Theory of the Magnetron

II. Oscillations in a Split-Anode Magnetron

LÉON BRILLOUIN*
University of Wisconsin, Madison, Wisconsin
(Received May 7, 1942)

A theoretical study is made of the possible oscillations in split-anode magnetrons. It is found that large oscillations can occur in certain frequency ranges, these ranges depending upon the number of segments. In Fig. 6 these frequency ranges are indicated with $y=21310/\lambda H$ as a function of the number of segments ($2n$). The large oscillations occur in the first and fourth region as tabulated at the top of the figure.

1. SPLIT-ANODE MAGNETRON—GENERAL REMARKS

THE behavior of a magnetron operated under steady conditions was studied in a previous paper[1] where the conditions of oscillation were also discussed for the case of a magnetron with one anode of cylindrical shape. This first paper will be referred to under the title "Magnetron I," and a numeral I will be added to its equation numbers to avoid confusion with the present paper.

The aim of the present work is a study of oscillation conditions in a split-anode magnetron. Such magnetrons are built with an anodic structure consisting of an even number of shells equally spaced so as to form an almost complete cylinder about the filament. The most popular type has two half-cylindrical anodes, but other models with as many as twelve or more anodes have also been successfully used. The oscillating circuit is connected by one of its terminals to the odd-numbered shells, the second terminal being connected to the even-numbered shells, as shown schematically in Fig. 1.

The exact field distribution in such magnetrons would be very difficult to compute, but a reasonable theoretical attempt is possible under the assumption of very small sinusoidal oscillations. This is illustrated in Fig. 2. Let us call b the radius of the anodic structure and $2n$ the number of anodes. The position of the anodes as a function of θ is indicated along the θ axis. Let V_c be the average constant potential of the anodes and $\pm U_a e^{i\omega t}$ the additional alternating potential, the $+$ sign corresponding to the odd anodes and the $-$ sign to the even ones. The potential distribution on the anode circle is represented by the curve in Fig. 2. Such a broken curve can easily be expanded in a Fourier series. Let α be the angle subtended by one anode while β is the angle between two successive anodes. Then $\alpha+\beta=\pi/n$. The potential on the radius $r=b$ is given by the series

$$V(b)=V_c(b)+U_a e^{i\omega t}\frac{8}{\pi n\beta}\times\sum_p \frac{1}{p^2}\sin\left(\tfrac{1}{2}pn\beta\right)\sin(pn\theta). \quad p \text{ odd} \quad (1)$$

The first term $p=1$ is the most important and the next term $p=3$ can be made zero by taking

$$\sin(3/2)n\beta=0,\quad \beta=2\pi/3n,\quad \alpha=\pi/3n,$$

which means that the distance between the anodes is twice the width of the anodes.

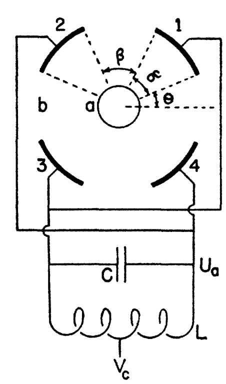

FIG. 1. Connections of terminals to oscillator.

* Now at Brown University, Providence, Rhode Island.
[1] L. Brillouin, Phys. Rev. 60, 385 (1941); also Elec. Communication 20, 112 (1941).

The calculations will be made under the assumption that the alternating term U_a is very small with an ϵ coefficient the highest power of which will be neglected. Furthermore, we shall keep only the first term of the Fourier expansion.

$$V(b)=V_c(b)+\epsilon V_a(b)e^{i\omega t}\sin(n\theta),\qquad \epsilon V_a(b)=U_a(8/\pi n\beta)\sin(\tfrac{1}{2}n\beta). \tag{2}$$

This is the assumed distribution of potential on the circle $r=b$, but one should keep in mind the fact that a magnetron built with $2n$ anodes may eventually work on one of the higher modes corresponding to $2n'$ anodes with $n'=3n, 5n, \cdots$. It is easily seen that these higher modes of oscillation will increase in probability if the interval β between the anodes is very small. At the limit $\beta=0$, $\alpha=\pi/n$, the amplitude of the pth term becomes

$$\lim_{\beta=0}\frac{8}{\pi n\beta}\frac{1}{p^2}\sin(\tfrac{1}{2}pn\beta)=\frac{4}{\pi p}. \tag{3}$$

2. FUNDAMENTAL EQUATIONS

The fundamental equations are Maxwell's relations which will be written in cylindrical coordinates r, θ, z, where z is the axis taken along the filament. It will be assumed that all quantities are independent of z. The current density components J_r, J_θ, including Maxwell's displacement current, are

$$J_r=\rho v_r+\frac{\epsilon_0}{4\pi}\frac{\partial E_r}{\partial t},\qquad J_\theta=\rho v_\theta+\frac{\epsilon_0}{4\pi}\frac{\partial E_\theta}{\partial t}. \tag{4}$$

v_r and v_θ are the components of the velocities of the electrons, ρ the space charge density, E_r and E_θ the components of the electric field, and $E_z=0$. The coefficients ϵ_0, μ_0 represent the dielectric power and the magnetic permeability in vacuum. Electrostatic c.g.s. units will be consistently used, making

$$\epsilon_0=1,\quad \mu_0=c^{-2}. \tag{5}$$

$J_r rd\theta$ is the radial current flowing through $rd\theta$, per unit of length z, while $J_\theta dr$ is the angular component of the current through dr, per unit z. The total radial current per unit of length of z is

$$I_r=\int_0^{2\pi}J_r rd\theta. \tag{6}$$

It is well known that the introduction of Maxwell's displacement terms secures the following relation:

$$\operatorname{div}J=\frac{1}{r}\frac{\partial}{\partial r}(rJ_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(J_\theta)=0. \tag{7}$$

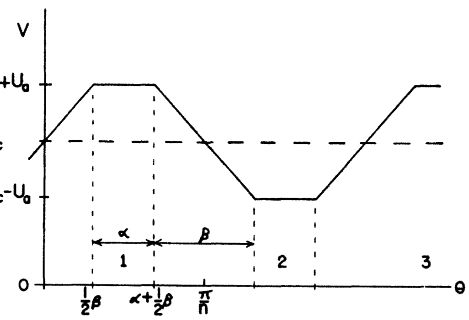

FIG. 2. Potential distribution on the anode circle.

