The Problem of the Rotating Disk

CARLTON W. BERENDA
College of the City of New York, New York, New York
(Received May 29, 1942)

In Part I of this paper, the spatial geometry of the surface of a rotating disk is examined from the standpoint of general relativity theory. Eddington's argument for a homaloidal surface is shown to be in error, and Einstein's "relative" geometry is correlated with the "intrinsic" geometry of the disk (i.e., the geometry as determined by an observer at rest on the rotating disk). The Gaussian measure of hypercurvature of the surface, at any point on the disk at radius r is found to be $-3\frac{\omega^2}{c^2}\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-2}$. In Part II, the temporal aspects of the rotating disk are examined and a new test of general relativity, by use of the cyclotron, is proposed: an artificially radioactive element of low atomic weight is revolved, as ions, within the cyclotron. Upon being brought to rest, the element should be found *more radioactive* than an equivalent sample of that element remaining at rest.

PART I. GEOMETRY

THE problem of the rotating disk, in relativity theory, is associated with the names of Ehrenfest, Einstein, Lorentz, and Eddington.

Einstein's Geometry

Einstein and Infeld have argued that a rigid disk under uniform angular velocity ω relative to a galilean frame, will exhibit a non-euclidean geometry in the definite sense that the circumference of the disk will no longer equal $2\pi r$, where r is the radius of the disk. More precisely, we fix small rigid rods along the entire disk's periphery at right angles to the radii of the disk. We also surround the edge of the disk with similar, small, rigid rods at rest, in galilean space, with respect to the disk's center. Let us measure such a rod $\bar{P}$ fixed on the disk at r as it passes any rod P' fixed in galilean space. Then relative to the G (galilean) observer, who uses the rod P', the measured length of the adjacent rod on the disk is

$$\bar{P}=P'\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}},$$

where $v=\omega r$. Hence $\bar{P}<P'$. This is in accordance with the special theory, and appears to be correct, even though, as we shall see, the complete analysis of the problem requires the formulae of the general theory and superimposes an *additional* effect upon the measurement of $\bar{P}$. Rods at right angles to P (i.e., rods directed along the radius) will not contract, by the special theory, since r is at right angles to the line of motion (v). Hence the measured radius is not affected by the rotation. Each G observer finds the adjacent rod on the disk to have the length $\bar{P}$. If we now add up the measuremental results of the G observers all around the disk, we find

$$\bar{C}=\sum\bar{P}=\sum P'\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}=C'\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}<C',$$

where

$$\sum P'=C'=2\pi r$$

which is the circumference of the disk when it is at rest in the G frame; and $\bar{C}$ is the circumference of the rotating disk relative to the G observers. Then

$$\bar{C}=2\pi r\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}<2\pi r,$$

so that the geometry of the rotating disk is apparently non-euclidean.[1]

Opposed to this point of view are A. S. Eddington, H. A. Lorentz, and H. Levy who contend that the geometry of the rotating disk remains euclidean.[2]

The first part of this paper will attempt to demonstrate the error in Eddington's argument, as well as to distinguish between the "relative"

[1] Cf. A. Einstein and L. Infeld, *Evolution of Physics*, pp. 240–42; also Infeld's review, in *Science and Society* **4**, 236 (1940), of H. Levy's *Modern Science* (p. 595).

[2] A. Eddington, *Mathematical Theory of Relativity* (1923), §4 and p. 36; *Space, Time and Gravitation*, p. 75. H. Lorentz, *Nature* **106**, 795; *Collected Papers*, Vol. 7 (1934), pp. 171–72.

geometry of Einstein and an "intrinsic" geometry obtained by an observer at rest on the rotating disk. It is shown that the intrinsic geometry of the disk is one whose Gaussian measure of surface hypercurvature is negative and variable. Superimposed on the intrinsic geometry there is, for a galilean observer, the relative geometry such that

$$\bar{C}=C_0\left(1-\frac{v^2}{c^2}\right),$$

where C_0 is the circumference of the rotating disk as measured by an observer at rest on the disk.

$$C_0=2\pi r\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}>2\pi r.$$

Transformation Equations

To deal with the problem systematically, we may proceed as follows: Allow a G observer to assign cartesian or polar coordinates fixed on the rotating disk, with the center O of the disk as the origin of the coordinates. To make this assignment, we have to consider the relation of our galilean measures x', y' to the rotating coordinates x, y. The constant angular velocity of the disk is $\omega=d\theta'/dt'$. Now $d\theta'$ is the measured angle element swept out by a radial coordinate on the disk, as measured from some base line fixed in G space and originating at O. The time dt' is the time required for sweeping out $d\theta'$, and is measured at O. We then take

$$\begin{aligned} x'&=x\cos\omega t-y\sin\omega t \\ y'&=x\sin\omega t+y\cos\omega t \quad t'=t, \end{aligned} \tag{1}$$

to which we may add $z'=z$, since the rotation is around the z axis at O. These equations *define* the cartesian coordinates on the disk. They are the equations of motion of a point fixed on the disk rotating in G space.

We must briefly consider the time coördinate. We may place a clock at O in G space. In assigning a coordinate time to events fixed on the disk, we do not correct for the time light requires to pass from the event to the clock at O. We simply assign time coordinate values to events on the disk as observed at O, using the G clock. If we call t the coordinate time of an event anywhere on the disk, then we identify the coordinate time t with the time t' of the event's observation as measured by the G clock at O. Hence the above equation $t'=t$. It follows from these equations that

$$x'^2+y'^2=x^2+y^2,$$

so that the radial cartesian coordinate vector r equals the galilean measured value r'.

