Electric Quadrupole Moments of Light and Heavy Nuclei

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Under the assumption that the asymmetrical nuclear charge distribution is due to a single proton, a general expression for the electric quadrupole moment as a function of nuclear spin is obtained. Using the plausible restriction for heavy nuclei that all allowed pairs of values of the orbital angular momentum of the proton and of the core occur with equal probability, we get as a result that the quadrupole moment is negative and that the value increases negatively as the nuclear spin increases. The calculation for light nuclei, taking into account the effect of all particles by using the Hartree model, shows that the addition of a proton to a nucleus, whose Q was originally zero, produces a negative quadrupole moment.

INTRODUCTION

THE existence of a nuclear electric quadrupole moment indicates a deviation from a spherically symmetrical average charge distribution within the nucleus. A positive value indicates a distribution extended along the axis of the nuclear spin (cigar-shaped nucleus) and a negative value implies a flattening toward the plane normal to this axis (watch-shaped nucleus).

With the exception of deuterium, the quadrupole moment has been measured only for nuclei of odd atomic weight. The regularities among the magnetic moments for these and other additional odd nuclei indicate very strongly that the resultant spin of the nuclear particles apart from the orbital angular momentum is $\frac{1}{2}$, i.e., $S=\frac{1}{2}$. Also to a first approximation the magnetic moment is due largely to the odd particle.[1] The question arises as to whether the quadrupole moment might be due to this odd particle. In other words we consider the nucleus, apart from the odd particle, to consist of a charge distribution which is spherically symmetrical and that the departure from this symmetry for the whole nucleus is caused by the odd particle. This means we must consider the odd particle as a proton.

If l is the orbital angular momentum of a single proton and if m is the z component of l, then the quadrupole moment is positive if $l(l+1)$ is greater than $3\,m^2$, otherwise it is negative. If the orbital angular momentum is directed along the z axis then the quadrupole moment is negative, but there are several orientations of $\mathbf{l}$ for which it is positive. The problem is therefore one of determining the effective orientation of the orbital angular momentum of the single proton.

As far as they have been observed, the large quadrupole moments are all positive and are found in the region of the periodic table between atomic numbers sixty and eighty, reaching a maximum at atomic number seventy-one.[1c] And in addition the magnitudes of nine of the moments are too large to be reasonably explained by the asymmetrical charge distribution of one proton. The deviation could be explained by the alpha-particle model. The problem of the energetically favorable packing of spherical alphas has been discussed by Wefelmeier,[1d] who has shown that the most elongated framework of alphas is to be expected in the neighborhood of atomic number 71. In addition Fano[2] has indicated briefly that the angular momentum of a single proton in a p state might well be oriented along the long axis of an elongated nucleus, thus contributing a small negative effect to an already large positive moment due to the alphas.

Of the nine nuclei with observed quadrupole moments that are small enough to be due to a single proton, three are negative, the rest positive. For these nine we can assume a symmetrical core and consider the effect of the single proton. In the first section a general expression for the quadrupole moment as a function of the

[1] (a) Th. Schmidt, Zeits. f. Physik **106**, 358 (1937); (b) H. Margenau and E. Wigner, Phys. Rev. **58**, 103 (1940); (c) D. R. Inglis, Phys. Rev. **60**, 837 (1941); (d) W. Wefelmeier, Naturwiss. **25**, 525 (1937); Zeits. f. Physik **107**, 332 (1937).

[2] See discussion of U. Fano, Naturwiss. **25**, 602 (1937), who has shown that it is at least plausible that the orbital angular momentum should be oriented along the long axis of an elongated nucleus, i.e., $m=l$.

total nuclear spin is derived. By use of a plausible assumption concerning the probability of allowed pairs of values of the orbital angular momentum of the proton and of the core, the expression is found to lead to the result that the quadrupole moment is negative and its value increases negatively as the spin increases. In the second section we shall calculate the quadrupole moment of light nuclei, using the Hartree model.

