

## Mobilities in Some Free Electron Gases

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Mobility coefficients for free electrons and mobilities of positive ions are reported for hydrogen, deuterium, and some mixtures of hydrogen and nitrogen, and of hydrogen and helium, for field strengths between 1 and 2 volts per cm per mm of mercury gas pressure. The relation between free electron drift velocity and field strength is found to be parabolic in this range for all these gases. Discussion of these observations shows that they can be explained qualitatively if electrons excite rotations of the hydrogen molecules by collision, and the cross sections for this process are estimated. A mathematical appendix states the theoretical results required in the discussion and derives those details not previously obtained.

### A. THE EXPERIMENTAL DATA

**D**URING the past year, measurements of the mobilities of electrons and positive ions in various free electron gases have been made with refinements of the technique reported previously.<sup>1</sup> This technique is applicable where the drift velocity,  $\bar{u}$ , of electrons (or ions) is related to the field intensity per unit pressure,  $E/p$ , by a relation of the form

$$\bar{u} = K(E/p)^n,$$

where  $n$  is a constant and  $K$  is a generalized mobility coefficient. In these gases the relation has been shown to be  $\bar{u} = K(E/p)^{\frac{1}{2}}$  for electrons within the range of field intensities used.

Further measurements have shown that the previous use of wire 0.0075 cm in diameter introduced a small uncertainty in the data for hydrogen which could be reduced by using wire 0.0025 cm in diameter. The finer wire had a tendency to vibrate and break, but it was found that tying the wire around a short segment of cord, near the supports, dampened out the vibrations completely. The finer wire was used in all the measurements being reported here.

Measurements were made in hydrogen, deuterium, mixtures of hydrogen and nitrogen, and mixtures of helium and hydrogen. The deuterium gas behaved similarly to hydrogen in conditioning and appearance of the discharge. Nitrogen containing less than 1 percent hydrogen when the wire is negative produces intensely hot spots

on the wire where the discharge is taking place and melts the tungsten wire in two. Helium containing 10 percent hydrogen or less has a much lower electron mobility than hydrogen, and the wire breaks, probably owing to an incipient arc.

Small pieces of rubber or traces of carbon tetrachloride vapor in the tube introduce a gaseous impurity which falsifies the results unpredictably and which is not eliminated by long continued discharge from a point-to-cylinder in a side arm of the tube. Such a discharge was used for cleaning up the gas and was quickly effective if sulphur—and chlorine—containing impurities were avoided.

Table I compares the mobility coefficients for hydrogen and deuterium. Although these values for  $K$  are probably not accurate to better than

TABLE I. Mobility in hydrogen and deuterium.

	$K[(\text{cm./sec.})/(\text{e.s.u./atmos.})^{\frac{1}{2}}]$	$C[(\text{cm./sec.})/(\text{e.s.u./atmos.})]$
H <sub>2</sub>	$7.8 \times 10^5$	$3.06 \times 10^3$
D <sub>2</sub>	$7.5 \times 10^5$	$2.26 \times 10^3$

TABLE II. Values of the mobility coefficient  $K$  for electrons and the positive ions mobility  $C$  for mixtures of gases.

	$K[(\text{cm./sec.})/(\text{e.s.u./atmos.})^{\frac{1}{2}}]$	$C[(\text{cm./sec.})/(\text{e.s.u./atmos.})]$
100% H <sub>2</sub>	$7.8 \times 10^5$	$3.06 \times 10^3$
30% H <sub>2</sub> +70% N <sub>2</sub>	$8.7 \times 10^5$	$1.13 \times 10^3$
20% H <sub>2</sub> +80% N <sub>2</sub>	$8.7 \times 10^5$	$0.99 \times 10^3$
10% H <sub>2</sub> +90% N <sub>2</sub>	$7.9 \times 10^5$	$0.92 \times 10^3$
1% H <sub>2</sub> +99% N <sub>2</sub>	$7.5 \times 10^5$	$0.90 \times 10^3$
100% N <sub>2</sub>	—	$0.78 \times 10^3$
30% H <sub>2</sub> +70% He	$8.7 \times 10^5$	$3.00 \times 10^3$
10% H <sub>2</sub> +90% He	$(\bar{u} \neq K(E/p)^{\frac{1}{2}})$	$2.77 \times 10^3$

<sup>1</sup> W. H. Bennett, Phys. Rev. **58**, 992 (1940).

5 percent, the repeated checks with the same tube make it likely that the relative values are correct to better than 2 percent.

Table II shows the changes in  $K$  and  $C$  (the positive ion mobility) produced by decreasing the amount of hydrogen in nitrogen and for mixtures of He and H<sub>2</sub>. As the concentration of hydrogen in nitrogen is decreased, the value of  $K$  increases, then decreases. A similar effect seems to be taking place when helium is introduced in hydrogen.