We now employ the rule that G space and time readings satisfy the relation

$$ds^2=dt'^2-\frac{1}{c^2}[dx'^2+dy'^2+dz'^2],$$

where $|ds|$ is an invariant with the dimensions of seconds. From our transformation equations we obtain:

$$\begin{aligned} dx'&=\cos\omega t\cdot dx-\sin\omega t\cdot dy-\omega(x\sin\omega t+y\cos\omega t)dt \\ dy'&=\sin\omega t\cdot dx+\cos\omega t\cdot dy+\omega(x\cos\omega t-y\sin\omega t)dt \\ dz'&=dz;\quad dt'=dt. \end{aligned}$$

Substituting these values in the space-time equation we find that

$$ds^2=\left[1-\frac{\omega^2}{c^2}(x^2+y^2)\right]dt^2-\frac{1}{c^2}[-2\omega ydxdt +2\omega xdydt+dx^2+dy^2+dz^2]. \tag{2}$$

This may also be written in centimeters as $d\bar{s}^2=c^2ds^2$.

Eddington's Geometry

I shall now present Eddington's argument apparently demonstrating that both the radius and the circumference of a rotating disk contract by the same amount; we therewith obtain, or assume, homaloidal space. The following argument appears legitimate up to a certain point where objections will be raised.

Consider a disk having absolute rigidity, i.e., it is incompressible under duress of force, or, its moduli of elasticity are infinitely great. Thus, such a disk, resting in a G frame, if subjected to any external forces upon its rim, would undergo no contraction, extension, or torsion.

We now apply an equation obtained from a theorem of Jacobi, showing that between fixed limits of integration we have

$$\int\int\int\int(-g')^{\frac{1}{2}}\cdot dx'_1dx'_2dx'_3dx'_4$$
$$=\int\int\int\int(-g)^{\frac{1}{2}}\cdot dx_1dx_2dx_3dx_4,$$

where $x_4=ct$ and g is negative and is the determinant of $g_{\lambda\mu}$ in the line element of four dimensions

$$ds^2=g_{\alpha\beta}dx_\alpha dx_\beta.$$

In infinitesimal regions this gives

$$dT=(-g)^{\frac{1}{2}}dx_1dx_2dx_3dx_4,$$

where dT is an invariant four-dimensional volume element that can be written in proper measures as

$$dT=dx_0dy_0dz_0d\bar{s}_0,$$

where the proper time, in centimeters, is

$$d\bar{s}_0=(g_{44})^{\frac{1}{2}}\cdot dx_4.$$

We then obtain

$$dx_0dy_0dz_0=(-g/g_{44})^{\frac{1}{2}}dx_1dx_2dx_3. \qquad \text{(A)}$$

Now in the expression (from which $d\bar{s}_0$ is obtained):

$$d\bar{s}^2=\left[1-\frac{\omega^2}{c^2}(x^2+y^2)\right]dx_4{}^2-\left[-\frac{2\omega y}{c}dxdx_4+\frac{2\omega x}{c}dydx_4+dx^2+dy^2+dz^2\right]$$

we have

$$g=\begin{vmatrix} -1 & 0 & 0 & \dfrac{\omega y}{c} \\ 0 & -1 & 0 & \dfrac{-\omega x}{c} \\ 0 & 0 & -1 & 0 \\ \dfrac{\omega y}{c} & \dfrac{-\omega x}{c} & 0 & \left[1-\dfrac{\omega^2}{c^2}(x^2+y^2)\right] \end{vmatrix}$$

from which $g=-1$ and

$$g_{44}=\left[1-\frac{\omega^2}{c^2}(x^2+y^2)\right]=\left(1-\frac{\omega^2r^2}{c^2}\right).$$

From (A) we then obtain

$$dx_0dy_0dz_0=\left(1-\frac{\omega^2r^2}{c^2}\right)^{-\frac{1}{2}}dxdydz. \qquad \text{(B)}$$

It is now convenient to transform to polar coordinates. The element of area is

$$dxdy=rd\theta dr.$$

Then in (B):

$$r_0d\theta_0dr_0dz_0=\left(1-\frac{\omega^2r^2}{c^2}\right)^{-\frac{1}{2}}rd\theta drdz.$$

Since there is no motion along the z axis, $z=z_0$. Hence

$$r_0d\theta_0dr_0=\left(1-\frac{\omega^2r^2}{c^2}\right)^{-\frac{1}{2}}rd\theta dr. \qquad \text{(C)}$$

It is at this point that our objection to Eddington's argument begins. Eddington sets

$$r_0dr_0=\left(1-\frac{\omega^2r^2}{c^2}\right)^{-\frac{1}{2}}rdr. \qquad \text{(C')}$$

Then

$$\int_0^{r_0}r_0dr_0=\int_0^r\left\{1-\frac{\omega^2r^2}{c^2}\right\}^{-\frac{1}{2}}rdr$$

or

$$\tfrac{1}{2}r_0{}^2=\frac{c^2}{\omega^2}\left\{1-\left(1-\frac{\omega^2r^2}{c^2}\right)^{\frac{1}{2}}\right\}$$

or finally

$$r=r_0\left(1-\frac{\omega^2r_0{}^2}{4c^2}\right)^{\frac{1}{2}}. \qquad \text{(C'')}$$

This gives, to a first approximation

$$r=r_0\left(1-\frac{\omega^2r_0{}^2}{8c^2}\right),$$

which is the result obtained by Eddington and Lorentz.