QUADRUPOLE MOMENTS OF HEAVY NUCLEI

Let us consider the wave function of the ground state of nuclei of odd atomic weight. We limit our discussion to odd nuclei as H^2 is the only even nucleus whose electric quadrupole moment has been measured. If a spin-independent symmetric Hamiltonian is used, then the normal states are doublets, $S=\frac{1}{2}$, and the total orbital angular momentum can assume the values $L_1=J-\frac{1}{2}$ and $L_2=J+\frac{1}{2}$, where J is the total angular momentum of the nucleus. If M is the z component of J, we are interested in the component $M=J$ in the calculation of the quadrupole moment. The total wave function with definite values of J and M can be written as the sum of two functions

$$\Psi(J,\,M)=a_1\psi(L_1,\,S,\,J,\,M)+a_2\psi(L_2,\,S,\,J,\,M),$$

where

$$|a_1|^2+|a_2|^2=1 \quad\text{and}\quad M=J.$$

Expanding each $\psi(L)$ in terms of functions which have sharp values of m_L and m_S, we have

$$\psi(L_1)=\Phi(L_1,\,m_L=L_1,\,m_S=\tfrac{1}{2})\equiv\Phi_+(L_1),$$

$$\psi(L_2)=-\frac{1}{(2J+2)^{\frac{1}{2}}}\Phi(L_2,\,m_L=L_2-1,\,m_S=\tfrac{1}{2})+\left(\frac{2J+1}{2J+2}\right)^{\frac{1}{2}}\Phi(L_2,\,m_L=L_2,\,m_S=-\tfrac{1}{2}),$$

$$\equiv-\frac{1}{(2J+2)^{\frac{1}{2}}}\Phi_+(L_2)+\left(\frac{2J+1}{2J+2}\right)^{\frac{1}{2}}\Phi_-(L_2).$$

If we define $Q=\sum_\pi(3z^2-r^2)$, an operator which acts only on the coordinates of the protons, then the quadrupole moment is

$$\int\Psi^*(J,\,M=J)Q\Psi(J,\,M=J)d\tau=|a_1|^2Q_{1,1,+}+|a_2|^2\left(\frac{1}{2J+2}Q_{2,2,+}+\frac{2J+1}{2J+2}Q_{2,2,-}\right)-\frac{2}{(2J+2)^{\frac{1}{2}}}a_1{}^*a_2Q_{1,2,+},$$

where

$$Q_{1,1,+}\equiv\int\Phi_+{}^*(L_1)Q\Phi_+(L_1)d\tau$$

and similarly for the other Q's. In obtaining this equation we have used the fact that Q is Hermitian and also that $\int\Phi_+{}^*Q\Phi_-d\tau=0$ because of the orthogonality of the spin functions.

We express each $\Phi(L,\,m_L,\,m_S)$ in terms of wave functions $\phi(l_\pi,\,l_\nu,\,L,\,m_L)$, where l_π and l_ν are, respectively, the total orbital angular momenta of the protons alone and the neutrons alone.

$$\Phi(L,\,m_L)=\sum_{l_\pi,\,l_\nu}C_{(l_\pi,l_\nu,\,L,\,m_L)}\phi(l_\pi,\,l_\nu,\,L,\,m_L),$$

where $\Sigma|C|^2=1$, and $|C_{(l_\pi,\,l_\nu,\,L,\,m_L)}|^2$ is the probability that the orbital angular momentum of the protons equals l_π, and that of the neutrons equals l_ν. The only condition on a particular pair of values of l_π and l_ν is that their vector sum is equal to L.

We can expand the wave function for a particular pair of l_π and l_ν in terms of wave functions, each of which has definite z components of l_π and l_ν.[3]

$$\phi(l_\pi,\,l_\nu,\,L,\,m_L)=\sum_{m_\pi,\,m_\nu}\Theta_{(l_\pi,\,l_\nu,\,m_\pi,\,m_\nu)}\times(l_\pi,\,l_\nu,\,m_\pi,\,m_\nu|l_\pi,\,l_\nu,\,L,\,m_L).$$

Since $m_\pi+m_\nu=m_L$, this summation is really a single one. The coefficients have been worked out only as far as $l_\pi=2$ if $l_\pi<l_\nu$, or $l_\nu=2$ if $l_\nu<l_\pi$. Each Θ can be expressed as the product of a function involving the coordinates of the protons

[3] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, 1935), pp. 76–77.

TABLE I. Values of the integral $I=\int\phi^*(l_\pi', l_\nu', L, m_L)Q\phi(l_\pi, l_\nu, L, m_L)d\tau$.