This may easily be verified, with the conservation equation for electric charge

$$\partial\rho/\partial t=-\operatorname{div}(\rho v)=-\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)-\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta), \tag{8}$$

together with one of Maxwell's equations:

$$\operatorname{div}E=\frac{1}{r}\frac{\partial}{\partial r}(rE_r)+\frac{1}{r}\frac{\partial}{\partial\theta}E_\theta=4\pi\rho/\epsilon_0. \tag{9}$$

Let us call H_r, H_θ, H_z the components of the oscillating magnetic field. The symmetry of the problem shows that H_r and H_θ are zero. Consequently

$$\operatorname{div}H=\partial H_z/\partial z=0, \tag{10}$$

which proves that H_z is independent of z. Maxwell's equations yield

$$\operatorname{rot}H=4\pi J\begin{cases}\partial H_z/r\partial\theta=4\pi J_r,\\ -\partial H_z/\partial r=4\pi J_\theta;\end{cases} \tag{11}$$

$$\operatorname{rot}E=-\mu_0\partial H/\partial t,\qquad \partial(rE_\theta)/r\partial r-\partial E_r/r\partial\theta=-\mu_0\partial H_z/\partial t. \tag{12}$$

This system reduces to three relations: (11.1), (11.2), and (12). Equation (7) shows that the relations (11.1) and (11.2) are compatible as it gives

$$\partial^2 H_z/\partial r\partial\theta = \partial^2 H_z/\partial\theta\partial r.$$

Hence we can write

$$H_z = 4\pi\int rJ_r d\theta = -4\pi\int J_\theta dr, \qquad (13)$$

and we are left with one relation

$$\frac{\partial(rE_\theta)}{r\partial r} - \frac{\partial E_r}{r\partial\theta} = -4\pi\mu_0\frac{\partial}{\partial t}\int rJ_r d\theta, \qquad (14)$$

which is to be used together with (9) to define the electric field. It is well known that any field E can be represented as the sum of an irrotational field (grad V) and a field E' with no divergence.

$$E_r = -\frac{\partial V}{\partial r} + E'_r, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} + E'_\theta, \qquad (15)$$

$$\operatorname{div} E' = \frac{\partial(rE'_r)}{r\partial r} + \frac{\partial E'_\theta}{r\partial\theta} = 0. \qquad (16)$$

Substituting in Eqs. (9) and (14) we obtain

$$\Delta V = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} = -\frac{4\pi\rho}{\epsilon_0}; \qquad (17)$$

$$\frac{\partial(rE'_\theta)}{r\partial r} - \frac{\partial E'_r}{r\partial\theta} = -4\pi\mu_0\frac{\partial}{\partial t}\int rJ_r d\theta. \qquad (18)$$

The assumption will now be made that the second field E' is very small compared to grad V and can be consistently neglected. This will be proved later on to be a reasonable hypothesis in most cases, provided the dimensions of the magnetron are small compared with the wavelength λ in vacuum.

$$b \ll \lambda. \qquad (19)$$

Some special cases where this assumption would lead to error will be discussed. Most of them lie outside the actual operation conditions of the magnetron. Equation (18) may be reduced by the use of (16). One differentiates (18) with respect to θ and replaces $\partial E'_\theta/\partial\theta$ by its value $-\partial rE'_r/\partial r$, which yields

$$\Delta(rE'_r) = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}rE'_r + \frac{1}{r}\frac{\partial^2 E'_r}{\partial\theta^2} = 4\pi\mu_0\partial(rJ_r)/\partial t. \qquad (20)$$

This equation will be used later in the discussion of the order of magnitude of E'_r and the conditions for its negligibility.

The preceding equations must be combined with the equations of motion for the electrons [Eqs. I.1) to (I.8)].

$$d^2r/dt^2 = \frac{e}{m}E_r + r\dot{\theta}^2 - 2r\omega_H\dot{\theta}, \quad d(r^2\dot{\theta})/dt = \frac{e}{m}rE_\theta + 2\omega_H r\dot{r}, \qquad (21)$$

in which

$$\omega_H = -\tfrac{1}{2}\mu_0(e/m)H \quad \text{Larmor's angular velocity.} \qquad (22)$$

The second equation in (21) may also be written

$$\frac{d}{dt}[r^2(\dot{\theta}-\omega_H)] = \frac{e}{m}rE_\theta. \qquad (23)$$

According to the assumption about the field, only grad V will be used in these equations.

