The Influence of an "Uneven" Anisotropy on the Path of Light Rays

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The refractive index of a medium may be $\mu(\mathbf{r}, \mathbf{s}) = \mu_0(\mathbf{r}) + \mu_1(\mathbf{r}, \mathbf{s})$ the vector $\mathbf{r}$ denoting the coordinates of a point and the unit vector $\mathbf{s}$ the direction of the light ray, while μ_1 may be any function of $\mathbf{s}$. The deflection of a ray pencil $\mathbf{s} = \mathbf{s}(\mathbf{r})$ by the anisotropy μ_1 is defined by the "curvature vector $\mathbf{k}$ of the deflected rays relative to the rays in the isotropic medium μ_0." We find $\mathbf{k} = \mathbf{s} \times \text{curl}\, \mathbf{A}(\mathbf{r}, \mathbf{s})$, $\mathbf{s}$ being $\mathbf{s}(\mathbf{r})$ and $\mathbf{A} = \mu_1 \mathbf{s} + \text{grad}_s\, \mu_1 - \mathbf{s}(\mathbf{s}\, \text{grad}_s\, \mu_1)$, grad_s which implies differentiation with respect to the components of $\mathbf{s}$. The simplest case, notwithstanding the anisotropy of crystals, is $\mu_1 = \mathbf{s}\mathbf{g}(\mathbf{r})$, $\mathbf{g}$ being an arbitrary function of $\mathbf{r}$. This case covers, for example, the deflection of light rays by the motion of the medium. We have only to put $\mathbf{g} = -(1/c^2)\mu_0^2 \mathbf{w}(\mathbf{r})$. Here μ_0 is the refractive index and $\mathbf{w}$ the velocity distribution in the medium. A second example is the deflection of electron rays in an electromagnetic field. We have to put $\mathbf{g} = (\epsilon/c)\mathbf{a}(\mathbf{r})$. Here $\mathbf{a}$ is the vector potential of the field.

IF the refractive index of a medium is dependent upon the direction of the light ray we speak of an "anisotropy" of the medium. The refractive index μ at a certain point may be an arbitrary function of the rectangular coordinates x, y, z and of the direction of the light ray

$$\mu = \mu(\mathbf{r}, \mathbf{s}). \tag{1}$$

The vector $\mathbf{r}$ has the components x, y, z and the vector $\mathbf{s}$ the components $\cos\alpha$, $\cos\beta$, $\cos\gamma$, α, β, γ being the angles between the light ray and the axes. If

$$\mu(\mathbf{r}, -\mathbf{s}) = \mu(\mathbf{r}, \mathbf{s}) \tag{2}$$

the anisotropy may be called an "even" one. All crystals have an even anisotropy. If for a certain medium Eq. (2) does not hold we speak of an "uneven" anisotropy.

Examples of "uneven" anisotropy are provided by the path of light rays through moving bodies or the path of electrons through an electromagnetic field. The formulae we are going to develop will embrace in one mathematical pattern all these problems.

According to Fermat's principle the path of a light ray gives to the integral

$$T = \int \mu(\mathbf{r}, \mathbf{s}) ds \tag{3}$$

a minimum value. In this formula ds is the length of a curve element. The differential equations of the light rays are then evidently the Lagrange differential equations of the variation problem of Eq. (3).

If the medium were isotropic, μ would only depend on $\mathbf{r}$ and the Lagrange equations could be expressed by the simple vector equation

$$d\mathbf{p}/ds = \text{grad}\, \mu(\mathbf{r}), \tag{4}$$

$$\mathbf{p} = \mu(\mathbf{r})\mathbf{s}. \tag{5}$$

If μ is dependent also on $\mathbf{s}$, the differential equation of the light rays keeps its simple form (4). However, the connection between $\mathbf{p}$ and μ is no longer given by the simple Eq. (5) but by[1]

$$\mathbf{p} = \mu(\mathbf{r}, \mathbf{s})\mathbf{s} + \mathbf{Q}(\mathbf{r}, \mathbf{s}). \tag{6}$$

