On Discharge Voltage and Return Voltage Curves for Absorptive Capacitors

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Formulas for the initial slope of the discharge voltage and return voltage curves in absorptive capacitors are given and it is shown that the discharge voltage curves are limited by two threshold curves, one of them being directly proportional to the slope of the other one. Some applications for the determination of the dielectric relaxation function, the dielectric loss, the time constant, and the electrical charge of absorptive capacitors are given.

P. BÖNING[1] has recently established some relations connecting the initial slope of the return voltage curves with the dielectric loss and other characteristic properties of absorptive capacitors. A brief account of some of his deductions and results can be found in a paper by Whitehead and Eager;[2] these authors have performed an experimental test of one of Böning's relations. Böning's analysis is based upon Maxwell's theory of the two-layer dielectric. It has been recognized that this theory yields qualitative results only—the behavior of real dielectrics is much too complex to be accounted for by so simple a model. Therefore Böning's formulas claim an approximate validity only. Yet by the application of the principle of superposition there can be obtained a rigorous treatment of the discharge voltage and return voltage curves observed in absorptive capacitors and of the way these curves may find useful applications in the technique of measurement. The validity of the principle of superposition for capacitors with solid dielectrics seems to be sufficiently established on experimental as well as theoretical grounds.

I. THE SUDDEN OPEN CIRCUIT

Consider a capacitor forming part of an electrical network. At a reference time $t=0$ there shall be a sudden open circuit in the branch containing the capacitor so that the terminals of the capacitor become completely isolated one from another and from the rest of the network. After the opening of the branch, the total current across the capacitor must be zero. If the principle of superposition is valid, this current can be expressed in terms of the geometric capacitance C, the ohmic resistance R, and the dielectric relaxation function $\varphi(t)$. Hence[3]

$$t>0: J(t)=C\frac{dU}{dt}+\frac{U}{R}+\int_0^t \frac{dU}{d\tau}\varphi(t-\tau)d\tau+i(t)=0. \quad (1)$$

The solution of (1) gives the voltage at the terminals of the capacitor after the sudden open circuit.

With the introduction of a linear operator $F[U]$ this equation can be written in the shortened form

$$F[U]+i(t)=0. \quad (2)$$

During the period of time previous to the sudden open circuit, i.e., for $t<0$, the capacitor shall have been energized by a given voltage $U(t)$. In an anomalous dielectric, every variation of electric stress produces a current which continues flowing even after the stress has ceased. The term $i(t)$ denotes the sum of the currents created during the period of energizing and still continuing after the insulating of the terminals; $i(t)$ is given as a function of $U(t)$ as follows:

$$i(t)=\int_{-\infty}^0 \frac{dU}{d\tau}\varphi(t-\tau)d\tau. \quad (3)$$

It is this term that causes the rise of voltage at the terminals of a capacitor which has been short-circuited temporarily. Equation (1) demonstrates that this voltage must be different from

[1] P. Böning, Zeits. f. tech. Physik 19, 241 (1938).
[2] J. B. Whitehead and G. S. Eager, J. App. Phys. 13, 43 (1942).
[3] B. Gross, Phys. Rev. 57, 57 (1940).

zero so long as $i(t)$ is different from zero. $U(t)$ can remain zero only if the short circuit has lasted so long that at its end $i(t)$ can be neglected.

The term $\int_0^t$ in Eq. (1) represents the absorption current which surges in the dielectric in consequence of the voltage variations which occur after the sudden open circuit.

Immediately before and after the sudden open circuit the voltage at the terminals must be the same, for it can never jump spontaneously. So for $t=0$ one obtains the condition of continuity

$$\lim_{\epsilon=0} U(0-\epsilon)=\lim_{\epsilon=0} U(0+\epsilon). \tag{4}$$

It may be stated that this equation is not valid for a circuit where an inductor is put in series with the capacitor; in this case there would be a voltage jump at $t=0$.

It is easily seen that (1) can also be written in the form

$$(dQ/dt)+U/R=0, \tag{5}$$

where

$$Q=CU+\int_{-\infty}^{t}\frac{dU}{d\tau}\int_0^{t-\tau}\varphi(\sigma)d\sigma d\tau. \tag{6}$$

With the introduction of the quantity Q, the equation of the absorptive capacitor has the same form as that of the non-absorptive capacitor.