The derivative of a function $f(r, \theta, t)$ is often to be taken following the motion of one electron. The symbol d/dt will be used on such occasions:

$$d/dt = \partial/\partial t + v_r\partial/\partial r + v_\theta\partial/r\partial\theta. \qquad (24)$$

Applying this definition, using Eqs. (4) and (17) and neglecting E', one obtains, since $\epsilon_0 = 1$ (e.s.u.),

$$\frac{d}{dt}(r\partial V/\partial r) = r\partial^2 V/\partial r\partial t + v_r\frac{\partial}{\partial r}(r\partial V/\partial r) + v_\theta\partial^2 V/\partial r\partial\theta = -4\pi rJ_r - v_r\partial^2 V/r\partial\theta^2 + v_\theta\partial^2 V/\partial r\partial\theta. \qquad (25)$$

In a similar way

$$\frac{d}{dt}\left(\frac{\partial V}{\partial\theta}\right) = \partial^2 V/\partial\theta\partial t + v_r\partial^2 V/\partial\theta\partial r + v_\theta\partial^2 V/r\partial\theta^2 = -4\pi rJ_\theta + v_r\partial^2 V/\partial\theta\partial r - v_\theta\frac{\partial}{\partial r}(r\partial V/\partial r). \qquad (26)$$

These two important formulae are the generalization of Eq. (I.41). They are very useful because they no longer contain the space charge density ρ. Our fundamental equations thus reduce to Eqs. (7), (21), (23), (25), and (26).

3. SMALL OSCILLATIONS IN A SPLIT-ANODE MAGNETRON

As in "Magnetron I" (Section 6) it is assumed that very small oscillations are superimposed on the static potential and current distributions. The smallness of these oscillations is secured by a parameter ϵ, the higher powers of which will be consistently neglected. We thus assume

$$\left.\begin{aligned} V(r,\theta,t)&=V_c(r)+\epsilon V_a(r,\theta,t),\\ J_r(r,\theta,t)&=I_c/2\pi r+\epsilon J_{ar}(r,\theta,t),\\ J_\theta(r,\theta,t)&=\rho_c r\dot{\theta}_c+\epsilon J_{a\theta}(r,\theta,t).\end{aligned}\right\} \quad (27)$$

The potential V_c and the current I_c are those defined in "Magnetron I" for the magnetron operated under steady conditions.

$$\dot{\theta}_c(r)=\omega_H(1-a^2/r^2). \quad (28)$$

In the new field of force electron trajectories are slightly modified. Let us follow one particular electron leaving the filament at a time t_0; its position at time t is

$$\begin{gathered} r_c,\ \theta_c \quad \text{in the unperturbed motion.}\\ r=r_c+\epsilon r_a(r_c,\theta_c,t),\quad \theta=\theta_c+\epsilon\theta_a(r_c,\theta_c,t) \quad (29)\\ \text{in the perturbed motion.}\end{gathered}$$

In the unperturbed motion the electron describes the static trajectory calculated in "Magnetron I," while in the perturbed motion its position at time t is at a distance ϵr_a, $\epsilon\theta_a$ from the unperturbed location. It is necessary to calculate the value of quantities such as V at the point where the electron is actually located. Let f be such a quantity. Then Eq. (27) would read

$$f(r,\theta,t)=f_c(r,\theta)+\epsilon f_a(r,\theta,t). \quad (30)$$

If this quantity be measured at the point $r_c+\epsilon r_a$, $\theta_c+\epsilon\theta_a$ where the electron is located, we must write

$$\begin{aligned} f(r_c+\epsilon r_a,\theta_c+\epsilon\theta_a,t)&=f_c(r_c,\theta_c)\\ &\quad+\epsilon[r_a\partial f_c/\partial r+\theta_a\partial f_c/\partial\theta+f_a], \quad (31)\end{aligned}$$

terms in ϵ^2, ϵ^3, $\cdots$ being consistently neglected.

The fundamental equations can now be written with the expansions (27) to (31). Equation (7) first yields a very simple relation

$$\partial(rJ_{ar})/\partial r=-\partial J_{a\theta}/\partial\theta. \quad (32)$$

Equation (21) gives

$$\begin{aligned} d^2r_c/dt^2+\epsilon d^2r_a/dt^2 &= -\frac{e}{m}[\partial V_c/\partial r+\epsilon r_a\partial^2V_c/\partial r^2+\epsilon\partial V_a/\partial r]\\ &\quad-2\omega_H[r_c\dot{\theta}_c+\epsilon(r_c\dot{\theta}_a+r_a\dot{\theta}_c)]\\ &\quad+r_c\dot{\theta}_c^2+\epsilon[r_a\dot{\theta}_c^2+2r_c\dot{\theta}_c\dot{\theta}_a]. \quad (33)\end{aligned}$$

Terms independent of ϵ correspond to the static formula (I.19). In ϵ terms we use (28) for $\dot{\theta}_c$ and get finally

$$d^2r_a/dt^2+r_a\left(\frac{e}{m}\partial^2V_c/\partial r^2+\omega_H^2-\omega_H^2a^4/r^4\right) = -\frac{e}{m}\partial V_a/\partial r-2\omega_Ha^2\dot{\theta}_a/r_c. \quad (34)$$

Equation (23) is treated in the same way. Writing only ϵ terms one finds

$$\frac{d}{dt}(r_c^2\dot{\theta}_a+2r_cr_a(\dot{\theta}_c-\omega_H)) = \frac{d}{dt}\left(r_c^2\dot{\theta}_a-2\omega_Hr_a\frac{a^2}{r_c}\right)=-\frac{e}{m}\partial V_a/\partial\theta. \quad (35)$$

Turning to Eq. (25) and applying (31), we get from the ϵ terms

$$\frac{d}{dt}[r_a\partial V_c/\partial r+r_cr_a\partial^2V_c/\partial r^2+r_c\partial V_a/\partial r] = -4\pi r_cJ_{ar}-\frac{\dot{r}_c}{r_c}\partial^2V_a/\partial\theta^2+r_c\dot{\theta}_c\partial^2V_a/\partial r\partial\theta. \quad (36)$$