Up to and including Eq. (C), there seems to be no valid objection to the argument, but in stating (C′), Eddington is assuming that $d\theta_0=d\theta$. This, however, is the whole point at issue, since the latter assumption is equivalent to the postulate that angular measures are unaffected by rotation, i.e., that the geometry remains euclidean. In fact, Eddington and Lorentz show that circumference and radius both contract in the same ratio relative to a G observer. Thus, an element of the relative circumference would be

$$dC = r d\theta.$$

Likewise, an element of the proper circumference would be

$$dC_0 = r_0 d\theta_0.$$

But from the Eddington argument $\theta = \theta_0$ and

$$C = r\int_0^{2\pi} d\theta,$$

then

$$C = \left(1 - \frac{\omega^2 r_0^2}{4c^2}\right)^{\frac{1}{2}} r_0 \int_0^{2\pi} d\theta_0 = \left(1 - \frac{\omega^2 r_0^2}{4c^2}\right)^{\frac{1}{2}} C_0.$$

Now it is true that the necessary and sufficient condition for "flat" or homaloidal space-time is the vanishing of the Riemann-Christoffel tensor, that is

$$R^{\tau}_{\mu\nu\sigma} = 0 \quad \text{where} \quad \mu, \nu, \sigma, \tau = 1, 2, 3, 4.$$

And this condition is satisfied when one can transform away the *entire* gravitational field. This is implicit in the *integral* transformation Eqs. (1) for x, y, z, t.[3]

However, this argument is valid only for the geometry of space-time—it is not necessarily valid for a sub-space such as the surface of the rotating disk. In general, the condition for homaloidality in n space, namely,

$$R^{\tau}_{\mu\nu\sigma} = 0 \quad \text{where} \quad \mu, \nu, \sigma, \tau = 1, 2, \cdots, n$$

is not always applicable to a space of $n - m$ dimensions, where $m \leqq n - 1$. Of course, the conditions

$$R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha} = 0$$

and

$$R = g^{\alpha\beta} R_{\alpha\beta} = 0$$

are also satisfied for the space-time ($n = 4$) of the disk, but not necessarily for the surface space ($n = 2$) of the disk.

Intrinsic Geometry

Another confusion that appears to arise, in arguing for homaloidal space on the disk, is the improper inference that, to obtain the proper spatial geometry, it is sufficient to take t as constant in the equation

$$d\bar{s}^2 = c^2\left[1 - \frac{\omega^2}{c^2}(x^2 + y^2)\right]dt^2 - [-2\omega y dx dt + 2\omega x dy dt + dx^2 + dy^2 + dz^2],$$

thereby obtaining

$$d\bar{s}_1^2 = -[dx^2 + dy^2 + dz^2].$$

Defining the proper spatial interval as

$$dl = i d\bar{s}_1,$$

we have

$$dl = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}},$$

therewith apparently indicating euclidean geometry once more.

The error in this argument was pointed out by M. von Laue.[4] In order to obtain the purely spatial geometry of the disk, one must select a vector at *right angles* to the world lines of points fixed on the disk. It is not sufficient, for this purpose, to set x_4 equal to a constant and therefore $dx_4 = 0$. The latter is permissible only when $g_{a4} = 0$ for $a = 1, 2, 3$—that is, *if the time axis is everywhere at right angles to the spatial extension.* Parenthetically, it should be remarked that while it is a quite natural geometrical requirement to obtain the three-dimensional sub-space from space-time through this orthogonality condition, it does not follow logically from this, that we may identify rigid metric rod readings on the disk with spatial dimensions and geometry derived from such orthogonality. The identification is a matter of assumption—a convenient one no doubt. This assumption appears related to a fundamental arbitrary feature in the definition of a rigid body that I have dealt with more fully elsewhere.[5]

To return to the basic argument, let us transform our line element $d\bar{s}$ through the equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

[3] See R. Tolman, *Relativity, Thermodynamics and Cosmology,* pp. 185–86; R. Lindsay and H. Margenau, *Foundations of Physics,* pp. 360–63.

[4] M. von Laue, *Die Relativitätstheorie,* Vol. 2, pp. 142–43.

[5] Carlton B. Weinberg, Phil. Sci. **8**, No. 4, pp. 506–532; 618–623.