To express $\mathbf{Q}$ in a simple way we introduce the vector $\text{grad}_s\, \mu$. Its components are the partial derivatives of μ with respect to the three components of $\mathbf{s}$. It has to be distinguished from $\text{grad}\, \mu$, the components of which are the derivatives with respect to x, y, z. Then[1]

$$\mathbf{Q} = \text{grad}_s\, \mu - \mathbf{s}(\mathbf{s}\, \text{grad}_s\, \mu), \tag{7}$$

$$\mathbf{Q}\mathbf{s} = 0, \tag{8}$$

$\mathbf{Q}$ is perpendicular to $\mathbf{s}$.

$\mathbf{s} = \mathbf{s}(\mathbf{r})$ may be a "bundle" of rays; through every point $\mathbf{r}$ passes just one ray of this bundle. $\mathbf{s} = \mathbf{s}(\mathbf{r})$ is called an "orthotomic bundle" if by substituting $\mathbf{s} = \mathbf{s}(\mathbf{r})$ into (6) we obtain a vector field $\mathbf{p}(\mathbf{r})$ that is met normally by the ∞' "wave surfaces" of the bundle $\mathbf{s}(\mathbf{r})$. There is a simple

[1] P. Frank, Zeits. f. Physik **80**, 4 (1933).

condition which a bundle of curves $\mathbf{s}$ has to meet in order to be an "orthotomic bundle" of light rays in a medium with the refractive index $\mu(\mathbf{r}, \mathbf{s})$.

$\mathbf{s}(\mathbf{r})$ has to satisfy the equation

$$\text{curl } \mathbf{p}(\mathbf{r}) \times \mathbf{s}(\mathbf{r}) = 0 \tag{9}$$

if we denote by $\mathbf{a} \times \mathbf{b}$ the vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

The length of $\mathbf{p}$ is, by the way, the ratio c/w, c meaning the vacuum speed of light and w the speed of the normal displacement of the wave surface.

The formulae (6), (7), (9) enable us to solve the problem of the deflection of light rays by an uneven anisotropy of the medium.

We start considering the path of light rays in an isotropic medium with the refractive index $\mu_0(\mathbf{r})$. We call this path the "undisturbed" path. Then we alter the medium in such a way that an anisotropy is produced. The refractive index is now

$$\mu(\mathbf{r}, \mathbf{s}) = \mu_0(\mathbf{r}) + \mu_1(\mathbf{r}, \mathbf{s}). \tag{10}$$

We obtain a new path of the rays: the "disturbed" path. If we substitute μ from (10) into (6) and (7), we obtain:

$$\mathbf{p} = \mu_0(\mathbf{r})\mathbf{s} + \mathbf{A}(\mathbf{r}, \mathbf{s}), \tag{11}$$

$$\mathbf{A} = \mu_1(\mathbf{r}, \mathbf{s})\mathbf{s} + \text{grad}_s\, \mu_1 - \mathbf{s}(\mathbf{s} \text{ grad}_s\, \mu_1). \tag{12}$$

If the medium is isotropic we have

$$\mu = \mu_0(\mathbf{r}), \quad \mathbf{p} = \mu_0(\mathbf{r})\mathbf{s}. \tag{13}$$

If $\mathbf{s}(\mathbf{r})$ is an arbitrary bundle of curves the left-hand term of (9) is not necessarily equal to zero. However, it is a measure for the departure of the bundle of curves $\mathbf{s}$ from the light rays in the medium with the refractive index $\mu(\mathbf{r}, \mathbf{s})$. If we put

$$\mathbf{k} = \text{curl } \mathbf{p}(\mathbf{r}) \times \mathbf{s}(\mathbf{r}) \tag{14}$$

the vector $\mathbf{k}$ may be called the "relative curvature of the bundle of curves $\mathbf{s}$ with respect to the light rays in the medium with the refractive index μ''. This is a generalization of what is called in the elementary mathematics the "curvature of a curve."