Equation (26) yields

$$\begin{aligned} \epsilon\frac{d}{dt}(\partial V_a/\partial\theta) &= -4\pi\left[\rho_cr_c^2\dot{\theta}_c+\epsilon r_a\frac{\partial}{\partial r_c}(\rho_cr_c^2\dot{\theta}_c) +\epsilon r_cJ_{a\theta}\right]+\epsilon\dot{r}_c\partial^2V_a/\partial r\partial\theta\\ &\quad-[r_c\dot{\theta}_c+\epsilon(r_c\dot{\theta}_a+r_a\dot{\theta}_c)]\frac{\partial}{\partial r}\Big[(r_c+\epsilon r_a)\\ &\quad\times(\partial V_c/\partial r+\epsilon r_a\partial^2V_c/\partial r^2+\epsilon\partial V_a/\partial r)\Big]. \quad (37)\end{aligned}$$

Terms independent on ϵ just cancel due to (I.21)

$$4\pi\rho_c r_c+\frac{\partial}{\partial r}(r_c\partial V_c/\partial r)=0, \qquad (38)$$

and ϵ terms result in

$$\frac{d}{dt}(\partial V_a/\partial\theta)=-4\pi r_a\frac{\partial}{\partial r_c}(\rho_c r_c^2\dot{\theta}_c)-4\pi r_c J_{a\theta}$$
$$+\dot{r}_c\partial^2 V_a/\partial r\partial\theta-r_c\dot{\theta}_c\frac{\partial}{\partial r}[r_a\partial V_c/\partial r$$
$$+r_c r_a\partial^2 V_c/\partial r^2+r_c\partial V_a/\partial r]$$
$$-[r_c\dot{\theta}_a+r_a\dot{\theta}_c]\frac{\partial}{\partial r}(r_c\partial V_c/\partial r), \qquad (39)$$

where the last term according to (38) is equal to

$$+4\pi\rho_c r_c(r_c\dot{\theta}_a+r_a\dot{\theta}_c).$$

4. TYPE OF ELEMENTARY SOLUTIONS

The new system of Eqs. (32) to (39) is linear with respect to all of the alternating quantities, a condition which enables us to superimpose elementary solutions and to make use of imaginary exponentials. Each alternating quantity is supposed to depend on θ and t in the following way

$$f_a(r,\theta,t)=f_a(r)e^{i(\omega t-n\theta)}. \qquad (40)$$

$f_a(r)$ is an imaginary expression including the phase angle factor.

A solution of type (40) represents one elementary solution, and the general solution of any particular problem may involve superposition of such elementary solutions in order to satisfy boundary conditions. One elementary solution corresponds to a sort of wave rotating about the filament with angular velocity ω/n. In the example shown in Fig. 2, the potential distribution on the anodic cylinder should be represented by Eq. (2). This result will be obtained by the superposition of two solutions $+n$ and $-n$ of type (40), provided these two solutions have the same amplitude factor f_a for $r=b$. These two $\pm n$ solutions will, however, depend on r in different ways so that the superposition will yield a potential (2) only on the cylinder $r=b$. This will be discussed later on.

An elementary solution (40) must be studied for

$$\omega>0,\quad n \text{ positive or negative.} \qquad (41)$$

One must remember, however, that a simultaneous change of the signs on ω and n does not affect the exponential in formula (40). Hence

ω, n and $-\omega, -n$ should yield the same results;

$\omega, -n$ and $-\omega, n$ should also be equivalent.

Every elementary solution must satisfy the boundary conditions on the filament

$$V_a(a)=0,\quad \partial V/\partial r=0,\quad \text{for}\quad r=a. \qquad (42)$$

From Eq. (40) one readily gets

$$\partial/\partial t=i\omega,\quad \partial/\partial\theta=-in. \qquad (43)$$

Our equation system contains derivatives d/dt taken along the path of one electron, as expressed in Eq. (24). On the other hand, making use of Eq. (31), we have reduced any quantity f, observed at the point where an electron is located in the unperturbed motion, to a function of the coordinates r_c, θ_c defining the position of the same electron in the unperturbed motion.[2] The result is that in order to follow the motion of the perturbed electron, we only need $r_c\theta_c$ to describe an unperturbed trajectory; this yields

$$d/dt=\partial/\partial t+\dot{r}_c\partial/\partial r_c+\dot{\theta}_c\partial/\partial\theta$$
$$=i(\omega-n\dot{\theta}_c)+\dot{r}_c\partial/\partial r_c. \qquad (44)$$

Calculations appear to be very complicated in the general case, so we make some simplifying assumptions:

A. very small filament $a=0$,
B. magnetron in the critical state $I_c=0$. (45)

Conditions A and B were discussed in "Magnetron I," Section 3, in which we found that the

[2] Referring to Eqs. (29) and (31), one should notice that they correspond to the use of Lagrange's definitions in hydrodynamics. The same problem could be discussed on a different basis with Euler's definitions. In (29) we compare the position of one particular electron in the unperturbed motion with the position of the *same* electron in the perturbed motion. Euler's method consists in comparing a certain electron located at r, θ (velocities v_r, v_θ) in the unperturbed motion with a *different* electron which, in the perturbed motion happens to come to the same position r, θ, but with different velocities $v_r+\epsilon u_r$, $v_\theta+\epsilon u_\theta$. Euler's method seems to be the one chosen by Blewett and Ramo [Phys. Rev. **57**, 635 (1940)]. With such definitions Eq. (31) must be cancelled, while Eq. (44) should be completed with additional terms because following the motion of one unperturbed electron means comparing different perturbed electrons.