Then we may write

$$-d\bar{s}^2=dr^2+r^2d\theta^2+dz^2+\frac{r^2\omega}{c}d\theta dct+\frac{r^2\omega}{c}dctd\theta-\left(1-\frac{\omega^2r^2}{c^2}\right)dc^2t^2$$

or

$$-d\bar{s}^2=dx_1{}^2+r^2dx_2{}^2+dx_3{}^2+\frac{2r^2\omega}{c}dx_2dx_4-\left(1-\frac{\omega^2r^2}{c^2}\right)dx_4{}^2$$
$$=g_{11}dx_1{}^2+g_{22}dx_2{}^2+g_{33}dx_3{}^2+2g_{24}dx_2dx_4+g_{44}dx_4{}^2.$$

Now the angles between the various coordinates are given by the direction cosines

$$\cos(\mu,\nu)=\frac{g_{\mu\nu}}{(g_{\mu\mu}\cdot g_{\nu\nu})^{\frac{1}{2}}},$$

where (μ, ν) is the angle between the x_μ and x_ν coordinates.[6] We can readily see that the angles $(\mu, 4)$ equal 90° except for $\mu=2$. Thus we have

$$\cos(2,4)=\frac{g_{24}}{(g_{22}\cdot g_{44})^{\frac{1}{2}}}=\frac{r^2\omega/c}{\left[r^2\left[-\left(1-\frac{\omega^2r^2}{c^2}\right)\right]\right]^{\frac{1}{2}}}$$

or

$$i\cos(2,4)=\frac{r\omega}{c}\left(1-\frac{\omega^2r^2}{c^2}\right)^{-\frac{1}{2}},$$

which gives an imaginary angle between x_2 and x_4 that is a function of r and is equivalent to 90° only when $r=0$.

Hence we find that in $d\bar{s}^2$ the time axis is not everywhere at right angles to the spatial dimensions. We must therefore choose a (four-component) vector which does satisfy this condition.

We take a contravariant vector Δx_μ and the covariant vector $g_{\nu\beta}dx_\beta$ at right angles to each other, so that their scalar product is zero:

$$g_{\alpha\beta}dx_\beta\Delta x_\alpha=0. \qquad (3)$$

The vector $g_{\nu\beta}dx_\beta$ is the vector in the world line element $(-d\bar{s}^2)$ where dx_ν defines the infinitesimal coordinate path of a point along its world line. Using the same coordinates, we define Δx_μ as the vector standing at right angles to the world lines of any two points fixed on the disk, so that Δx_μ is an infinitesimal difference in the coordinates between two points at rest on the rotating disk. In other words, this is the vector of the spatial extension. Then, by our assumption, we have

$$g_{ab}\Delta x_a dx_b+g_{a4}\Delta x_a dx_4+g_{4a}\Delta x_4 dx_a+g_{44}\Delta x_4 dx_4=0 \qquad a,b=1,2,3.$$

But, for any points fixed on the disk

$$\frac{dx_a}{d\bar{s}}=\frac{dx_b}{d\bar{s}}=0,$$

then

$$(g_{a4}\Delta x_a+g_{44}\Delta x_4)dx_4=0$$

or

$$\Delta x_4=-\frac{1}{g_{44}}\cdot g_{a4}\Delta x_a.$$

The invariant (proper) *spatial* interval between two points at rest on the disk is therefore obtained from the scalar product

$$dl^2=g_{\alpha\beta}\Delta x_\alpha\Delta x_\beta,\quad \alpha,\beta=1,\cdots,4 \qquad (4)$$

where dl is the proper length of the spatial element between points whose coordinates are separated by the coordinate vector Δx_μ. Then

$$dl^2=g_{ab}\Delta x_a\Delta x_b+g_{a4}\Delta x_a\Delta x_4+g_{4a}\Delta x_4\Delta x_a+g_{44}\Delta x_4{}^2$$
$$=g_{ab}\Delta x_a\Delta x_b+g_{a4}\Delta x_a\left(-\frac{1}{g_{44}}\cdot g_{a4}\Delta x_a\right)+g_{a4}\Delta x_a\left(-\frac{1}{g_{44}}\cdot g_{a4}\Delta x_a\right)+g_{44}\cdot\left(-\frac{1}{g_{44}}\right)^2\cdot(g_{a4}\Delta x_a)^2.$$

Now write $\Delta x_a=\delta x_a$. Then

$$dl^2=g_{ab}\delta x_a\delta x_b-\frac{1}{g_{44}}(g_{a4}\delta x_a)^2-\frac{1}{g_{44}}(g_{a4}\delta x_a)^2+\frac{1}{g_{44}}(g_{a4}\delta x_a)^2$$
$$=g_{ab}\delta x_a\delta x_b-\frac{1}{g_{44}}\cdot(g_{a4}\delta x_a)^2.$$

[6] Cf. T. Levi-Civita, *The Absolute Differential Calculus* (1929), p. 128.

But we may write

$$(g_{a4}\delta x_a)^2 = g_{a4}\delta x_a g_{b4}\delta x_b.$$

Then

$$dl^2 = \frac{1}{g_{44}}\{g_{44}g_{ab} - g_{a4}g_{b4}\}\,\delta x_a\delta x_b, \quad a, b = 1, 2, 3. \quad (5)$$

It will be noted that when $g_{a4}=0$,

$$dl^2 = g_{ab}\delta x_a\delta x_b,$$

which is, of course, the ordinary expression for the spatial line element.

Now in the rotating disk, we obtain from the polar coordinate expression for $-d\bar{s}^2$:

$$g_{11} = +1; \quad g_{33} = +1; \quad g_{24} = g_{42} = +\frac{r^2\omega}{c}$$

$$g_{22} = +r^2; \quad g_{44} = -\left(1 - \frac{\omega^2 r^2}{c^2}\right);$$

$$g_{12} = g_{13} = g_{14} = g_{23} = g_{24} = 0.$$

With these values, we have

$$dl^2 = \frac{1}{g_{44}}\{(g_{11}g_{44} - g_{14}g_{14})\delta x_1^2 + (g_{22}g_{44} - g_{24}g_{24})\delta x_2^2 + (g_{33}g_{44} - g_{34}g_{34})\delta x_3^2\},$$

$$dl^2 = dr^2 + \left(1 - \frac{\omega^2 r^2}{c^2}\right)^{-1} r^2 d\theta^2 + dz^2, \quad (6)$$

which is the spatial line element for the rotating disk.