In the case of a leakfree capacitor ($R=\infty$), no loss of electricity could occur after the open circuit because there would be no ohmic conduction. The quantity of electricity held by the capacitor at the moment of the open circuit would remain constant so long as the terminals remain insulated. If in (5) we put $U/R=0$, we have $Q=\text{const.}=Q(0)$. In the case of a leakfree capacitor, Q would obey a law of conservation. We therefore identify $Q(t)$ with the total amount of electricity stored at any moment by the capacitor. CU is the quantity of electricity contained by the geometric capacitance and $\int_{-\infty}^t$ the quantity of electricity absorbed by the dielectric.

Consideration should be given to the discontinuous variations of the applied voltage. At the moment of a voltage jump, dU/dt becomes infinite. In consequence, the integrals (1), (3), and (6) apparently become indeterminate. But it is easy to see what values are to be attributed to the integrals in these cases. Every voltage jump $\Delta U(\tau_i)$ is followed by an absorption current $\Delta U\varphi(t-\tau_i)$. In the present case, for every discontinuous variation of $U(t)$, the value of an integral of the type $\int dU/d\tau\varphi(t-\tau)d\tau$, as calculated by ordinary methods, must be increased by a term $\Delta U_i\varphi(t-\tau_i)$. It is in this sense we are using the symbol $\int$. It is easily seen that the definition of the integral to which we are led by empirical considerations coincides with the definition of Stieltjes' integral. Indeed, as it has been pointed out to us by F. M. de Oliveira Castro, the treatment can be made rigorous from a mathematical point of view if one writes the principle of superposition in the form of Stieltjes' integral

$$\int\varphi(t-\tau)dU.$$

II. THE INITIAL SLOPE OF THE DISCHARGE VOLTAGE AND RETURN VOLTAGE CURVES AND THE MEASUREMENT OF THE RELAXATION FUNCTION

For the limit $t=0$, i.e., just at the moment when the sudden open circuit occurs and the terminals become insulated, the integral in (1) disappears. U has still the value it possessed at the end of the energizing period according to (3). Thus

$$\left(\frac{dU}{dt}\right)_0=-\frac{U(0)}{RC}-\frac{i(0)}{C}. \tag{7}$$

Now consider the curve of return voltage. The capacitor has been charged during a very long time under a constant voltage E and then short-circuited during an interval T. The voltage jump of magnitude $-E$, occurring at the beginning of the short circuit, is followed by an absorption current which at an instant t is given by $i(t)=-E\varphi(t+T)$. $U(0)$ is zero. After the insulating of the terminals, the voltage rises again. The initial slope of this *return voltage* curve is therefore

$$\left(\frac{dU}{dt}\right)_0=+\frac{E}{C}\varphi(T). \tag{8}$$

One can just as easily obtain the initial slope of the discharge voltage curve. In this case, before the terminals have become insulated the capacitor has been charged under a constant voltage E for a period T. The voltage jump at the beginning of the energizing period, of magnitude $+E$, produces an absorption current given by $i(t)=E\varphi(t+T)$. Furthermore, $U(0)=E$. After the terminals have been insulated, the voltage begins to drop steadily. The initial slope of this *discharge voltage curve* is given by the equation

$$\left(\frac{dU}{dt}\right)_0=-\frac{E}{RC}-\frac{E}{C}\varphi(T). \tag{9}$$

The initial slope of the discharge curve obtained after a complete charge ($T=\infty$; $\varphi(T)=0$) is expressed by $-E/RC$. Its determination enables one therefore to obtain the characteristic time constant RC of the capacitor. The time it takes to charge completely an absorptive capacitor is generally of the order of many hours, even for low loss materials like mica and sulfur.

The initial slope of the discharge voltage curve observed after the capacitor has been charged during an extremely short time ($T\rightarrow 0$) gives a measure of the initial value of the relaxation function.