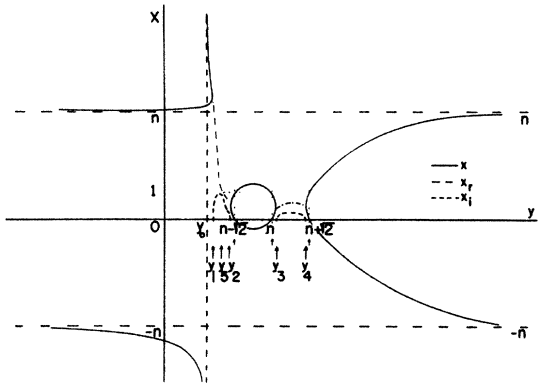

FIG. 3. The roots of Eq. (56).

electrons build a cloud with constant space charge density ρ_0 and rotating about the filament with a constant angular velocity ω_H.

$$\rho_0 = m\omega_H^2/2\pi e, \quad \dot{\theta}_c = \omega_H, \quad \dot{r}_c = 0. \tag{46}$$

Hence Eq. (44) reduces to

$$d/dt = i\alpha, \quad \alpha = \omega - n\omega_H. \tag{47}$$

These simple relations must be introduced into the general equations of Section 3. The problem to be discussed here is very similar to the one treated in "Magnetron I," Section 6. There we had to discuss the role which some damping terms should play in a more complete theory, and the same remarks apply here. The use of Eqs. (43) and (47) is equivalent to taking account only of forced vibrations and neglecting proper vibrations of the electronic system, as such proper vibrations must be damped and practically disappear after a short time.

Equation (34) yields

$$\left[-\alpha^2 + \omega_H^2 + \frac{e}{m}\partial^2 V_c/\partial r^2\right] r_a = -\frac{e}{m}\partial V_a/\partial r. \tag{48}$$

V_c is given by Eq. (I.23)

$$V_c = V_0 = -\frac{m}{2e}\omega_H^2 r^2. \tag{49}$$

Hence

$$r_a = (e/m\alpha^2)\partial V_a/\partial r. \tag{50}$$

Turning now to Eq. (35), one finds

$$\dot{\theta}_a = enV_a/mr_c^2\alpha. \tag{51}$$

By using (49) and (36) the following relation is obtained:

$$2\frac{m}{e}\alpha\omega_H^2 r_a r_c - \omega r_c \partial V_a/\partial r = -i4\pi r_c J_{ar}, \tag{52}$$

while Eq. (32) results in

$$\frac{\partial}{\partial r}(rJ_{ar}) = inJ_{a\theta}. \tag{53}$$

Equations (50) and (52) give

$$J_{ar} = (i/4\pi)\partial V_a/\partial r(2\omega_H^2/\alpha - \omega). \tag{54}$$

Equation (39) may now be greatly simplified by using (47) and (53):

$$\begin{aligned} n\alpha V_a = &-4\pi r_a \frac{\partial}{\partial r_c}(m\omega_H^3 r_c^2/2\pi e) \\ &+(4\pi i/n) r_c \frac{\partial}{\partial r_c}(r_c J_{ar}) \\ &- r_c\omega_H \frac{\partial}{\partial r_c}\left[-\frac{2m}{e}\omega_H^2 r_a r_c + r_c \partial V_a/\partial r\right] \\ &+2\frac{m}{e}\omega_H^2 r_c(r_c\dot{\theta}_a + r_a\omega_H). \end{aligned} \tag{55}$$

Referring now to (50), (51), and (54), one finds a homogeneous differential equation for V_a:

$$\begin{aligned} &Ar\frac{\partial}{\partial r}r\frac{\partial V_a}{\partial r} + Br\frac{\partial V_a}{\partial r} + CV_a = 0, \\ &A = \frac{1}{n}(2\omega_H^2/\alpha - \omega) + \omega_H(1 - 2\omega_H^2/\alpha^2), \\ &B = 2\omega_H^3/\alpha^2, \\ &C = n(\alpha - 2\omega_H^2/\alpha). \end{aligned} \tag{56}$$

Let us introduce a variable $y = \omega/\omega_H$, $\alpha = \omega_H(y-n)$ and new coefficients

$$\begin{aligned} A' &= (y-n)^2 A/\omega_H = -2 - \zeta/n, \\ B' &= (y-n)^2 B/\omega_H = 2, \\ C' &= (y-n)^2 C/\omega_H = n\zeta, \\ \zeta &= (y-n)^3 - 2(y-n). \end{aligned}$$

A solution may be found by taking

$$\begin{aligned} V_a &= Kr^x, \\ A'x^2 + B'x + C' &= 0. \end{aligned} \tag{57}$$

We thus obtain two solutions:

$$\left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} = -\frac{B'}{2A'} \pm \frac{1}{2A'}(B'^2 - 4A'C')^{\frac{1}{2}}. \tag{58}$$

It should be immediately noticed that A', B', and C' are functions of y and n, so that the solutions x_1, x_2 will depend only upon these two variables. Furthermore we should remember the conditions (42) stating that V_a and $\partial V_a/\partial r$ have to be zero on the filament. This results in the

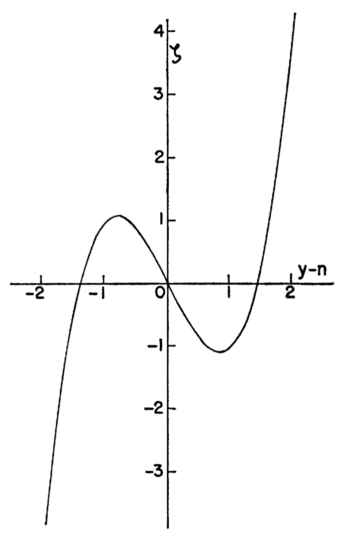

FIG. 4. ζ as a function of $y-n$.