Starting from these remarks it will be easy to give a concise formula for the departure of the "disturbed" light rays from the "undisturbed" ones, or in other words for the deflection of the light rays produced by the anisotropy. We take as a measure of this departure the relative curvature $\mathbf{k}_0$ of the light rays in the medium with the refractive index $\mu(\mathbf{r}, \mathbf{s})$ with respect to the light rays in the medium with the refractive index $\mu_0(\mathbf{r})$.

If $\mathbf{s}(\mathbf{r})$ is a bundle of light rays in the anisotropic medium with the refractive index $\mu(\mathbf{r}, \mathbf{s})$ we conclude from (14) and (13)

$$\mathbf{k}_0 = \text{curl } \mu_0(\mathbf{r})\mathbf{s} \times \mathbf{s}(\mathbf{r}). \tag{15}$$

Then we conclude from (9) and (11)

$$\text{curl } \mu_0\mathbf{s} \times \mathbf{s} + \text{curl } \mathbf{A} \times \mathbf{s} = 0 \tag{16}$$

and therefore from (15) and (16)

$$\mathbf{k}_0(\mathbf{r}) = \mathbf{s} \times \text{curl } \mathbf{A}. \tag{17}$$

If we replace in (12) $\mathbf{s}$ by a specific bundle $\mathbf{s}(\mathbf{r})$, $\mathbf{A}(\mathbf{r}, \mathbf{s})$ becomes a function of merely $\mathbf{r}$.

We are now going to discuss a simple case of "uneven" anisotropy which is of particular importance in the application of our formulae to problems of physics.

We assume a medium with a particular refractive index $\mu(\mathbf{r}, \mathbf{s})$ of the form (10), for which $\mathbf{s}$ cancels in the right-hand term of (12). Then $\mathbf{A}$ becomes a function of $\mathbf{r}$ only.

$$\mathbf{A}(\mathbf{r}, \mathbf{s}) = \mathbf{g}(\mathbf{r}). \tag{12a}$$

Multiplying (12) by $\mathbf{s}$ we obtain

$$\mu_1(\mathbf{r}, \mathbf{s}) = \mathbf{g}(\mathbf{r})\mathbf{s}, \quad \mu = \mu_0(\mathbf{r}) + \mathbf{g}(\mathbf{r})\mathbf{s}. \tag{18}$$

Then the refractive index μ becomes a linear function of $\mathbf{s}$. This is the simplest case of "uneven" anisotropy. μ can be expressed by means of *one* vector field $\mathbf{g}(\mathbf{r})$ which describes the characteristic optical properties of our medium. The deflection produced by this anisotropy is then after (17), (18);

$$\mathbf{k}_0(\mathbf{r}) = \mathbf{s}(\mathbf{r}) \times \text{curl } \mathbf{g}(\mathbf{r}). \tag{19}$$

APPLICATIONS

We examine as a first example the anisotropy produced by stationary motion in an isotropic medium. The distribution of velocity in the medium may be given by the vector field $\mathbf{w}(\mathbf{r})$. We denote the speed of light in vacuum by c and in our medium, when at rest, by $v_0(\mathbf{r})$, the velocity by the vector $\mathbf{v}_0$. The velocity of light in the medium, when at motion, may be $\mathbf{v}$. We assume that the velocity $\mathbf{w}$ of the medium is such that we have

$$\mathbf{v}=\mathbf{v}_0+\mathbf{w}, \quad \mathbf{v}=v\mathbf{s}. \tag{20}$$

If we denote by $\mu_0(\mathbf{r})$ the refractive index of the medium, when at rest, and by μ, when at motion, we obtain:

$$\mu_0(\mathbf{r})=c/v_0, \quad \mu=c/v. \tag{21}$$

If we assume that the velocities of the medium are small relative to the velocity of light we get as the first approximation

$$\mu(\mathbf{r}, \mathbf{s})=\mu_0(\mathbf{r})-(1/c)\mu_0^2\mathbf{sw}. \tag{22}$$

This is a special case of the uneven anisotropy described by (18).