We may note in passing that

$$\Delta x_4 = \frac{r\omega/c}{\left(1 - \frac{\omega^2 r^2}{c^2}\right)} \cdot rd\theta = \frac{v/c}{1 - \frac{v^2}{c^2}} \cdot rd\theta$$

or approximately

$$\Delta x_4 = \frac{v}{c} \cdot rd\theta,$$

which gives us a measure of the fourth component of the spatial vector Δx_μ.

Manifestly, the element dl exhibits a non-euclidean geometry on the surface of the disk. Setting z as a constant, we have

$$dl^2 = dr^2 + \left(1 - \frac{r^2\omega^2}{c^2}\right)^{-1} r^2 d\theta^2. \quad (7)$$

If we identify dl with actual measures (with a rigid rod element) by an observer D at rest on the rotating disk, then the proper (measured) length along the radial coordinate element, determined by a D observer, is $dl_r = dr$. The proper length along the disk at right angles to r is

$$dl_c = \frac{rd\theta}{\left(1 - \frac{r^2\omega^2}{c^2}\right)^{\frac{1}{2}}} \quad \text{so} \quad rd\theta < dl_c,$$

where dr and $rd\theta$ are the coordinate lengths of elements. The proper circumference is

$$C_0 = \int_0^{C_0} dl_c = \frac{r}{\left(1 - \frac{\omega^2 r^2}{c^2}\right)^{\frac{1}{2}}} \int_0^{2\pi} d\theta = \frac{2\pi r}{\left(1 - \frac{\omega^2 r^2}{c^2}\right)^{\frac{1}{2}}},$$

where $2\pi r = C$ is the coordinate circumference. Then

$$C_0 = C\left(1 - \frac{\omega^2 r^2}{c^2}\right)^{-\frac{1}{2}} \quad \text{or} \quad C_0 > 2\pi r.$$

Correlation of the Geometries

Returning to the original Einstein argument, and noting that r is the same radius of the rotating disk as in the above equations, we can relate the measures of a G observer directly to those of a D observer. We have $C = C' = 2\pi r$, then

$$C' = C_0\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = \bar{C}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

whence

$$\bar{C} = C_0\left(1 - \frac{v^2}{c^2}\right),$$

where $\bar{C}$ is the circumference of the rotating disk measured by a G observer and C_0 is its circumference as measured by a D observer. We should note that *if it were not for the change in the intrinsic geometry* of the disk, as given by the relation

$$C_0 = C\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

the Einstein relation

$$\bar{C} = C'\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

would be a purely relative one, in that an observer D on the disk would find a disk (of radius r) at rest in galilean space (and therefore moving relative to the D observer) to have a non-euclidean geometry satisfying the equivalent relation

$$\bar{C}_D = C_D'\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}},$$

where $\bar{C}_D$ would be the circumference of the galilean disk relative to the D observer and $C'_D = 2\pi r$ would be its circumference when at rest in D's space (we assume the galilean disk to have the same axis of relative rotation as the D disk). *But actually* we would have

$$C'_D = C_0 = 2\pi r\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

from which we must conclude that

$$\bar{C}_D = 2\pi r$$

or the geometry of the galilean disk, relative to a D observer, would remain euclidean.

In connection with Eddington's argument we should note that, in our notation $dr_0 = dl_r = dr$ and

$$r_0 d\theta_0 = dl_c = \left(1-\frac{\omega^2 r^2}{c^2}\right)^{-\frac{1}{2}} r d\theta,$$

from which we obtain

$$d\theta_0 = \left(1-\frac{\omega^2 r^2}{c^2}\right)^{-\frac{1}{2}} d\theta$$

rather than Eddington's $d\theta_0 = d\theta$. Our equation merely shows that, in measuring angles originating from the disk's center O, the measured ratio of an element of arc length to the radius changes as one proceeds outward along the radii.

Hypercurvature of the Surface

We may now inquire into the values of the Riemann-Christoffel tensor $R^{\kappa}_{\lambda\mu\nu}$, the contracted symmetrical form $R_{\lambda\mu} = R^{\alpha}_{\lambda\mu\alpha}$, and the scalar invariant $R = g^{\alpha\beta}R_{\alpha\beta}$, for the rotating disk where we confine our attention to two dimensions (λ, μ, ν, $\kappa = 1, 2$). Taking

$$dl^2 = dr^2 + \left(1-\frac{\omega^2 r^2}{c^2}\right)^{-1} r^2 d\theta^2$$
$$= g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2,$$

where $dx_1 = dr$; $dx_2 = d\theta$ and

$$|g_{\mu\nu}| = \begin{vmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{vmatrix} = \begin{vmatrix} +1 & 0 \\ 0 & +r^2\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-1} \end{vmatrix}$$

we find

$$R_{11} = \frac{3\omega^2}{c^2}\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-2},$$
$$R_{12} = R_{21} = 0,$$
$$R_{22} = \frac{3\omega^2 r^2}{c^2}\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-3}.$$