necessity of choosing the root with a real part greater than one.

$$\mathbf{Re}(x) \geqslant 1. \tag{59}$$

For each elementary solution (57) in V_a we shall obtain the corresponding current densities from Eqs. (53) and (54).

$$\begin{aligned} &V_a = Kr^x, \\ &J_{ar} = ixV_a/4\pi r[2\omega_H{}^2/(\omega - n\omega_H) - \omega], \\ &J_{a\theta} = -\frac{i}{n}\frac{\partial}{\partial r}(rJ_{ar}) \\ &\qquad = (x^2V_a/4\pi nr)[2\omega_H{}^2/(\omega - n\omega_H) - \omega]. \end{aligned} \tag{60}$$

5. DISCUSSION OF THE RESULTS

A discussion of the practical results must be based on a numerical calculation of the two roots x_1 and x_2 of Eq. (56). These roots, x_1 and x_2, must be calculated as functions of $y = \omega/\omega_H$ for different values of n. The solutions corresponding to $+\omega$ and $\pm n$ may also be obtained with $+n$ and $\pm\omega$ according to a remark on Eq. (41). Figure 3 shows a typical graph of x as a function of y as found for $n = 2, 3, \cdots$. The case $n = 1$ is exceptional. Figure 3 is self-explanatory and shows the behavior of the roots as functions of y. There are two regions, y_1y_2 and y_3y_4, which give imaginary solutions $x = x_r \pm ix_i$. Curves representing x and x_rx_i have been drawn separately. The limits of these regions are given by

$$\begin{aligned} \tfrac{1}{4}B'^2 - A'C' = 1 + \zeta(\zeta + 2n) = 0, \\ \zeta = -n \pm (n^2 - 1)^{\frac{1}{2}}. \end{aligned} \tag{61}$$

Inside these regions of imaginary solutions we are interested in defining the point y_5 for which $x_r = 1$:

$$x_r = -B'/2A' = 1/(2 + \zeta/n) = 1, \quad \zeta = -n.$$

Other points of interest are y_0, where $A' = 0$ ($\zeta = -2n$) and the points $\zeta = 0$ ($y = n$ or $n \pm \sqrt{2}$) where the x roots are 1 and 0. All these solutions are easily obtained by the use of Fig. 4 relating ζ to $y - n$. The case $n = 1$ is exceptional since the regions y_1y_2 and y_3y_4 both disappear and the x roots are always real, one of them being 1 for any y value (Fig. 5).

We now wish to discuss these results in relation to condition (59). The following cases are to be distinguished:

I. Two real roots less than 1. None of these roots can be used for our problem. This occurs for

$$y_2 < y < n - \sqrt{2}, \quad n < y < y_3, \quad y_4 < y < n + \sqrt{2}. \tag{62}$$

II. Two real roots, one greater and the other

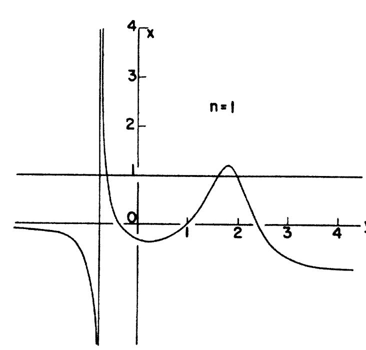

FIG. 5. Roots of Eq. 56 for $n = 1$.

less than 1. The first root can be used. This occurs for

$$y<y_0, \quad n-\sqrt{2}<y<n, \quad n+\sqrt{2}<y.$$

III. Two real roots greater than 1, both yielding possible solutions. This occurs for

$$y_0<y<y_1.$$

IV. Two imaginary roots with $x_r<1$. No useful root. This occurs for

$$y_5<y<y_2, \quad y_3<y<y_4.$$

V. Two imaginary roots with $x_r>1$, yielding two possible solutions. This occurs for

$$y_1<y<y_5.$$

These results are summarized on Fig. 6, which shows the distribution of the different regions corresponding to cases I, II, III, IV, and V in the y, n plane. Curves have been drawn considering n as a continuous variable but the results have physical meaning only for integral values of n. In addition to the previously defined curves, the curve corresponding to $x=n-2$ is also plotted. This is interesting in view of the approximation introduced in Section 2, neglecting the additional field E'. (See Section 7.)

In case II we obtain one possible solution (57) and there will be no difficulty in solving completely our problem. We shall prove that the magnetron operated under such conditions behaves like a pure imaginary impedance and should be unable to sustain oscillations.

In cases III and V we obtain two distinct solutions, yielding a potential

$$V_a=K_1r^{x_1}+K_2r^{x_2}. \tag{63}$$

This leaves an arbitrary constant, K_1/K_2 for instance, which we are unable to determine. It is thus impossible to draw any precise conclusions in these cases. This difficulty seems to be connected with the simplifying assumption made in Section 4 [Eq. (45.A)] about the filament having a radius zero. A special discussion of a magnetron with finite filament radius should be made to answer the question.