In these measurements of  $K$ , the value of  $E/p$  varied between about 1 and 2 volts per cm per mm of mercury pressure at 0°C, so that the consistency of the values of  $K$  for each gas or mixture for the various currents<sup>2</sup> shows that the electron drift velocity follows  $\bar{u} = K(E/p)^{1/2}$  in that range.

## B. DISCUSSION OF THE OBSERVATIONS

### 1. The Positive Ion Mobilities

The positive ion mobilities may be discussed with the aid of Langevin's formula

$$C = \frac{3}{16Y} \cdot \left\{ \frac{1+m/M}{(D-1)d} \right\}^{1/2},$$

where  $m$  is the mass of a molecule,  $M$  is the mass of an ion,  $D$  is the polarizability of the gas,  $d$  is the density, and  $Y$  is a function of pressure, collision area, and polarizability. Of these,  $m$  should equal  $M$  for hydrogen and for deuterium, and  $Y$  and  $D$  should be the same for both. The only

TABLE III. Calculated and observed values of positive ion mobilities.  $C/10^3[(\text{cm}/\text{sec.})/(\text{e.s.u.}/\text{atmos.})]$ .

	Observed	Calculated	
		H <sub>2</sub> <sup>+</sup> in N <sub>2</sub>	N <sub>2</sub> <sup>+</sup> in N <sub>2</sub>
30% H <sub>2</sub> +70% N <sub>2</sub>	1.13	1.14	1.00
20% H <sub>2</sub> +80% N <sub>2</sub>	0.99	1.04	0.93
10% H <sub>2</sub> +90% N <sub>2</sub>	0.92	0.97	0.84
1% H <sub>2</sub> +99% N <sub>2</sub>	0.90	—	0.79

<sup>2</sup> For test of data see W. H. Bennett, reference 1.

The following relations are used in this paper between units or notations. Mobility:

$$1 \text{ (cm/sec.)}/\{(\text{volt/cm})/\text{atmos.}\} = 300 \text{ (cm/sec.)}/(\text{e.s.u. of field/atmos.}).$$

Mobility coefficient ( $\mu = \frac{1}{2}$ ):

$$1.59 \text{ (cm/sec.)}/\{(\text{volt/cm})/\text{mm Hg}\}^{1/2} = 1 \text{ (cm/sec.)}/(\text{e.s.u. of field/atmos.})^{1/2}.$$

Cross section:

$$p\alpha = NQ.$$

Drift velocity:

$$(e/m)^{1/2} = 4.20 \times 10^7 \text{ (cm/sec.)}/(\text{volt})^{1/2}.$$

difference should appear in the density,  $d$ , so the mobilities should be related by

$$C_H = \sqrt{2} C_D.$$

The ratio of the mobilities measured is 1.36 instead of the value 1.41 expected.

The positive ion mobility in nitrogen is

$$0.78 \times 10^3 \text{ (cm/sec.)}/(\text{e.s.u.}/\text{atmos.})$$

or

$$2.61 \text{ (cm/sec.)}/\{(\text{volt/cm})/\text{atmos.}\},$$

which compares favorably with

$$2.67 \text{ (cm/sec.)}/\{(\text{volt/cm})/\text{atmos.}\},$$

the value usually cited.<sup>3</sup>

The values of the positive ion mobilities for mixtures of hydrogen and nitrogen are compared in Table III with those calculated from a formula given by Loeb.<sup>4</sup>

$$C = \frac{1}{x_1/C_1 + x_2/C_2},$$

where  $x_1$  and  $x_2$  are the respective fractional concentrations of the components whose mobilities are  $C_1$  and  $C_2$ . In the second column we calculated with  $C_2$  from 1 percent H<sub>2</sub> in N<sub>2</sub> and in the third column we used  $C_2$  from pure N<sub>2</sub>.

Tyndall<sup>5</sup> does not report a mobility for H<sub>2</sub><sup>+</sup> in He but the result for the 10 percent hydrogen in helium mixture can be used to calculate mobility for hydrogen ions in helium. The result is  $2.75 \times 10^3 \text{ (cm/sec.)}/(\text{e.s.u.}/\text{atmos.}).$

### 2. Electron Drift Velocities

The drift of electrons through gas under an applied field is well understood theoretically from the smallest up to large fields of the order of 1000 volts per cm per mm of mercury pressure at 0°C.<sup>6</sup> Up to such fields the velocity distribution of the electrons remains nearly isotropic, and Lorentz' method of expansion in spherical har-

<sup>3</sup> J. H. Mitchell and K. E. W. Ridler, Proc. Roy. Soc. **146**, 911 (1934).

<sup>4</sup> L. B. Loeb, *Fundamental Processes of Electrical Discharge in Gases* (Wiley, 1939), p. 42.

<sup>5</sup> A. M. Tyndall, *The Mobility of Positive Ions in Gases* (Cambridge University Press, 1938).