It is interesting to note that *at the beginning* of the discharge the voltage at the terminals of an absorptive capacitor decreases more rapidly than does the voltage of that non-absorptive capacitor which has the same ohmic resistance and geometric capacitance as the first one. Afterwards the discharge of the first one becomes slower. The initial slopes of the discharge voltage and return voltage curves are related one to another in a very simple way. Their sum is constant and given by $-E/RC$.

Once the value of the geometric capacitance is known, the measurement of the initial slopes of the return voltage or discharge voltage curves permits the calculation of the relaxation function by the formulas (8) and (9). In many cases for the direct determination of this function obtained by recording the discharge current of a short-circuited capacitor, this method may be substituted advantageously. When concerned with large values of t, $\varphi(t)$ becomes so small that only an electrometric method can give satisfactory results.

The dielectric loss, characterized by the tangent of the loss angle δ, can be split up into two parts, one due to the ohmic conductivity and another one due to the absorption current. The first part is given by $1/\omega RC$. According to Schweidler[4] the second one can be calculated if the relaxation function is known. It is

$$\tan\delta=\frac{1/\omega R+\int_0^\infty \varphi(u)\sin\omega u du}{C+\int_0^\infty \varphi(u)\cos\omega u du}. \tag{10}$$

The relations (8) and (9) give the connection between the dielectric relaxation function and the initial slopes of the return voltage or discharge voltage curves. We are therefore able to correlate the dielectric loss with these functions, substituting in (10) for $\varphi(u)$ its values obtained, respectively, from (8) or (9). The formulas so obtained do not depend on any model circuit but suppose only the validity of the principle of superposition.

For very high frequencies, the peak value of the component of the current in phase with the applied voltage is given by $A\varphi(0)+A/R$, and the peak value of the component in quadrature with the applied voltage is given by ωAC where A is the peak value of the applied voltage. Then we have Böning's formula

$$\tan\delta=\frac{\varphi(0)+\frac{1}{R}}{\omega C}=\frac{\tan\alpha}{\omega E}+\frac{1}{\omega RC},$$

where $\tan\alpha$ denotes the initial slope of the return voltage curve observed after a short circuit of extremely small duration. The critical value of ω, for which this relation becomes valid, can be inferred from the condition

$$\varphi\left(\frac{2\pi}{\omega}\right)\backsimeq\varphi(0).$$

[4] E. v. Schweidler, Ann. d. Physik **24**, 711 (1907).

III. THE THRESHOLD VALUES FOR THE DISCHARGE VOLTAGE AND RETURN VOLTAGE CURVES AND THE MEASUREMENT OF THE TIME CONSTANT

The application of the relations (8) and (9) requires the knowledge of the time constant RC and the geometric capacitance C. There are many other cases where one needs to know RC, which is perhaps the most important quantity for characterizing the behavior of a capacitor.

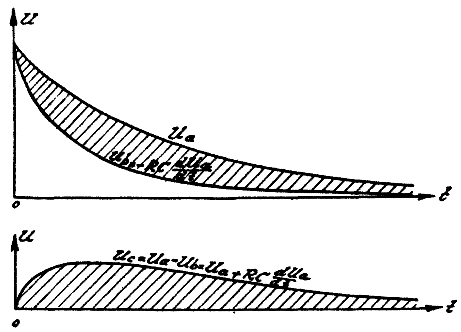

FIG. 1.

A conventional laboratory method for measuring RC consists in observing the discharge voltage curve of the capacitor. If the voltage during the time t drops from V_1 to V_2, RC is calculated by the well-known formula $t/\ln(V_1/V_2)$. In absorptive capacitors this formula cannot be applied because there the simple exponential law, upon which it is based, is no longer valid. One of the first tasks of all theories concerning the general behavior of absorptive capacitors has to be the deduction of a sufficiently precise method for determining RC. The method referred to above does not fulfill this requirement about precision because it implies the derivation of an experimentally given curve. In the following, there are developed some simple and general laws which provide a method of the kind needed, and help at the same time to come to a closer understanding of the behavior of absorptive capacitors.

The discharge voltage curve of an absorptive capacitor depends to a great extent on the duration of the previous charging period, decreasing rapidly if this period has been short and decreasing slowly if it has been long. A similar relation exists between the duration of the short circuit and the rise of the return voltage, which increases with decreasing duration of the short circuit. In both cases there exist threshold values.