In cases I or IV no solution is obtained. Both roots are smaller than 1, which means that condition (59) cannot be fulfilled. If we try any one of the roots, we obtain an infinite oscillating electric field $-\partial V/\partial r$ on the filament ($r=0$). Instead of a magnetron working far from saturation and showing small oscillations around its steady conditions, we obtain large oscillations where the current emitted by the filament is zero (half of the time) or the saturation current for the other half periods. This shows that cases I and IV should represent the conditions under which a magnetron is able to oscillate and to

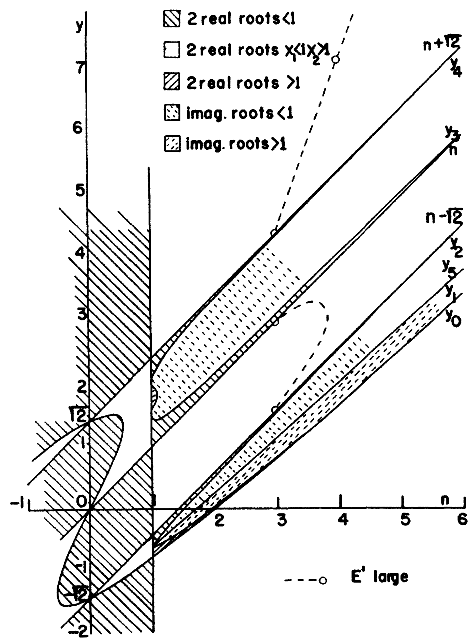

FIG. 6. Summary of results.

sustain high frequency oscillations in an outer circuit. The frequency is given by the corresponding y values:

$$\begin{aligned} y_5<y<n-\sqrt{2}, \\ n<y<n+\sqrt{2}. \end{aligned} \tag{64}$$

We must now see how to build up the complete solution for the potential distribution inside a magnetron with $2n$ anodes, operated at the frequency $\omega=y\omega_H$. According to the remarks at the beginning of Section 4 this complete solution will result from the superposition of elementary solutions corresponding to n and $-n$. Let us first suppose the corresponding roots x_+ and x_- both to be real and greater than 1 (case II).

$$V_a(r,\theta,t)=K_+r^{x_+}e^{i(\omega t-n\theta)}+K_-r^{x_-}e^{i(\omega t+n\theta)}, \tag{65}$$

where the K_+ and K_- constants must be adjusted so as to reproduce, for $r=b$, the assumed potential distribution (2) on the anodes

$$V_a(b, \theta, t) = V_a(b)e^{i\omega t}\sin n\theta. \quad (66)$$

This gives

$$K_+ = -\frac{1}{2i}V_a(b)b^{-x_+}, \quad K_- = \frac{1}{2i}V_a(b)b^{-x_-},$$

and

$$V_a(r, \theta, t) = \frac{1}{2i}V_a(b)e^{i\omega t} \times\left(-\left(\frac{r}{b}\right)^{x_+}e^{-in\theta}+\left(\frac{r}{b}\right)^{x_-}e^{in\theta}\right), \quad (67)$$

which gives the complete potential distribution inside the magnetron.

6. INTERNAL RESISTANCE OF THE MAGNETRON

Having obtained the distribution of potential in the magnetron, we are now in a position to compute the current densities and to calculate the current reaching the anodes. This might involve some difficulties when the anodes are narrow and far apart from each other, but if the magnetron is built with wide anodes and small intervals between them ($\alpha \gg \beta$), we may assume that each anode collects all the current flowing radially throughout an angle π/n. Thus the total current flowing to the first anode will be

$$I_{1r} = \epsilon\int_0^{\pi/n} J_{ar}rd\theta, \quad r=b, \quad (68)$$

but there are n anodes (numbers $1, 3, \cdots, 2n-1$) connected in parallel and the total current is n times greater

$$I_{ar} = \epsilon n\int_0^{\pi/n} J_{ar}rd\theta. \quad (69)$$

J_{ar} will consist of two terms corresponding to $\pm n$, each of which is related to the corresponding voltage term by Eq. (60). Let us first consider the case of two real roots x_+ and x_- with a potential (67). The relation (60) has to be applied separately on the two terms of (67) which gives

$$J_{ar} = (V_a(b)/8\pi r)e^{i\omega t} \times\left[-x_+(r/b)^{x_+}\left(\frac{2\omega_H^2}{\omega-n\omega_H}-\omega\right)e^{-in\theta} +x_-(r/b)^{x_-}\left(\frac{2\omega_H^2}{\omega+n\omega_H}-\omega\right)e^{in\theta}\right]. \quad (70)$$

Now

$$\int_0^{\pi/n} e^{\pm in\theta}d\theta = \pm 2i/n. \quad (71)$$

Making $r=b$, and using (69) to (71), we obtain

$$I_{ar} = \epsilon\frac{i}{4\pi}V_a(b)e^{i\omega t}\left[x_+\left(\frac{2\omega_H^2}{\omega-n\omega_H}-\omega\right) +x_-\left(\frac{2\omega_H^2}{\omega+n\omega_H}-\omega\right)\right]. \quad (72)$$

And according to Eq. (2), the potential of the odd anodes is

$$U_ae^{i\omega t} = \epsilon V_a(b)\frac{\pi n\beta}{8\sin(n\beta/2)}e^{i\omega t} \approx \epsilon\frac{\pi}{4}V_a(b)e^{i\omega t}. \quad \beta \ll \alpha. \quad (73)$$

In addition to radial currents, we have found that there are also angular components of the current density inside the magnetron. Thus we may have to add a correction term representing the current flowing to the edges of the anode; this would be something like

$$I_{a\theta} = \epsilon n[J_{a\theta}(\theta=0) - J_{a\theta}(\theta=\pi/n)]\delta r, \quad \text{for} \quad r=b, \quad (74)$$