<sup>6</sup> L. B. Loeb, reference 4, Chapters 4 and 5; S. Chapman and T. E. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, 1939), pp. 345 ff. M. J. Druyvesteyn and F. M. Penning, Rev. Mod. Phys. **12**, 90-99 (1940). Many references, especially to the earlier work, will be found in the above.

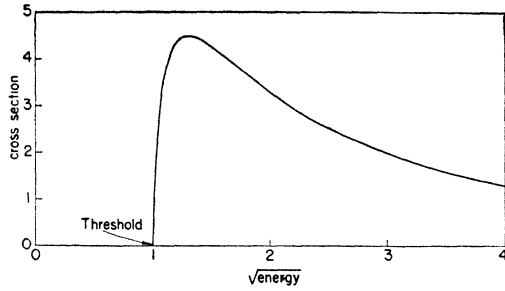


FIG. 1. Cross section for inelastic collision with a dipole (Massey).

monics about the direction of the field and drift can be used. The most likely collision process between electrons and gas molecules is elastic scattering, and transfer to the gas molecules of the momentum gained by the electron in the direction of the field between collisions is almost entirely due to this. Owing to the small mass of an electron relative to an ion, however, transfer to the gas molecules by elastic scattering of the energy gained between collisions is slow; as a result the electrons acquire large energy not in general distributed according to Maxwell's law, and the first two equations of the Lorentz expansion must be solved simultaneously to give the velocity dependence of the isotropic part and of the term in "cos  $\theta$ " in the distribution; further, in the first (energy transfer) equation the effects of inelastic collisions become as important as, or more important than, those of elastic collisions, except for lower fields in monatomic gases when all collisions are elastic.

In the absence of knowledge of the collision cross sections for elastic and inelastic collisions it is not possible to compute the drift velocities when inelastic collisions occur; it may, however, be possible to infer some information about the collision cross sections from the observed drift velocities. The drift velocity observed as a function of field strength is plainly inadequate to give both the cross section for momentum transfer in elastic impact as a function of speed and the cross sections for various losses of energy in inelastic impact. We have, however, knowledge of the former from direct observation of angle scattering<sup>7</sup> for electrons of energy more than

<sup>7</sup> C. Ramsauer and R. Kollath, *Handbuch der Physik*, Vol. 22, Part 2 (1933), p. 243; C. E. Normand, *Phys. Rev.* **35**, 1217 (1930).

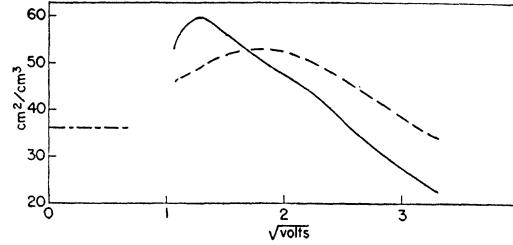


FIG. 2. ——— Value indicated for  $\alpha$  assumed constant; ——— Observed values for  $\alpha$  (cross section for momentum transfer) (Ramsauer and Kollath); - - - The usual collision cross section (Ramsauer and Kollath).

0.6 electron volt; moreover the sideways diffusion observed by Townsend<sup>8</sup> as a function of field strength gives us an independent set of observational data, while the data may be observed as functions of fractional concentration in gas mixtures as well as of the field.

### 3. Electron Mobility Coefficients

In the range of field strengths of these observations, Townsend's results show that in pure H<sub>2</sub> the average energy of the electrons varies from 0.35 to 0.54 electron volt. This is less than the threshold for excitation of the first vibrational state of H<sub>2</sub> at 0.54 eV. If the variation of drift velocity as the square root of the field strength is produced, as it could be, by collision cross section for momentum transfer  $\alpha$ , nearly independent of speed (up to energies of 0.5 eV) and average fractional loss of energy per such collision (by amounts small compared to 0.1 eV)  $f$ , also independent of speed, we have<sup>9</sup> (Section C (2.4) below)

$$\bar{\epsilon} = 0.604 \frac{E}{p\alpha} f^{-\frac{1}{2}}, \quad (3.1)$$

$$\bar{u} = 3.17 \times 10^7 \left( \frac{E}{p\alpha} f^{\frac{1}{2}} \right), \quad (3.2)$$

where  $E$  is field in volts per cm and  $p$  is gas pressure in mm Hg at 0°C,  $\alpha$  is expressed as an "absorption coefficient" in (cm<sup>2</sup>/[cm<sup>3</sup>·mm Hg]), and  $\bar{\epsilon}$  and  $\bar{u}$  are the mean electron energy in electron volts and drift velocity in cm/sec., respectively. Using  $\bar{\epsilon} = 0.35$  eV (from Townsend's

<sup>8</sup> J. S. E. Townsend, *J. Frank. Inst.* **200**, 563 (1934).

<sup>9</sup> See M. J. Druyvesteyn and F. M. Penning, reference 6, for references.

