ_{Poisson}
Present calculation	16.12	309.4	49±16	3.0±1	243.7	16.12
Previous calculation[a]	17.0	442	153	9	272	17

[a] W. T. Scott and G. E. Uhlenbeck, Phys. Rev. 57, 1061A (1940).

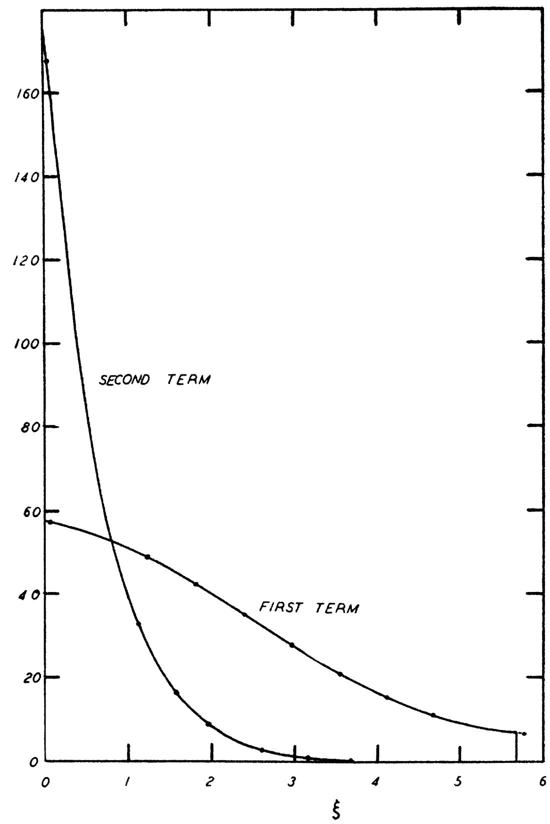

FIG. 5. The integrand for the cosmic-ray problem, analogous to the function $I(z, x, \xi)$ for the Furry problem. The two terms refer to the two terms of $k(s, t, x)$ in Eq. (42) which have been considered.

where we have introduced the new abbreviations:

$$S(s, t)=S(t, s)=2\int_0^1 d\xi(1-\xi)^s\xi^t\bar{\gamma}(\xi)$$

$$=\frac{2}{3}\frac{\Gamma(s+1)\Gamma(t+1)}{\Gamma(s+t+1)}\times\left[3-\frac{4(s+1)(t+1)}{(s+t+2)(s+t+3)}\right], \quad (41a)$$

$$T(s, t)=\int_0^1 d\xi(1-\xi)^s\xi^t\bar{\pi}(\xi)$$

$$=\frac{1}{3}\frac{\Gamma(s+1)\Gamma(t)}{\Gamma(s+t+1)}\left[4-\frac{t(4s+t+5)}{(s+t+1)(s+t+2)}\right].$$

These equations must now be solved with the initial condition that each of the functions k, h, and j vanish for $x=0$; g and θ are given by (33a).

The solution is quite cumbersome and we shall only indicate the result for $k(s, t, x)$:

$$k(s, t, x)=\frac{E_0^{s+t}}{r(s)r(t)r(s+t)}\times\{S(s, t)C(s+t)[\lambda(s)-D][\lambda(t)-D]$$
$$+T(s, t)B(t)[\lambda(s+t)-D][\lambda(s)-D]$$
$$+T(t, s)B(s)[\lambda(s+t)-D][\lambda(t)-D]\}$$
$$\times\int_0^x d\xi \exp\{(\xi-x)[\lambda(s)+\lambda(t)]-\xi\lambda(s+t)\}$$
$$+\text{seven similar terms.} \quad (42)$$

The seven additional terms are formed by replacing in the first term $\lambda(s+t)$ by $\mu(s+t)$, or $\lambda(s)$ by $\mu(s)$, or $\lambda(t)$ by $\mu(t)$, or by any combination of such replacements; $r(s)$ is an abbreviation for $\mu(s)-\lambda(s)$.

From $k(s, t, x)$ one finally finds $\langle N^2\rangle_{\text{Av}}$ according to the equation:

$$\langle N^2\rangle_{\text{Av}}=\bar{N}+\frac{1}{(2\pi i)^2}\int\int dsdt\frac{e^{(s+t)z}}{stE_0^{s+t}}k(s, t, x), \quad (43)$$

where as before $z=\log(E_0/\epsilon)$. The integrals must again be performed by the saddle point method. A precise calculation becomes quite involved, and we have therefore tried to simplify it by the following two considerations.

1. Only the exponential parts of the integrand and the factor $1/st$ are taken as "rapidly varying." From the experience gained in §3, one may expect that this will give reliable results.