	$L=M_L=0$		$L=M_L=1$			$L=M_L=2$			$L=M_L=3$			$L=M_L=4$			$L=M_L=5$		
	$l_\pi l_\nu$	D	$l_\pi l_\nu$	D	X	$l_\pi l_\nu$	D	X	$l_\pi l_\nu$	D	X	$l_\pi l_\nu$	D	X	$l_\pi l_\nu$	D	X
	00	0	01	0	0.28	02	0	0.48	03	0	0.59	04	0	0.64	05	0	0.68
	11	0	10	−0.40		11	−0.40	.14	12	−0.40	.26	13	−0.40	?	14	−0.40	?
	22	0	11	.20		20	− .56		21	− .56	.08	22	− .56	.16	23	− .56	?
	33	0	12	− .04	.29	12	.40	.34	30	− .67		31	− .67	.05	32	− .67	.11
	44	0	21	− .20		21	− .28		13	.50	?	40	− .73		41	− .73	.04
			22	.20		13	− .12	?	22	− .14	.23	14	.56	?	50	− .77	
			23	− .057	?	22	.12	.22	31	− .50		23	− .06	?	15	.60	?
			32	− .16		31	− .46		14	− .16	?	32	− .40	.16	24	0	?
						23	.33	?	23	.26	?	41	− .62		33	?	?
						32	− .12		32	− .21	.15	15	− .20	?	42	− .55	.11
						24	− .16	?	41	− .59		24	.34	?	51	− .69	
						42	− .41		24	.36	?	42	− .49	.11	25	.38	?
									42	− .36		51	− .68		52	− .53	.05
												25	.36	?	61	− .74	
												52	− .51		26	.36	?
															62	− .61	
															27	− .33	?
															72	− .71	
Value of Q for single free proton	0			−0.40			−0.56			−0.67			−0.73			−0.77	
Average of D's	0			−0.057			−0.14			−0.19			−0.27			−0.34	
Number of D's	∞			8			12			13			15			18	
$\frac{\text{Average of } D\text{'s}}{Q \text{ for s.f.p.}}$				14%			25%			28%			37%			44%	
$\bar{D}_{2,2}$ *	0			0			−0.098			−0.16			−0.25			−0.32	

$* \bar{D}_{2,2}=\frac{(L_2-1)(L_2+3/2)}{L_2(L_2+1/2)}\bar{D}_{2,2,-}.$

times a function containing only neutron coordinates:

$$\Theta(l_\pi, l_\nu, m_\pi, m_\nu)=\theta(l_\pi, m_\pi)\Omega(l_\nu, m_\nu).$$

It can easily be proved that

$$\int\Omega^*(l_\nu', m_\nu')\Omega(l_\nu, m_\nu)d\tau=\delta(l_\nu', l_\nu)\delta(m_\nu', m_\nu).$$

The magnetic moment of an odd nucleus seems to be due largely to the single odd particle.[1] One immediately wonders whether the quadrupole moment can be understood on the basis of this simple picture also. We will therefore assume in this part of our analysis that the nucleus consists of a core in which the charge distribution is spherically symmetric, and outside is a single proton. The motion of this proton is strongly affected by the other particles, including the other protons, but the latter protons do not contribute to the quadrupole moment. Hence these core protons will act like neutrons in the calculation of Q and l_ν will be the total orbital angular momentum of the core. For the single proton we shall use the normalized associated Legendre function for $\theta(l_\pi, m_\pi)$ to express the angular dependence. Then

$$\int\theta^*(l_\pi', m_\pi')Q\theta(l_\pi, m_\pi)d\tau=\delta(m_\pi', m_\pi)\Bigg\{\delta(l_\pi', l_\pi)2\left[\frac{l_\pi(l_\pi+1)-3m_\pi^2}{(2l_\pi+3)(2l_\pi-1)}\right]$$
$$+\delta(l_\pi', l_\pi+2)\frac{3}{2l_\pi+3}\left[\frac{(l_\pi+m_\pi+2)(l_\pi+m_\pi+1)(l_\pi-m_\pi+2)(l_\pi-m_\pi+1)}{(2l_\pi+1)(2l_\pi+5)}\right]^{\frac{1}{2}}$$
$$+\delta(l_\pi', l_\pi-2)\frac{3}{2l_\pi-1}\left[\frac{(l_\pi+m_\pi)(l_\pi+m_\pi-1)(l_\pi-m_\pi)(l_\pi-m_\pi-1)}{(2l_\pi-3)(2l_\pi+1)}\right]^{\frac{1}{2}}\Bigg\}\langle r^2\rangle_{\text{Av}}$$

as can be shown by the use of the recurrence relation for different orders of the associated Legendre function.[4] An expression for the average value of the square of the radius vector will be given later.