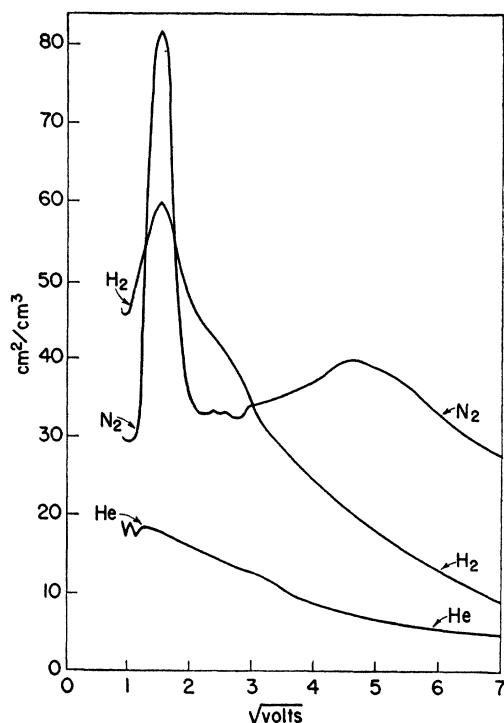


FIG. 3. Total collision cross section after Normand.

observations) and  $\bar{u} = 12.4 \times 10^5$  cm/sec. (from the observations reported here) for

$$E/p = 1 \text{ (volt/cm)/mm Hg,}$$

we obtain

$$\alpha = 32 \text{ (cm}^2\text{/cm}^3 \cdot \text{mm Hg),} \quad (3.3)$$

$$f = 0.0026. \quad (3.4)$$

This value of  $f$  is five times that ( $2m/M = 0.00055$ )<sup>10</sup> which would arise from the elastic collisions, and must be assigned to excitation of the rotational energy states (0.015 ev for the first state of H<sub>2</sub>). Such an inelastic collision would have to occur about once in five elastic collisions. This is entirely possible, the only difficulty being to explain why the loss of energy per cm,  $f\alpha\epsilon p$ , should increase with the speed. If all the H<sub>2</sub> molecules were in their zero rotational state, and if only the first state could be excited, we should expect the loss of energy per cm to fall off inversely as the energy, as the speed (well above the threshold) increased<sup>11</sup> (Fig. 1). Many H<sub>2</sub> molecules, however, will be, at

<sup>10</sup> L. B. Loeb, reference 4, p. 212.

<sup>11</sup> H. S. W. Massey, Proc. Camb. Phil. Soc. 28, 99 (1932).

300°K, in excited rotational states, and we might well expect interchange of angular momentum by two or more quanta with a symmetrical molecule. Indeed, if there were collision cross sections for losses of energy  $\epsilon_r$  to various excitations.

$$\alpha_r(\epsilon) = 2f\alpha(\epsilon_{r+1} - \epsilon_r)/\epsilon \quad \epsilon > \epsilon_r, \\ = 0 \quad \epsilon < \epsilon_r, \quad (3.5)$$

(close together compared to 0.1 ev, but covering the whole range from 0 to 0.5 ev) we would obtain (Section C, (3.2 below)

$$\bar{\epsilon} = 0.612 \frac{E}{p\alpha} f^{-\frac{1}{2}}, \quad (3.11)$$

$$\bar{u} = 3.50 \times 10^7 \left( \frac{E}{p\alpha} f^{\frac{1}{2}} \right)^{\frac{1}{2}}, \quad (3.21)$$

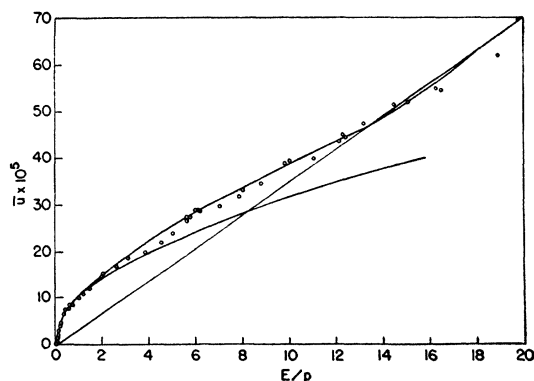
differing little from (3.1) and (3.2), and agreeing with the observations if

$$\alpha = 37 \text{ (cm}^2\text{/cm}^3 \cdot \text{mm Hg),} \quad (3.31)$$

$$f = 0.0022. \quad (3.41)$$

This value of  $\alpha$  cannot be compared directly with observed values since the latter<sup>12</sup> do not extend below 0.5 ev; however, it is not out of line with them (Fig. 2).

For D<sub>2</sub> we should expect the same value of  $\alpha$  as for H<sub>2</sub>, and to a first approximation the same chance of excitation of a rotational level. If only the first rotational level could be excited there would be only half the loss of energy in D<sub>2</sub> as in

FIG. 4. Bradbury and Nielsen's drift velocities in H<sub>2</sub>. Parabola, straight line, and theoretical interpolation.