$$\begin{aligned}\mu_1(\mathbf{r}, \mathbf{s})&=-(1/c^2)\mu_0^2(\mathbf{r})\mathbf{sw}(\mathbf{r}),\\ \mathbf{g}&=-(1/c^2)\mu_0^2(\mathbf{r})\mathbf{w}(\mathbf{r}).\end{aligned} \tag{23}$$

Then we obtain by substituting $\mathbf{g}$ from (23) into (19):

$$\mathbf{k}_0=(1/c^2)\text{ curl }(\mu_0^2\mathbf{w})\times\mathbf{s}. \tag{24}$$

This formula gives us the deflection of light rays $\mathbf{s}$ which is produced by a velocity distribution $\mathbf{w}(\mathbf{r})$ in an isotropic medium with the refractive index μ_0. Equation (24) gives us immediately the classical results of Stokes and Fresnel concerning the optical phenomena in moving bodies.

A second example is the influence of wind upon the quickest path of an airplane. If there is no wind the quickest path is, of course, the straight line. If there is a distribution $\mathbf{w}(\mathbf{r})$ of wind velocity in the region the plane has to pass there is a deviation from the rectilinear path. The straight line is no longer the "shortest." As for the mathematical formulation this problem is identical with the problem of the path of light rays in a moving medium. If we assume that the wind speed w is small relative to the speed v_0 impressed on the plane by the power of its motor we can apply directly (24). If we take into account (21) and the constancy of v_0 we conclude from (24):

$$\mathbf{k}_0=(1/v_0^2)\text{ curl }\mathbf{w}\times\mathbf{s}. \tag{25}$$

In this equation $v_0\mathbf{k}_0$ is the "curvature" in the ordinary sense of this word. It is obvious that the straight path remains the best to pursue if there are no whirls in the wind distribution.

As a third example we examine the "electron rays," the path of an electron with the charge ϵ through an electromagnetic field which is given by the scalar potential $\phi(\mathbf{r})$ and the vector potential $\mathbf{a}(\mathbf{r})$. We denote by m the mass of the electron, by c the speed of light in the vacuum, and by E the initial energy of the electron. According to K. Schwarzschild's *Minimum Principle of Electrodynamics* the electron rays are geometrically light rays in a medium with the refractive index.[2]

$$\mu(\mathbf{r}, \mathbf{s})=[2m(E-\epsilon\phi(\mathbf{r})]^{\frac{1}{2}}+(\epsilon/c)\mathbf{as}. \tag{26}$$

This is obviously a special case of (18) with

$$\mu_0(\mathbf{r})=[2m(E-\epsilon\phi)]^{\frac{1}{2}}, \quad \mathbf{g}(\mathbf{r})=(\epsilon/c)\mathbf{a}. \tag{27}$$

By substituting $\mathbf{g}$ from (27) into (19) we obtain:

$$\mathbf{k}_0=(\epsilon/c)\mathbf{s}\times\text{curl }\mathbf{a}. \tag{28}$$

If we denote the intensity of the magnetic field by $\mathbf{h}$, we have

$$\text{curl }\mathbf{a}=\mathbf{h} \tag{29}$$

and therefore

$$\mathbf{k}_0=(\epsilon/c)\mathbf{s}\times\mathbf{h}. \tag{30}$$

We see from (27) that the rays traverse the electric field like an isotropic medium. The influence of the magnetic field is the influence of an anisotropy characterized by the vector $(\epsilon/c)\mathbf{a}$. The vector $\mathbf{k}_0$ in (30) means therefore the deflection of the electron rays produced by the magnetic force. $\mathbf{k}_0$ is the "relative curvature" of the electron rays in the electromagnetic field with respect to the path of the rays in the purely electric field. Equation (30) is, of course, a form of the law of Biot-Savart or of the "Lorentz force." The speed of the electron does not occur in this form of the law. It gives exactly the "relative curvature" produced by the magnetic field.

[2] Frank-Mises, *Differentialgleichungen der Mechanik und Physik* (1935), Vol. 2, Chap. 1, p. 3 (61); W. Glaser, Zeits. f. Physik **80**, 451 ff (1933).