The *slowest of all discharge voltage curves* $U_a(t)$ is observed after the capacitor has been charged under a continuous voltage E during a time T so long that any further charging would no more increase the quantity of electricity held by this capacitor. There

$$\int_T^\infty \varphi(t)dt \simeq 0; \quad i(t)=0$$

and

$$F[U_a]=0, \tag{11a}$$

$$U_a(0)=E. \tag{11b}$$

The *fastest of all discharge voltage curves* $U_b(t)$ is observed if the voltage E has been applied during a time T so short that there has not yet been any absorption of electricity in the dielectric. The quantity of electricity held by the capacitor is still equal to EC. There

$$\int_0^T \varphi(t)dt \simeq 0; \quad i(t)=E\varphi(t)$$

and

$$F[U_b]+E\varphi(t)=0, \tag{12a}$$

$$U_b(0)=E. \tag{12b}$$

The *fastest of all return voltage curves* $U_c(t)$ is observed if the capacitor, after having been charged under the continuous voltage E during an extremely long time, is short-circuited during a time T so short that it has lost only the quantity of electricity CE contained by the geometric capacitance, whereas the quantity of electricity stored in the dielectric remains unchanged. There

$$\int_0^T \varphi(t)dt \simeq 0; \quad i(t)=-E\varphi(t)$$

and

$$F[U_c]-E\varphi(t)=0, \tag{13a}$$

$$U_c(0)=0. \tag{13b}$$

To obtain a direct relationship between these voltage curves, differentiate (11a). Noting that

$$\frac{d}{dt}\int_0^t \frac{dU_a}{d\tau}\varphi(t-\tau)d\tau$$
$$=\int_0^t \frac{d^2U_a}{d\tau^2}\varphi(t-\tau)d\tau+\left(\frac{dU_a}{dt}\right)_0\varphi(t) \quad (14)$$

and taking into account (9), we obtain

$$F\left[-RC\frac{dU_a}{dt}\right]+E\varphi(t)=0, \quad (15a)$$

$$-\left(RC\frac{dU_a}{dt}\right)_0=E. \quad (15b)$$

Comparing Eqs. (12) and (15) one arrives at the relationship between U_a and U_b

$$U_b=-RC\frac{dU_a}{dt}. \quad (16)$$

Comparing (11), (12), and (13) one observes[5]

$$U_c=U_a-U_b \quad (17)$$

and therefore

$$U_c=U_a+RC\frac{dU_a}{dt}. \quad (18)$$

All discharge voltage curves are limited by two threshold curves, one of them being directly proportional to the slope of the other one. The difference of these curves gives the threshold for all the return voltage curves. (See Fig. 1.)

The relations (16) and (18) are generalizations of the differential equation of the non-absorptive capacitor

$$U=-RC(dU/dt), \quad (19)$$

which is obtained as a particular case for $U_a=U_b$ and, consequently, $U_c=0$.

Integrating (16), we have

$$RC=\frac{\int_0^t U_b d\tau}{U_a(0)-U_a(t)}. \quad (20)$$

This relationship seems to be very suitable for an experimental determination of RC and of R and C alone if the measurements are repeated with a loss-free capacitor of known capacity connected in parallel to the test capacitor.

Closely related to the measurement of the geometric capacitance is the determination of the quantity of electricity held by the capacitor. For this purpose, Eq. (15) suggests a simple method. By integrating (15) from 0 to ∞ we obtain

$$Q(0)=\frac{1}{R}\int_0^\infty Udt. \quad (21)$$

$1/R$ times the integral over the whole discharge voltage (or return voltage) curve gives directly the quantity of electricity the capacitor held at the beginning of these curves. On the other hand, Q could be calculated if one performs an integration over the absorption current. Because it is simpler to determine

$$\int_0 Udt$$

than it is to determine

$$\int_0 \varphi dt,$$

the first method seems to be more suitable.

I am greatly indebted to Professor E. L. da Fonseca Costa who made this study possible.

[5] B. Gross, reference 3, Eq. (12).