<sup>12</sup> C. Ramsauer and R. Kollath, Ann. d. Physik 12, 538 (1932).

$H_2$ , and  $\bar{u}$  should be smaller by the factor  $2^{-\frac{1}{2}}=0.841$ ; if, on the other hand, (3.5) were correct  $\alpha_r(\epsilon)$  should be halved, but so should  $\epsilon_{r+1}-\epsilon_r$ , and so  $\alpha$  and  $\bar{u}$  would be unchanged. (Moreover the threshold for excitation of the first vibrational state in  $D_2$  is only 0.38 ev.) It is thus not unreasonable that  $\bar{u}$  for  $D_2$  is 0.96 that for  $H_2$ .

In the range of field strengths used, Townsend's results show that in pure  $N_2$ ,  $\bar{\epsilon}$  varies from 0.80 to 1.13 ev and in pure He from 2.0 to 3.9 ev. In the former this range contains the minimum at which the total cross section falls to 28 ( $cm^2/cm^3 \cdot mm \text{ Hg}$ ); in the latter (as indeed in the same range of energy for  $H_2$  and  $N_2$ ), the total cross section is falling off<sup>13</sup> (Fig. 3). In all these cases and ranges the rate of loss of energy in  $H_2$  is considerably greater for electrons of the same energies than in either  $N_2$  or He (*viz.* 7 and 14 times, respectively). Thus up to 80 percent of  $H_2$  or of He the loss of energy is almost entirely to  $H_2$  molecules, and we may expect, as observed, nearly the same form of dependence of  $\bar{u}$  on  $E/p$  with a coefficient varying with the concentration. If both  $\alpha$  and  $f$  varied in the same way with speed for both gases, the form of dependence of mobility coefficient on concentration could be predicted, e.g., for  $\alpha$  and  $f$  constant at values  $\alpha_1$  and  $f_1$  and  $\alpha_2$  and  $f_2$  for gases with concentrations  $x_1$  and  $x_2$ ,

$$\bar{u} = 3.17 \times 10^7 \left( \frac{E}{p} \right)^{\frac{1}{2}} \frac{(x_1 f_1 \alpha_1 + x_2 f_2 \alpha_2)^{\frac{1}{2}}}{(x_1 \alpha_1 + x_2 \alpha_2)^{\frac{1}{2}}}. \quad (3.5)$$

Although  $\alpha$  and  $f$  probably do not vary in this way we should still expect a slight gradual rise of  $\bar{u}$  to a maximum as the concentration of  $H_2$  diminishes, followed at last by a rather sudden fall to the drift velocity for pure  $N_2$  or He as the average energy of the electrons approaches that for pure  $N_2$  or He only as the last of the hydrogen is withdrawn.

It may be noted that if the collision cross section for momentum transfer  $\alpha(\epsilon)$  is proportional inversely to the speed, as is nearly true at above 1.5 ev in  $H_2$  or He or above 16 ev in  $N_2$ , the drift velocity (Section C, (1.6) below)

$$\bar{u} = 3.0 \times 10^7 \frac{E}{p} / \{ \epsilon^{\frac{1}{2}} \alpha(\epsilon) \}, \quad (3.6)$$

<sup>13</sup> C. E. Normand, reference 7.

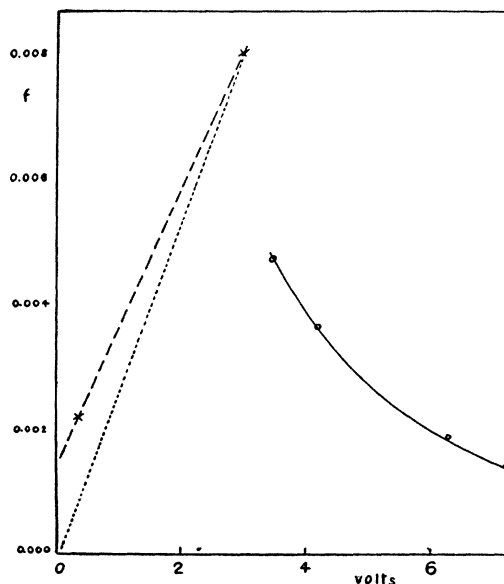


FIG. 5. Fraction of energy lost per collision. ——— Ramien; — — — Used in this paper; ······· Used by Nielsen and Bradbury.

no matter how energy is lost. (If  $f$  were constant, we should then have a Maxwell distribution with

$$\bar{\epsilon} = \frac{1}{2f} \frac{(E/p)^{\frac{1}{2}}}{\{ \epsilon^{\frac{1}{2}} \alpha(\epsilon) \}^2},^{14}$$

if  $f$  varied as  $\epsilon$ , we should have a Druyvesteyn distribution with

$$\bar{\epsilon} = 0.604 \frac{(E/p)}{\alpha(\epsilon) f(\epsilon)^{\frac{1}{2}}}.$$