We also find that

$$R = g^{11}R_{11} + g^{22}R_{22} = \frac{6\omega^2}{c^2}\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-2}.$$

The tensor $R_{\lambda\mu\nu\kappa} = g_{\kappa\beta}R^{\beta}_{\lambda\mu\nu}$ has certain components and relations of interest to us. We find that $R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$ and all others vanish for two dimensions. Now $R = g^{\alpha\beta}R_{\alpha\beta}$ (conversely, we have $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$) and $R_{\mu\nu} = g^{\gamma\delta}R_{\gamma\mu\nu\delta}$ whence

$$R = g^{\gamma\delta}g^{\alpha\beta}R_{\gamma\alpha\beta\delta} = g^{12}g^{21}R_{1212}$$
$$+ g^{21}g^{12}R_{2121} + g^{11}g^{22}R_{1221} + g^{22}g^{11}R_{2112},$$
$$\therefore \quad R = g^{12}g^{12}R_{1212} + g^{12}g^{12}R_{1212} - g^{11}g^{22}R_{1212}$$
$$- g^{11}g^{22}R_{1212} = 2(g^{12}g^{12} - g^{11}g^{22})R_{1212}.$$

But we can show that

$$g^{11}g^{22} - g^{12}g^{12} = g^{-1},$$

then

$$R_{1212} = -\frac{g}{2}R.$$

For the rotating disk, we find that

$$R_{1212} = -\frac{3\omega^2 r^2}{c^2}\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-3}.$$

Now Gauss shows R_{1212}/g to be the measure of curvature of the surface at any point x_1, x_2, so that $R_{1212}/g = 1/r_1r_2 = -R/2$ where r_1 and r_2 are the principal radii of curvature at that point. Then

$$\frac{1}{r_1r_2} = -\frac{3\omega^2}{c^2}\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-2}.$$

Hence, the surface hypercurvature of the rotating disk is negative and variable.

PART II. TIME

This brings us to the last part of our paper: the behavior of clocks at rest on the rotating disk. It seems that in this connection, a new experimental test is now theoretically possible for Einstein's general theory—a test which I shall now discuss.

Events in a Gravitational Field

It has been shown that the periods of clocks or events are, in general, related as follows:[7]

$$\frac{\delta t_2^0}{\delta t_1^0}=\frac{dt_2\left[g_{\alpha\beta}\dfrac{dx_\alpha}{dx_4}\dfrac{dx_\beta}{dx_4}\right]^{\frac{1}{2}}_{(x_1)_2(x_2)_2(x_3)_2(x_4)_2}}{dt_1\left[g_{\alpha\beta}\dfrac{dx_\alpha}{dx_4}\dfrac{dx_\beta}{dx_4}\right]^{\frac{1}{2}}_{(x_1)_1(x_2)_1(x_3)_1(x_4)_1}},$$

where $x_4=t=$coordinate time, where

$$\delta t_1^0=dt_1\left[g_{\alpha\beta}\frac{dx_\alpha}{dx_4}\frac{dx_\beta}{dx_4}\right]^{\frac{1}{2}}_{(x_1)_1(x_2)_1(x_3)_1(x_4)_1}$$

is the proper period of a light emitting source measured by an observer at rest with respect to that source which has a coordinate velocity $(dx_1/dx_4,\ dx_2/dx_4,\ dx_3/dx_4)_1$ at the point $(x_1)_1$, $(x_2)_1$, $(x_3)_1$ at the time $(x_4)_1$, and where

$$\delta t_2^0=dt_2\left[g_{\alpha\beta}\frac{dx_\alpha}{dx_4}\frac{dx_\beta}{dx_4}\right]^{\frac{1}{2}}_{(x_1)_2(x_2)_2(x_3)_2(x_4)_2}$$

is the proper period of the light from the source as observed by an observer located at the point $(x_1)_2$, $(x_2)_2$, $(x_3)_2$ at the time $(x_4)_2$, and moving with a coordinate velocity $(dx_1/dx_4,\ dx_2/dx_4,\ dx_2/dx_4)_2$. Also dt_1 is the coordinate period of light emitting source, while dt_2 is that period for the observer at $(x_1)_2$, $(x_2)_2$, $(x_3)_2$. Since we have for light, $ds=0$, we can easily show that $t_2=f(t_1)$ which gives the coordinate time t_2 of the observer's reception of the light signal from the source in terms of the coordinate time t_1 of the signal's emission. Whence

$$\frac{dt_2}{dt_1}=\frac{df(t_1)}{dt_1}.$$

Since $f(t_1)$ is a function of the $g_{\mu\nu}$ values, then where $g_{\mu\nu}$ is not a function of t, we will have $dt_2=dt_1$ or the coordinate periods of emission and observation of successive signals are equal in their intervals.

We should note that while this deduction (used for the generalized Doppler effect) is made in terms of light frequency from radiating atoms, it can be applied to proper times of periodic processes other than radiating atoms. Light will still be the means of observation, but the periods will be those of "clocks" in various parts of the gravitational field, and the time (t_2-t_1) will be the coordinate interval required for a light signal to pass over the distance to the observer, while dt_1 will be the coordinate interval between clock beats at the source and dt_2 will be such an interval for the observer. δt_1^0 and δt_2^0 will be the corresponding proper values.