Now it readily follows that

$$\int \phi^*(l_\pi', l_\nu', L, m_L) Q \phi(l_\pi, l_\nu, L, m_L) d\tau$$
$$= \delta(l_\nu', l_\nu) \sum (l_\pi', l_\nu', m_\pi', m_\nu' | l_\pi', l_\nu', L, m_L)$$
$$\times (l_\pi, l_\nu, m_\pi, m_\nu | l_\pi, l_\nu, L, m_L)$$
$$\times \int \theta^*(l_\pi', m_\pi') Q \theta(l_\pi, m_\pi) d\tau.$$

The values of the integral on the left-hand side of this equation in units of r^2 have been listed in Table I for $m_L = L$. The values of the integral for $m_L = L-1$ have not been tabulated since they are equal to $(L-3)/L$ times the corresponding integral for $m_L = L$. This general relation is apparent after the integrals have been evaluated for the two cases. The column headed D (direct) gives the sum of the $\delta(l_\pi', l_\pi)$ terms and the column headed X (cross term) gives the sum of the $\delta(l_\pi', l_\pi+2)$ terms coming from $\int \theta^*(l_\pi') Q \theta(l_\pi) d\tau$. As the function $(l_\pi'=3)$ combines with the function $(l_\pi=1)$ to give the same value as $(l_\pi'=1)$ combining with $(l_\pi=3)$ in the $\delta(l_\pi', l_\pi-2)$ term, we can multiply the latter by 2 and consider only $l_\pi' > l_\pi$ in the summation below. So we obtain

$$\int \Phi^*(L, m_L) Q \Phi(L, m_L) d\tau = \sum |C(l_\pi, l_\nu)|^2 D + 2 \sum C^*(l_\pi+2, l_\nu) C(l_\pi, l_\nu) X,$$

where the first sum is taken over all pairs of l_π and l_ν that can give L, and the second sum is taken likewise with the additional restriction that $l_\pi+2$ and l_ν can also give L. So

$$\frac{1}{2J+2} Q_{2,2,+} + \frac{2J+1}{2J+2} Q_{2,2,-}$$
$$= \sum |C(l_\pi, l_\nu)|^2 \left\{ \frac{1}{2J+2} D_{2,2,+} + \frac{2J+1}{2J+2} D_{2,2,-} \right\}$$
$$+ 2 \sum C^*(l_\pi+2, l_\nu) C(l_\pi, l_\nu)$$
$$\times \left\{ \frac{1}{2J+2} X_{2,2,+} + \frac{2J+1}{2J+2} X_{2,2,-} \right\}.$$

[4] See, for example, reference 3, Eq. 4^3, 2.

On writing $D_{2,2}$ for the first { } term and $X_{2,2}$ for the second { } term, the last equation reads

$$\frac{1}{2J+2} Q_{2,2,+} + \frac{2J+1}{2J+2} Q_{2,2,-} = \sum |C(l_\pi, l_\nu)|^2 D_{2,2} + 2 \sum C^*(l_\pi+2, l_\nu) C(l_\pi, l_\nu) X_{2,2}.$$

By use of the relation mentioned above, i.e.,

$$D_{2,2,+} = \frac{L_2 - 3}{L_2} D_{2,2,-}$$

the value of $D_{2,2}$ is found to be

$$D_{2,2} = \frac{(L_2-1)(L_2+\frac{3}{2})}{L_2(L_2+\frac{1}{2})} D_{2,2,-}$$

and this can be computed from Table I. An exactly similar relation exists between $X_{2,2}$ and $X_{2,2,-}$.