In fact any values of  $\alpha(\epsilon)$  agreeing with Normand's values (above 0.5 ev) and of  $f(\epsilon)$  giving with these values of  $\alpha(\epsilon)$  the observed values of  $\bar{u}(E/p)$  between  $E/p=1$  and 2 (volt/cm)/mm Hg, will give also values of  $\bar{u}$  indistinguishable from those observed over the whole range (up to  $E/p=20$ ).<sup>15</sup> (Fig. 4.) The curve computed (from Section C, 2.52 below) agrees with the observations simply because it smoothly joins the parabola given by 3.2 or 3.21 for small  $E/p$  (for small  $\epsilon$  depending on  $\alpha/f^{\frac{1}{2}}$  only) to the straight line through the origin for large  $E/p$  given by 3.6 (for large  $\epsilon$  depending on  $\epsilon^{\frac{1}{2}} \alpha(\epsilon)$  only). That the curve approaches the

<sup>14</sup> Reference 9, p. 90.

<sup>15</sup> N. E. Bradbury and R. A. Nielsen, Phys. Rev. **49**, 391 (1936).

straight line from below is to be expected theoretically.

(Because they use Townsend's values of  $\bar{\epsilon}$  together with Normand's values of  $\alpha$  and  $f=2m/M$  to obtain their curve (B) (Fig. 5, reference 15), Bradbury and Nielsen reach the conclusion that the large values of  $f$  are necessary to explain the drift velocity values. The discrepancy between their curve (B) and the observed values of  $\bar{u}$  arises from the inconsistency of Normand's  $\alpha$  and  $f=2m/M$  with Townsend's values of  $\bar{\epsilon}$ , not from any inconsistency between these values of  $\alpha$  and  $f$  and the observed values of  $\bar{u}$ , and the conclusion that the values of  $f$  are large can only be obtained if Townsend's values of  $\bar{\epsilon}$  are used.)

Townsend's value  $\bar{\epsilon}=2.9$  ev for  $E/p=20$  ((volt/cm/mm Hg), thus implies that  $f$  increases to 0.008 for an electron energy of about 3 ev,<sup>15</sup> so that while the collision cross section for momentum transfer is decreasing, that for loss of energy is still increasing as the energy increases. While this value of  $f$  fits in with Ramien's direct observations<sup>16</sup> for energy 4 ev, the fairly steady increase from 0.0022 at 0.35 ev to 0.008 at 3 ev does not agree with his assumption that the only important process is the excitation of the first vibrational state, nor does the trend fit in with his observation that at 7 ev  $f$  has fallen to 0.0015. The trend seems indeed difficult to explain unless, as sufficient energy becomes available, molecular vibrational states of higher and higher energies may be excited (Fig. 5).

It should be emphasized that these conclusions about  $f$  might be greatly modified if it should turn out that Townsend's energies are unreliable.

### C. MATHEMATICAL APPENDIX

#### 1. Druyvesteyn's Equation

We shall adopt the notation used by Morse<sup>17</sup> with an electron distribution function

$$f(v, \theta) = f_0(v) + f_1(v) \cos \theta + \dots \quad (1.1)$$

Supposing that momentum transfer takes place almost entirely by elastic collisions, we can express  $f_1$  in terms of the derivatives of  $f_0$ .<sup>18</sup> When effects of diffusion on

<sup>15</sup> Hans Ramien, Zeits. f. Physik **70**, 353 (1931).

<sup>17</sup> P. M. Morse, W. P. Allis, and E. S. Lamar, Phys. Rev. **48**, 412 (1935).

<sup>18</sup> Reference 17, p. 414.

$f(v, \theta)$  are to be ignored,

$$f_1(v) = \frac{1}{NvQ(v)} \frac{eE}{m} \frac{\partial f_0(v)}{\partial v}. \quad (1.2)$$

(Here  $f(v, \theta)v^2 dv d\omega$  is the number of electrons per unit volume of charge  $-e$  and mass  $m$  moving with speed in range  $dv$  about  $v$  in direction within solid angle  $d\omega$  about a direction making angle  $\theta$  with an applied electric field  $E$ .  $N$  is the number of gas molecules per unit volume with "collision cross section for momentum transfer"  $Q(v)$  for electrons of speed  $v$ ; so that  $NQ(v)$  is the same as  $p\alpha(v)$  above.)