2. Only two of the eight terms in $k(s, t, x)$ are taken into account, namely the one with $\lambda(s)$, $\lambda(t)$, $\lambda(s+t)$ (the first term) and the one with $\lambda(s)$, $\lambda(t)$, $\mu(s+t)$. The reason for this becomes clear if one examines the exponential parts of the different terms, remembering that $r(s)=\mu(s)-\lambda(s)$ is of the order unity or greater. For small values of ξ only the two terms mentioned are of the same order of magnitude; all other terms are much smaller. For $\xi\cong x$, some of them will become comparable with the first term, but then this term itself is quite small, so that the error made in neglecting the six additional terms all together is probably insignificant.

The calculation has been carried out for $z=5.67$ and only for $x=z$; with $\beta=90$ Mev for air, this corresponds to an initial energy of

11.5×10^9 ev and to a depth slightly beyond the maximum of the multiplication curve $\bar{N}(E_0, x)$ as given by (37).[11] Table III summarizes the result and Fig. 5 shows the integrands as functions of ξ for the two terms considered. The previous calculations referred to in Table III are results reported at the Washington meeting of 1940. At that time we did not know that it was important to include the factors $1/s$ in (35) and $1/st$ in (43) in the rapidly varying part of the integrand. Just as in I we found as a result far too great a value for the fluctuation. The accuracy of the present calculations is hard to estimate and the limits given are more or less a guess. However, it seems sure that also in the cosmic-ray case the fluctuations are much smaller than the Furry value (2) and of the order of a few times the Poisson value (2a).

One of us wishes to acknowledge the assistance of the Horace H. Rackham fund, without which much of this work would not have been possible.

[11] The maximum occurs at $x \cong 4.8$ and has the value 17.5.

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Relation of the Cosmic Radiation to Geomagnetic and Heliophysical Activities[1]

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(Received October 9, 1942)

Relations between 28-day fluctuations of intensity of the cosmic radiation and both terrestrial magnetic activity and sunspot areas were investigated. Definite pulses, both in the magnetic character and in sunspot areas, were found to be associated with the primary pulses in the cosmic radiation at Boulder, obtained by Chree's "superposed-epoch" method. They were in general phase opposition to the cosmic-ray pulses, but the tip of the magnetic-character pulse preceded the tip of the opposite cosmic-ray pulse by one day; the lead was three or four days in the case of the opposed sunspot pulses. Similar relations were not found among secondary pulses, although a 34-day periodicity in sunspot-area pulses referred to days selected on the basis of cosmic-ray intensity was displayed. Direct application of Chree's method to the magnetic character and sunspot areas, individually, indicated a 27-day periodicity in the former and a 34-day periodicity in the latter. A second method of investigation, used by Graziadei, Kolhörster, and others, was also employed. This yielded results in some respects contradictory to the first. In particular, it indicated 27-day fluctuations in sunspot areas in phase with the cosmic-ray fluctuations and out of phase with changes in magnetic character. However, it also indicated the 34-day periodicity in sunspot areas for the period of the investigation was more pronounced than the 27-day periodicity. Among other possibilities, the possible effects of sunspots through the agency of their magnetic fields were considered.

INTRODUCTION

In a recent paper[2] the author reported a statistical investigation of cosmic-ray intensity fluctuations at Boulder (lat. 40° N; long. 105°16′ W; alt. 5440 ft.) by Chree's "superposed-epoch" method of analysis. This provided evidence for the existence of secondary pulses at about 28-day intervals both preceding and subsequent to the primary cosmic-ray pulses. These secondary pulses represented deviations from the mean amounting to about 0.2 percent of the general average cosmic-ray ionization rate (corrected[3] for barometric variations) of 38.19 ions per cc per sec. in a heavily shielded, high pressure chamber.

[1] Preliminary reports on some portions of this investigation were made in a Letter to the Editor, Phys. Rev. **59**, 678 (1941), and at: the Lubbock, Texas, meeting of the Southwestern Division of the A.A.A.S., April 29, 1941; the Golden, Colorado, meeting of the Colorado-Wyoming Acad. Sci., Nov. 8, 1941; and the Detroit, Michigan, meeting of the Am. Phys. Soc., Feb. 20, 1942.

[2] J. W. Broxon, Phys. Rev. **59**, 773 (1941).

[3] In the paper of reference 2, this general average was incorrectly given as 38.16. It is not supposed that the difference exceeds the error of measurement of the absolute value of the ionization. However, 38.19 is nearer the average of the values used in the statistical investigation. Consequently, the upper pairs of curves in Figs. 1 and 2 of reference 2, and the cosmic-ray curves of Figs. 1, 2, 6, and 7 of this paper, are drawn 0.09 percent too high. Confidence that few errors in the statistical work have gone undiscovered is due to the fact that all tabulations have been checked, and each step in the computations has been performed at least twice except in the case of a few of the probable error computations.