If the product of three Hermitian functions $H(n_1, q^{\frac{1}{2}}x) H(n_2, q^{\frac{1}{2}}y) H(n_3, q^{\frac{1}{2}}z)$ is used as the radial part of the single proton wave function, it can readily be shown that the average value of r^2 is $(N+\frac{1}{2})/q$ where $N = n_1+n_2+n_3+1$, and with the further relation $N = 2n-l-1$ where n is the principal quantum number and l the azimuthal quantum number.[5] The relation for r^2 holds not only for $l_\pi' = l_\pi$, $n' = n$ but also for $l_\pi' = l_\pi+2$, $n' = n+1$, i.e., corresponding to the same N. For the light elements, the single proton is in the $2p$ shell and $N = 2$, while for the heavy elements at the end of the periodic table, $N = 5$.[6] As a value for $1/q$ we choose 4.2×10^{-26} cm^2 which has been used in calculations of the binding energy of Li^6,[7] and we will multiply this value by a scale factor equal to the square of the ratio of the nuclear radius of the element in question to the nuclear radius of Li^6, i.e., by (atomic weight/6)$^{\frac{2}{3}}$.

Since the X terms in any column of the tables, i.e., for a particular value of L, are all of the same sign, and since the C's are of both signs, the net effect of summing over the distribution of pairs of values of l_π and l_ν is probably small for these X terms. A similar argument may be applied to the terms involved in $Q_{1,2,+}$. So the

[5] See, for example, H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 172 (1936), Eq. (187e).
[6] H. Margenau, Phys. Rev. **46**, 613 (1934).
[7] H. Margenau and K. G. Carroll, Phys. Rev. **54**, 705 (1938).

main contribution for heavy nuclei is likely to come from

$$|a_1|^2 \sum C(l_\pi, l_\nu)^2 D_{1,1,+} + |a_2|^2 \sum C(l_\pi, l_\nu)^2 D_{2,2}.$$

Case I. J = 1/2

There should be no appreciable quadrupole moment due to the single proton. Cf. N^{13} and N^{15}, both of which have a nuclear spin of $\frac{1}{2}$ and have one proton lacking from a full $2p$ shell. Casimir[8] has shown that there should be no quadrupole moment for those nuclei that have $J=0$ or $\frac{1}{2}$. Experimental evidence on this point is absent for the quadrupole moment is calculated from the departures of the atomic hyperfine structure from the Landé interval rule. Hence for $J=\frac{1}{2}$, each energy level is split into only two levels and therefore no departures from the interval rule could be computed.

Case II. J = 3/2

In this case and the following ones some assumption about the probabilities of the different pairs of values of l_π and l_ν must be made. For simplicity one might assume that all allowable pairs occurred with equal probability, i.e., $|C(l_\pi, l_\nu)|^2 = \text{constant}$ for all $\mathbf{l}_\pi + \mathbf{l}_\nu = \mathbf{L}$. The average of the D terms must then be used and

$$Q = [|a_1|^2(-0.057) + |a_2|^2(-0.098)]r^2 = -0.057(1+0.72|a_2|^2)\langle r^2\rangle_{\mathrm{Av}}$$

for the number of terms computed. It is worthwhile to note that as more D terms are taken the average approaches more nearly the value of the quadrupole moment of a free proton with $l_\pi = L$ and $m_\pi = m_L$. This is indicated at the bottom of the tables.

If one arranges in order of increasing energy, the energy levels of a particle in a potential hole with finite walls, one gets the levels $1s$, $2p$, $3d$, $2s$; $4f$, $3p$, $5g$, $4d$, $6h$, $3s$.[6] The last shell is filled with the 92nd proton, and the $4f$ shell starts to be filled with the 21st proton. While the exact ordering of the levels for this simple picture has little meaning for heavy elements, it does indicate that the values of l_π may range from 0 to 5 in a random fashion, and our assumption of $|C(l_\pi, l_\nu)|^2 = \text{constant}$ gains a little more plausibility.

Similarly under the previous assumptions we find for

$$J=5/2,\quad Q=-0.14(1+0.14|a_2|^2)\langle r^2\rangle_{\mathrm{Av}},$$
$$J=7/2,\quad Q=-0.19(1+0.31|a_2|^2)\langle r^2\rangle_{\mathrm{Av}},$$
$$J=9/2,\quad Q=-0.27(1+0.2|a_2|^2)\langle r^2\rangle_{\mathrm{Av}}.$$

So we see that no matter what the value of a_2, the value of Q increases negatively as J increases.