We neglect ionization and suppose that the cross section for an inelastic collision of an electron of speed  $v$  in which energy  $\delta_r$  is lost (or a superelastic collision in which  $-\delta_r$  is gained) is  $q_r(v)$ , so that the probability of such a collision per unit time is  $Nvq_r(v)$  and the net gain to the distribution function at  $v$  per unit time due to this is

$$(N/v) \{v'^2 q_r(v') f_0(v') - v^2 q_r(v) f_0(v)\}, \quad (1.3)$$

where  $v'^2 = v^2 + (2/m)\delta_r$ , (since the number of electrons of speed in  $dv'$  about  $v'$  per unit volume per unit solid angle is  $f_0(v')v'^2 dv'$ , and since  $v' dv' = v dv$ ).

The function  $f_0(v)$ , in order that it shall not change appreciably over a free path, must satisfy Druyvesteyn's equation in the form

$$-\frac{e^2 E^2}{m^2} \frac{4}{3v} \frac{\partial}{\partial v^2} \left\{ \frac{v^2}{NQ(v)} \frac{\partial}{\partial v^2} f_0(v) \right\} = \sum_r \frac{N}{v} \{v'^2 q_r(v') f_0(v') - v^2 q_r(v) f_0(v)\}, \quad (1.4)$$

with

$$\frac{v^2}{NQ(v)} \frac{\partial}{\partial v^2} f_0(v) \rightarrow 0 \quad \text{as } v \rightarrow 0.$$

The mean drift velocity of the electrons

$$\bar{u} = \frac{1}{3} \int_0^\infty f_1(v) v^3 dv / \int_0^\infty f_0(v) v^2 dv = \frac{eE}{m} \int_0^\infty \frac{1}{3} \frac{\partial}{\partial v} \left( \frac{v^2}{NQ(v)} \right) f_0(v) dv / \int_0^\infty f_0(v) v^2 dv \quad (1.5)$$

in this case.

If  $vQ(v)$  is constant, then the mean drift velocity is

$$\bar{u} = eE/mNvQ(v), \quad (1.6)$$

no matter how energy is lost.

#### 2. Solutions for Small Energy Loss per Collision

We shall suppose that the electric field is so large that the electronic energy is large compared with the thermal molecular energy. The contribution to the right-hand side of (1.4) from elastic collisions with molecules of mass  $M$  is then

$$\frac{2m}{M} \frac{1}{v} \frac{\partial}{\partial v^2} (v^4 NQ(v) f_0(v)). \quad (2.1)$$

If the energy,  $\delta_r$ , lost in a type of inelastic collision is only a small fraction of the total energy, the contribution from this inelastic scattering can be written

$$\frac{2}{m} \frac{\delta_r}{v} \frac{\partial}{\partial v^2} \{v^2 Nq_r(v) f_0(v)\}. \quad (2.2)$$

Thus Druyvesteyn's equation becomes

$$-\frac{e^2 E^2}{m^2} \frac{4}{3v} \frac{\partial}{\partial v^2} \left\{ \frac{v^2}{NQ(v)} \frac{\partial}{\partial v^2} f_0(v) \right\} = \frac{1}{v} \frac{\partial}{\partial v^2} [v^4 Nq(v) f_0(v)], \quad (2.3)$$

where

$$q(v) = 2 \frac{m}{M} Q(v) + \sum_r \frac{\delta_r}{\frac{1}{2} m v^2} q_r(v). \quad (2.31)$$

$q(v)/Q(v)$  is often<sup>19</sup> called the "fraction of energy lost per collision" and denoted by  $f$ .

If  $Q(v)$  and  $q(v)$  are known (2.3) can be integrated by quadrature.<sup>20</sup> In particular, if  $Q(v) = Q$  and  $q(v) = fQ$  are constant, we have the original case of Druyvesteyn:

$$f_0(v) = A \exp \left( -\frac{3}{8} \frac{m^2 N^2 f Q^2}{e^2 E^2} v^4 \right), \quad (2.41)$$

so that

$$\bar{u} = \frac{\pi^{\frac{1}{2}}}{3\Gamma(\frac{3}{4})} \left( \frac{3f}{2} \right)^{\frac{1}{4}} \left( \frac{2eE}{mNQ} \right)^{\frac{1}{2}} = 0.7546 f^{\frac{1}{4}} \left( \frac{eE}{mNQ} \right)^{\frac{1}{2}}, \quad (2.42)$$

while the mean energy of the electrons

$$\bar{\epsilon} = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} \left( \frac{2e^2 E^2}{3fN^2 Q^2} \right)^{\frac{1}{2}} = 0.6039 f^{-\frac{1}{2}} \frac{eE}{NQ}. \quad (2.43)$$