Clocks on the Disk

In the case of two clocks at rest at different places on the rotating disk, our equation reduces to

$$\frac{\delta t_2^0}{\delta t_1^0}=\frac{[g_{44}]^{\frac{1}{2}}_{(x_1)_2(x_2)_2}}{[g_{44}]^{\frac{1}{2}}_{(x_1)_1(x_2)_1}}$$

but, in polar coordinates, this is

$$\frac{\delta t_2^0}{\delta t_1^0}=\frac{[g_{44}]^{\frac{1}{2}}_{r_2,\,\theta_2}}{[g_{44}]^{\frac{1}{2}}_{r_1,\,\theta_1}}$$

and since

$$g_{44}=1-\frac{\omega^2r^2}{c^2},$$

then

$$\delta t_2^0=\frac{\left(1-\dfrac{\omega^2r_2^2}{c^2}\right)^{\frac{1}{2}}}{\left(1-\dfrac{\omega^2r_1^2}{c^2}\right)^{\frac{1}{2}}}\delta t_1^0.$$

If the observer is at $r_2=0$, and the clock at r_1, then

$$\delta t_2^0=\left(1-\frac{\omega^2r_1^2}{c^2}\right)^{-\frac{1}{2}}\delta t_1^0.$$

We note that δt_1^0 corresponds to the proper time ds_0 in our notation.

The point $r_2=0$ is a singular point in the

---

[7] R. Tolman, *Relativity, Thermodynamics and Cosmology*, pp. 288–290.

gravitational field of the rotating disk and is at rest in some galilean space. If a clock is placed anywhere in galilean space and the observer is anywhere on the disk, the G clock will appear to have an angular velocity equal but opposite to that of the disk relative to the observer. We then have

$$\frac{\delta t_2{}^0}{\delta t_1{}^0}=\frac{[g_{44}]^{\frac{1}{2}}_{r_2}}{\left[g_{22}\frac{d\theta^2}{dt^2}+\cdots+g_{44}\right]^{\frac{1}{2}}_{r_1}},$$

where

$$ds^2=\left[1-\frac{\omega^2 r^2}{c^2}\right]dt^2-\frac{1}{c^2}[\omega r^2 d\theta dt+\omega r^2 dt d\theta+dr^2+r^2 d\theta^2],$$

which gives us the $g_{\mu\nu}$ values. But $d\theta/dt=-\omega$, then

$$\frac{\delta t_2{}^0}{\delta t_1{}^0}=\frac{\left[1-\frac{\omega^2 r_2{}^2}{c^2}\right]^{\frac{1}{2}}}{\left[-\frac{r_1{}^2}{c^2}\omega^2-\frac{\omega r_1{}^2}{c^2}(-\omega)-\frac{\omega r_1{}^2}{c^2}(-\omega)+1-\frac{\omega^2 r_1{}^2}{c^2}\right]^{\frac{1}{2}}}$$

(the G clock is assumed at rest relative to the z axis of rotation), hence

$$\delta t_2{}^0=\left[1-\frac{\omega^2 r_2{}^2}{c^2}\right]^{\frac{1}{2}}\delta t_1{}^0.$$

From all of this we may infer that clocks on the disk will run slow relative to clocks in G space at rest with respect to the rotational axis of the disk. Such slowing down is absolute, not in the sense that the principle of relativity of motion is not followed, for it can be shown that it is, but we see that the presence of the gravitational field on the disk introduces a gravitational potential for clocks on the disk—a potential not existing for the G clocks. This is another form of the well-known "clock paradox" dealt with so carefully by Richard Tolman as well as by Kopff.[8]

Our conclusion is that a clock, after being isochronized with a galilean clock and then set in rotation with respect to the latter clock, will, after rotating for some time and then being brought to rest again, no longer be isochronous, but will indicate earlier time than the G clock. It is assumed that the time required to bring the clock up to the curvilinear speed ωr is short compared with the time during which it remains at that speed. Similarly, the deceleration period should be short.

In the case of a disk of uniform angular acceleration $d\omega/dt=k$, the line element (for a system initially at rest) would be

$$ds^2=\left[1-\frac{k^2t^2(x^2+y^2)}{c^2}\right]dt^2-\frac{1}{c^2}[-2ktydxdt+2ktxdydt+dx^2+dy^2]$$

$$=\left[1-\frac{k^2t^2r^2}{c^2}\right]dt^2-\frac{1}{c^2}[2ktr^2d\theta dt+dr^2+r^2d\theta^2]$$

and where t is small, the $g_{\mu\nu}$ values obviously assume approximately their galilean values.

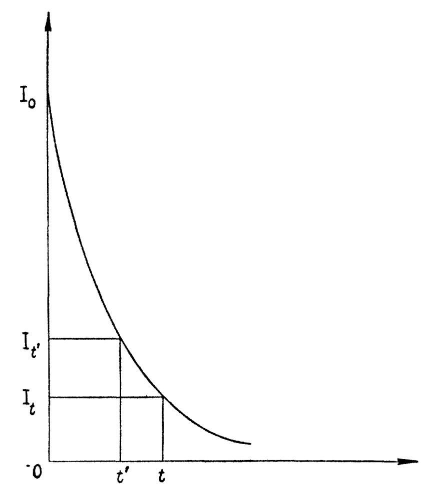

FIG. 1. The relationship between the radioactivities of two identical samples of an element of initial activity I_0, one sample remaining at rest in our space for the time t and the other being revolved in the cyclotron at constant curvilinear speed for the same time t, where t' is the corresponding relative time of the revolving sample. Then I_t is the activity of the resting sample and $I_{t'}$ is that of the revolving sample.