The only observed quadrupole moments of heavy nuclei which are negative are those of ${}_{29}Cu^{63} = -0.1$, ${}_{29}Cu^{65} = -0.1$, and ${}_{83}Bi^{209} = -0.4 \times 10^{-24}$ cm^2.[9] For both isotopes of copper $J=\frac{3}{2}$, and the magnetic moments[10] indicate that a_2 is probably small. For the Bi nucleus $J=9/2$, a_2 is probably large. So we obtain from the single particle picture a value of

$$Q=-0.05(1+0.72|a_2|^2)\times 10^{-24}\ \mathrm{cm}^2 \text{ for Cu}$$

and

$$Q=-0.33(1+0.2|a_2|^2)\times 10^{-24}\ \mathrm{cm}^2 \text{ for Bi}.$$

These results indicate that for these three nuclei the quadrupole moment may well be due to a single proton outside of a symmetrically charged spherical core.

The statement of Inglis[1] that "a single proton moving in any state but an s state has a quadrupole moment which is negative" of course applies only to a proton with $m=l>0$,[2]

$$Q=\frac{2[l(l+1)-3m^2]}{(2l+3)(2l-1)}=-\frac{2l}{2l+3}.$$

When $l(l+1)>3m^2$ this is no longer true. The combination of the different states with $m_\pi < l_\pi$ with the neutron functions gives rise to many positive terms, as seen in Table I. On the "average," however, these are more than offset by the negative terms.

The three odd nuclei discussed above are the only ones which have negative Q; all the rest have positive values. In these cases the quadrupole moment must be due to the cooperation of more than one proton outside the spherical core or to a different kind of coupling than that corresponding to our assumption. Among the heavy

[8] H. B. G. Casimir, Physica 2, 719 (1935).

[9] H. Schüler and Th. Schmidt, Zeits. f. Physik 111, 165 (1938); 99, 717 (1936).

[10] See reference 1(c) Fig. 3. A curve through the upper points would give the Schmidt curve $L_1 = J - \frac{1}{2}$, the bottom points would give $L_2 = J + \frac{1}{2}$.

TABLE II. Values of the quadrupole moments of light nuclei and the quantities used in their calculation.

Element	$25Q/\langle r^2\rangle_{Av}$	C_1	C_2	C_3	$\frac{\langle r^2\rangle_{Av}}{25}\times 10^{26}$ cm²	Q in units of 10^{-26} cm²
Li^7	$-(C_1^2+10C_2^2)$	0.681	0.732		0.467	−2.7
N^{13}	0					0
Be^7	$-(8\sqrt{5}\,C_1C_2+7C_1^2)$	0.681	0.732		0.467	−5.7
C^{13}	0					0
Li^8	$(5C_1^2-C_2^2)$	0.785	0.619		0.511	1.4
B^{12}	$-6\sqrt{15}\,C_1C_2$	0.785	0.619		0.670	−7.6
Be^9	$(-8\sqrt{5}\,C_1C_2-7C_2^2+7C_3^2)$	0.731	−0.344	−0.589	0.553	3.4
C^{11}	$-(-8\sqrt{5}\,C_1C_2-7C_2^2+7C_3^2)$	0.731	−0.344	−0.589	0.632	−3.9
B^9	$-6\sqrt{3}\,C_2C_3$	0.731	−0.344	−0.589	0.553	−1.2
B^{11}	$6\sqrt{3}\,C_2C_3$	0.731	−0.344	−0.589	0.632	1.3
N^{13}, N^{15}	0					0

nuclei, moreover, the magnitude of Q is much too large to be caused by a single proton.

QUADRUPOLE MOMENTS OF LIGHT NUCLEI

Some confirmation of the results of the foregoing section as well as further insight into the problem may be gained by calculating the quadrupole moments of light nuclei on the basis of the Hartree model. Feenberg and Wigner[11] have determined the ground state terms in this model and Rose and Bethe[12] have determined the nuclear spin, thus the a's and C's are known. The ground state of each odd nucleus is a P state and for Li^7, Be^7, Be^9, B^{11}, C^{11} the nuclear spin is $J=\frac{3}{2}$, hence $|a_1|^2=1$ and $|a_2|^2=0$. For N^{13}, N^{15}, C^{13}, O^{15} the nuclear spin is $J=\frac{1}{2}$ hence $|a_1|^2=0$ and $|a_2|^2=1$. In the case of N^{13} and N^{15} we have one p proton lacking from a complete shell which would have zero Q. But a hole acts like a proton with a negative charge in determining Q. Hence we can apply our previous result, since $J=\frac{1}{2}$, the quadrupole moment must be zero.