More generally, if  $Q(v) = Q_0/(1+(v/v_0)^4)^{\frac{1}{2}}$ , and  $q(v) = fQ(v)$ , where  $Q_0$ ,  $v_0$ , and  $f$  are constant, so that the cross sections are constant for small  $v$  and vary inversely as  $v$  for large  $v$ , but remain in the same ratio;

$$f_0(v) = A \exp [-\gamma v_0^4 \{1 + (v/v_0)^4\}^{\frac{1}{2}} - 1], \quad (2.51)$$

and we obtain

$$\bar{u} = \frac{\pi^{1/2}}{3\Gamma(\frac{3}{4})} \left( \frac{3f}{2} \right)^{1/4} \left( \frac{2eE}{mNQ_0} \right)^{1/2} \frac{K_{1/4}(\gamma)^2 + 3K_{3/4}(\gamma)K_{5/4}(\gamma)}{8(\pi/4\gamma)^{1/2} K_{5/4}(2\gamma)}, \quad (2.52)$$

and

$$\bar{\epsilon} = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} \left( \frac{2e^2 E^2}{3fN^2 Q_0^2} \right)^{1/2} \frac{K_{1/4}(2\gamma)}{K_{5/4}(2\gamma)}, \quad (2.53)$$

where  $\gamma = \frac{3}{8} (m^2 N^2 f Q_0^2 / e^2 E^2) v_0^4$ , and  $K_n(\gamma)$  is the Bessel function of order  $n$  and of pure imaginary argument,  $\sim (\pi/2\gamma)^{\frac{1}{2}} e^{-\gamma}$  as  $\gamma \rightarrow \infty$ .

### 3. Solutions if the Loss of Energy in a Collision May be Large

If energy can be lost by an electron of speed  $v'$  by amounts  $\delta_r$  with cross sections  $q_r(v')$  distributed nearly

continuously from 0 to  $\frac{1}{2} m v'^2$ , and if we write

$$q_r(v') = q(v', v) \frac{\delta_{r+1} - \delta_r}{\frac{1}{2} m v'^2} \quad v'^2 > \frac{2}{m} \delta_r$$

$$= 0 \quad v'^2 < \frac{2}{m} \delta_r$$

so that

$$\sum_r \left( \frac{\delta_r}{m} \right)^2 (\delta_{r+1} - \delta_r)$$

becomes  $\int ( ) 2v dv$ , we obtain for the right-hand side of (1.4)

$$\frac{1}{v} \left[ \int_0^\infty Nq(v', v) f_0(v') 2v' dv' - \int_0^v Nq(v, v') 2v' dv' f_0(v) \right]. \quad (3.1)$$

If we neglect loss of energy in other ways, (1.4) can be reduced to a linear differential equation of the second order when  $q(v, v')$  is of the form  $\alpha(v')\beta(v)$ . In particular, if  $Q(v) = Q$  and  $q(v', v) = 2fQ$  (so that  $f$  is still the "fraction of energy lost per collision") are constant, the distribution is Maxwellian;

$$f_0(v) = A \exp \left( -\left( \frac{3}{2} f \right)^{\frac{1}{2}} \frac{mNQ}{eE} v^2 \right), \quad (3.21)$$

and

$$\bar{u} = \frac{2^{\frac{1}{2}}}{3\pi^{\frac{1}{2}}} \left( \frac{3}{2} f \right)^{\frac{1}{4}} \left( \frac{2eE}{mNQ} \right)^{\frac{1}{2}} = 0.8325 f^{\frac{1}{4}} \left( \frac{eE}{mNQ} \right)^{\frac{1}{2}}, \quad (3.22)$$

$$\bar{\epsilon} = \frac{3}{4} \left( \frac{2e^2 E^2}{3fN^2 Q^2} \right)^{\frac{1}{2}} = 0.6123 f^{-\frac{1}{2}} \frac{eE}{NQ}. \quad (3.23)$$

### 4. Townsend's Data

Townsend's experiments lead to values for other averages over the electron distribution than  $\bar{u}$  and  $\bar{\epsilon}$ .<sup>21</sup> In fact his

$$U^2 = \int_0^\infty \frac{v}{NQ(v)} f_0(v) v^2 dv / \int_0^\infty \frac{1}{3} \frac{\partial}{\partial v} \left( \frac{v^2}{NQ(v)} \right) f_0(v) dv, \quad (4.1)$$

and

$$W = \frac{eE}{m} \int_0^\infty \frac{\partial}{\partial v} \left( \frac{v}{(NQ(v))^2} \right) f_0(v) dv / \int_0^\infty \frac{\partial}{\partial v} \left( \frac{v^2}{NQ(v)} \right) f_0(v) dv, \quad (4.2)$$

and  $W = \bar{u}$  in general only if  $Q(v) \propto 1/v$ ;  $\frac{1}{2} m U^2 = \bar{\epsilon}$  only if  $Q(v) \propto 1/v$  or if the distribution is Maxwellian. However, in all the cases we have considered, the ratios  $W/\bar{u}$  and  $\frac{1}{2} m U^2/\bar{\epsilon}$  differ inappreciably from unity.

<sup>19</sup> Reference 9, p. 93.

<sup>20</sup> Reference 17, p. 416.

<sup>21</sup> W. P. Allis and H. W. Allen, Phys. Rev. 52, 703 (1937).