Experimental Test through the Cyclotron

Now it is clear that the highest curvilinear velocities to be obtained on a solid rotating disk must be limited by the elastic constants and

[8] R. Tolman, reference 7, pp. 194–197; A. Kopff, *Mathematical Theory of Relativity* (1921), pp. 125–27.

breaking strengths of the materials of such disks. It can be shown that the limiting speed, under these conditions, is about 10^5 cm/sec. Obviously this is too small when we realize that we are dealing with a factor of v^2/c^2. There remains, however, the use of the cyclotron with (artificial) radioactive ions utilized as our moving clocks. The degree of radioactivity would be a measure of proper time. We know, of course, that the radioactivity of an element is not modified by mechanical, electric, or magnetic forces, or thermal effects. The force holding these ions fixed at a particular radius from the center of the cyclotron would be simply the magnetic field force, once the ions were accelerated out to that radius by the electric field forces (which would then be removed while the ions continued to revolve at fixed radii). The ions at the limiting radius of the cyclotron would eventually (after a period of constant revolution) be brought to rest once more and tested for their activity per unit mass. If our deductions are correct, these ions will be more radioactive than a sample of such atoms which remained at rest—the atoms remaining at rest would have less radioactivity per unit mass.

In a cyclotron, the force exerted by the magnetic field on a charged particle revolving with a speed of v cm/sec. is

$$F = BQv = mv^2/r,$$

where F is in dynes, B is the magnetic field flux density (gauss), Q is the charge (e.m.u.) on the particle of mass m (grams), and r is the radius of path curvature (cm). Whence $BQr = mv$. Where v approaches light speed c, we should introduce the relativistic mass change

$$m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

where m_0 is the proper mass of the particle. Then

$$\frac{BQr}{m_0} = \frac{v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \varphi,$$

whence

$$v = \varphi\left(1 + \frac{\varphi^2}{c^2}\right)^{-\frac{1}{2}}$$

or approximately

$$v = \varphi\left(1 - \frac{1}{2}\frac{\varphi^2}{c^2}\right).$$

We see, therefore, that to obtain speeds approaching that of light we need ions of highest

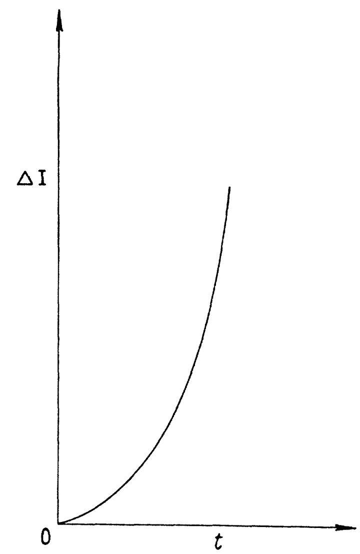

FIG. 2. The relationship between the fractional difference ΔI in activity of a resting and revolving sample of a radioactive element and the duration of revolution t determined by a resting clock.

charge and lowest mass possible in a cyclotron of large radius and high magnetic flux density. The duration of acceleration out to ωr and the corresponding deceleration of the ions of low atomic weight radioactive elements must be as brief as possible, while the duration of constant $v = \omega r$ for the ions should approximate their half-life.

Taking two samples of the same radioactive element having the same initial activity I_0, we may write for the sample at rest

$$I_t = I_0 e^{-\lambda t}$$

and for the sample at radius r revolving in the cyclotron

$$I_{t'} = I_0 e^{-\lambda t'},$$

where, approximately,

$$t-t'=\frac{1}{2}\frac{v^2}{c^2}t'$$

and t is the duration of the experiment in our time. See Fig. 1. This result may be written

$$\Delta I=\frac{I_{t'}-I_t}{I_t}=\exp\left(\frac{1}{2}\frac{v^2}{c^2}\lambda t\right)-1.$$

See Fig. 2. To obtain velocities approaching c with the large cyclotron now under construction in California appears possible even for radioactive ions of atomic weights as high as ten. But the actual testing of our deductions must await further investigations.

In conclusion, it is worthwhile noting that such an experiment, aside from being a new test of the general theory of relativity, would provide an application of that theory to nuclear physics of the atom.[9]

I am grateful to Professor H. P. Robertson of Princeton University for his suggestions, and to Professors H. Semat, J. D. Shea, H. C. Wolfe, and M. W. Zemansky of New York City College for their helpful discussions. To my colleague, Dr. A. Wundheiler, I am deeply indebted for his careful criticisms, especially on differential geometry.

[9] Of course, as Professor Einstein has kindly pointed out to me, the mesotron decomposition rate, at high speeds, gives results equivalent to those that would be obtained in the cyclotron for radioactive ions *while they are in motion*. One obviously can use the special theory to obtain the relation between t and t', but this formula is here shown to be a logical consequence of the general theory in the non-trivial sense that it makes unequivocal what the special theory alone would leave equivocal. Tolman's argument for the effect of the gravitational potential difference $d\Psi=v^2r^{-1}dr$ is here operative.