The wave functions for one to five identical p particles have been listed below. These are used in the wave functions of the odd nuclei, which have also been included here in order to show the correct phases which are of importance in obtaining the proper sign for the cross terms. In addition the wave functions for the two even nuclei Li^8 and B^{12}, whose ground state is a P state, have been included. All the other even nuclei have as a ground state an S state and hence will have no quadrupole moment, as can be readily shown.

In order that C_3 for B^9 and B^{11} will have the same sign, it is necessary to use for B^{11} the negative of the $P^1(^2D_\nu\,^2D_\pi)$ function used for B^9. This change of signs, when we pass from a nucleus of n_1 protons and n_2 neutrons to one of $6-n_1$ protons and $6-n_2$ neutrons, occurs only when $l_\pi+l_\nu-L$ is odd. The values of the C's for B^9, B^{11}, Be^9, and C^{11} are the same because of the model used, in which the interaction between protons is equal to the interaction between neutrons, and because of the results of the theory of holes.[13] The C's are not very sensitive to the values of the force constants.[14] From the above considerations it follows, for example, that B^9 and B^{11} will have Q's approximately equal in magnitude but opposite in signs. Similarly for Be^9 and C^{11}.

Using the values of Q calculated for the different light nuclei, we see that the substitution of a proton for a neutron makes Q more negative (exception B^{12}). There is no apparent relation for odd proton, even neutron nuclei as two are negative, two are zero, and one is positive. Also for odd neutron, even proton nuclei, two are negative, one is zero, and one is positive. Most interesting is the case of the addition of a proton to a nucleus whose Q was originally zero. This addition results in a negative Q until the p shell is lacking two protons, then the addition has no effect. This confirms the results of the previous section, that a single proton added to a nucleus with spherically symmetric charge distribution should produce a negative quadrupole moment.

The presence of a large cross term in the cal-

[11] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937).
[12] M. E. Rose and H. A. Bethe, Phys. Rev. **51**, 205 (1937).
[13] See reference 11, Eq. (11).
[14] See reference 12, Note 12.

culated value of Q for light nuclei in many cases might at first glance suggest that we are not justified in neglecting the effect of cross terms in the previous treatment. But the fault lies rather in the Hartree model. The ground state wave function consists of the sum of only two or three "parent" functions and it is well known that this yields a value of the binding energy which is too small. As more functions are used, the agreement becomes better. This would introduce more cross terms for Q which will, as we have indicated above, probably give a small net contribution.

The author is happy to express his thanks and appreciation to Professor Henry Margenau for suggesting this work and for his helpful interest in its progress.

APPENDIX I

Wave functions for:

One p particle

$P^1 = -a$
$P^0 = b$
$P^{-1} = c.$

Two p particles

$D^2 = aa$
$D^1 = -(ba+ab)/2^{\frac{1}{2}}$
$D^0 = (-ca+2bb-ac)/6^{\frac{1}{2}}$
$D^{-1} = (cb+bc)/2^{\frac{1}{2}}$
$D^{-2} = cc$
$P^1 = (-ab+ba)/2^{\frac{1}{2}}$
$P^0 = (-ac+ca)/2^{\frac{1}{2}}$
$P^{-1} = (bc-cb)/2^{\frac{1}{2}}$
$S^0 = -(ca+bb+ac)/3^{\frac{1}{2}}.$

Three p particles

$D^2 = aab$
$D^1 = (-abb+aac)/2^{\frac{1}{2}}$
$D^0 = -(acb+2abc+bac)/6^{\frac{1}{2}}$
$D^{-1} = (-acc+bbc)/2^{\frac{1}{2}}$
$D^{-2} = bcc$
$P^1 = -(abb+aac)/2^{\frac{1}{2}}$
$P^0 = (-acb+bac)/2^{\frac{1}{2}}$
$P^{-1} = -(acc+bbc)/2^{\frac{1}{2}}.$

Four p particles

$D^2 = aabb$
$D^1 = (aacb+aabc)/2^{\frac{1}{2}}$
$D^0 = (-bacb+2aacc-abbc)/6^{\frac{1}{2}}$
$D^{-1} = -(bacc+abcc)/2^{\frac{1}{2}}$
$D^{-2} = bbcc$
$P^1 = (-aacb+aabc)/2^{\frac{1}{2}}$
$P^0 = (bacb-abbc)/2^{\frac{1}{2}}$
$P^{-1} = (bacc-abcc)/2^{\frac{1}{2}}$
$S^0 = -(bacb+aacc+abbc)/3^{\frac{1}{2}}.$

Five p particles

$P^1 = aabbc$
$P^0 = aabcc$
$P^{-1} = -abbcc.$

APPENDIX II

Wave functions for:

Li^7 (2 neutrons, 1 proton) and N^{13} (4 neutrons, 5 protons).

$$C_1({}^1D_\nu\, {}^2P_\pi)+C_2({}^1S_\nu\, {}^2P_\pi),$$

where

$$P^1({}^1D_\nu\, {}^2P_\pi)=\{D_\nu{}^0P_\pi{}^1-3^{\frac{1}{2}}D_\nu{}^1P_\pi{}^0+6^{\frac{1}{2}}D_\nu{}^2P_\pi{}^{-1}\}/10^{\frac{1}{2}}$$

and

$$P^1({}^1S_\nu\, {}^2P_\pi)=S_\nu{}^0P_\pi{}^1.$$

For Be^8 (2 protons, 1 neutron) and C^{13} (4 protons, 5 neutrons) the subscripts ν and π are interchanged. $C_1=0.681$, $C_2=0.732$.

Li^8 (3 neutrons, 1 proton).

$$C_1({}^2P_\nu\, {}^2P_\pi)+C_2({}^2D_\nu\, {}^2P_\pi),$$

where

$$P^1({}^2P_\nu\, {}^2P_\pi)=\{P_\nu{}^0P_\pi{}^1-P_\nu{}^1P_\pi{}^0\}/2^{\frac{1}{2}}$$

and

$$P^1({}^2D_\nu\, {}^2P_\pi)=\{D_\nu{}^0P_\pi{}^1-3^{\frac{1}{2}}D_\nu{}^1P_\pi{}^0+6^{\frac{1}{2}}D_\nu{}^2P_\pi{}^{-1}\}/10^{\frac{1}{2}}.$$

For B^8 (3 protons, 1 neutron) the subscripts are interchanged. For N^{12} (3 neutrons, 5 protons) the negative of the $P^1({}^2P_\nu\, {}^2P_\pi)$ must be used, and for B^{12} (3 protons, 5 neutrons) the subscripts in the N^{12} functions are interchanged. $C_1=0.785$, $C_2=0.619$.

B^9 (2 neutrons, 3 protons).

$$C_1({}^1S_\nu\, {}^2P_\pi)+C_2({}^1D_\nu\, {}^2P_\pi)+C_3({}^1D_\nu\, {}^2P_\pi),$$

where

$$P^1({}^1S_\nu\, {}^2P_\pi)=S_\nu{}^0P_\pi{}^1,$$
$$P^1({}^1D_\nu\, {}^2P_\pi)=\{D_\nu{}^0P_\pi{}^1-3^{\frac{1}{2}}D_\nu{}^1P_\pi{}^0+6^{\frac{1}{2}}D_\nu{}^2P_\pi{}^{-1}\}/10^{\frac{1}{2}},$$
$$P^1({}^1D_\nu\, {}^2D_\pi)=\{-2^{\frac{1}{2}}D_\nu{}^{-1}D_\pi{}^2+3^{\frac{1}{2}}D_\nu{}^0D_\pi{}^1-3^{\frac{1}{2}}D_\nu{}^1D_\pi{}^0+2^{\frac{1}{2}}D_\nu{}^2D_\pi{}^{-1}\}/10^{\frac{1}{2}}.$$

For Be^9 (2 protons, 3 neutrons) the subscripts are interchanged. For B^{11} (4 neutrons, 3 protons) the negative of $P^1({}^1D_\nu\, {}^2D_\pi)$ must be used and for C^{11} (4 protons, 3 neutrons) the subscripts in the B^{11} functions are interchanged. $C_1=0.731$, $C_2=-0.344$, $C_3=-0.589$.