δr representing the apparent thickness of the electrode and the n factor taking care of the fact that there are n anodes connected in parallel. The edges of the first anode correspond to $\theta=0$ and π/n. $J_{a\theta}$ is known from Eq. (60) which must be applied separately to each elementary V_a solution, yielding for each solution an equation of the form

$$J_{a\theta} = -\frac{i}{n}\frac{\partial}{\partial r}(rJ_{ar}) = \frac{x^2}{4\pi nr}\left(\frac{2\omega_H^2}{\omega-n\omega_H}-\omega\right)V_a(r, \theta, t).$$

Hence, for the superposition of the $\pm n$ solutions,

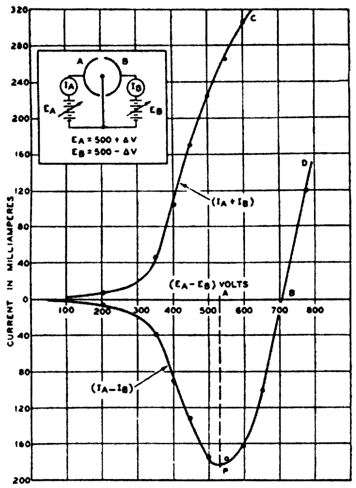

FIG. 7. Static characteristics of a two-segment magnetron.

as given by (67)

$$J_{a\theta}=\frac{V_a(b)e^{i\omega t}}{8\pi i r}\times\left[-\frac{x_+^2}{n}\left(\frac{2\omega_H^2}{\omega-n\omega_H}-\omega\right)\left(\frac{r}{b}\right)^{x_+}e^{-in\theta}-\frac{x_-^2}{n}\left(\frac{2\omega_H^2}{\omega+n\omega_H}-\omega\right)\left(\frac{r}{b}\right)^{x_-}e^{in\theta}\right]. \quad (75)$$

Exponentials are 1 for $\theta=0$ and -1 for $\theta=\pi/n$ so that the two terms in (74) are equal and one obtains

$$I_{a\theta}=\frac{\epsilon i}{4\pi b}V_a(b)e^{i\omega t}\left\langle x_+^2\left(\frac{2\omega_H^2}{\omega-n\omega_H}-\omega\right)+x_-^2\left(\frac{2\omega_H^2}{\omega+n\omega_H}-\omega\right)\right\rangle\delta r. \quad (76)$$

x_+ and x_- are of the order of magnitude of n, so that the additional contribution is of the order

$$I_{a\theta}\approx I_{ar}n\frac{\delta r}{b}, \quad (76a)$$

and will probably be small if the apparent thickness of the anodes is small. The important fact is that it comes out with the same phase angle as I_{ar} when both roots x_+ and x_- are real (case II). In such cases the currents I_{ar} and $I_{a\theta}$ retain pure imaginary amplitudes, while the potential keeps a real amplitude (73). This means that magnetrons operated under such conditions are equivalent to pure impedances without any (either positive or negative) resistance and seem to be unable to sustain oscillations in an outer circuit.

It should be noticed that in the present paper the currents have been written with a sign convention opposite to the usual one in circuit theory. Hence the internal impedance Z_a of the split anode magnetron should be defined as

$$Z_a=\frac{-U_ae^{i\omega t}}{I_{ar}}=i\pi^2\left[x_+\left(\frac{2\omega_H^2}{\omega-n\omega_H}-\omega\right)+x_-\left(\frac{2\omega_H^2}{\omega+n}-\omega\right)\right]^{-1} \quad (77)$$

from Eqs. (72) and (73). This impedance is observed in the devices shown on Fig. 1 or Fig. 7.

7. VALIDITY OF THE ASSUMPTION MADE IN SECTION 2, NEGLECTING THE ADDITIONAL FIELD E'

As noticed in Section 2, just after Eq. (18), the assumption has been made that the additional electric field E', induced by the alternating magnetic fields, could be neglected. It is now possible to compute the order of magnitude of this E' field, which is determined by Eq. (20). Because the $e^{i(\omega t-n\theta)}$ factor appears in this field, as in all alternating quantities, Eq. (20) reads

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}rE_r'-\frac{n^2}{r}E_r'=4\pi\mu_0 i\omega rJ_r, \quad (78)$$

where J_r is the sum of terms (60) corresponding to each elementary term r^x in V_a, as may be seen from Eqs. (67) and (70). We may, for instance, write (70) this way

$$J_{ar}=J_{ar}(+n)+J_{ar}(-n),$$

$$J_{ar}(\pm n)=\mp\frac{V_a(b)}{8\pi r}e^{i(\omega t\mp n\theta)}\times\left(\frac{r}{b}\right)^{x(\pm n)}x(\pm n)\omega_H\left(\frac{2}{y\mp n}-y\right). \quad (79)$$

And our Eq. (78) will read

$$E_r' = E_r'(+n) + E_r'(-n),$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} r E_r'(+n) - \frac{n^2}{r} E_r'(+n) = 4\pi\mu_0 i\omega r J_{ar}(+n). \tag{80}$$

The solution is obvious since J_{ar} depends upon r by r^{x-1}.

$$E_r'(+n) = \frac{4\pi i\mu_0\omega r^2 J_{ar}(+n)}{(x(+n)+2)^2-n^2} = \frac{8\pi^2 i}{c}\frac{r}{\lambda}\frac{rJ_{ar}(+n)}{(x(+n)+2)^2-n^2}, \tag{81}$$

as $\mu_0 = c^{-2}$ and $\omega/c = 2\pi/\lambda$. This shows that the additional field E' is very small because of the factor $(1/c)(r/\lambda)$, so long as the dimensions of the magnetron are small compared with the wave-length, which corresponds to statement (19). The only case for trouble would be when one of the roots x should come near to $\pm n-2$, a condition to be discussed in a moment.

The additional potential V' on the anode $r=b$ corresponding to E_r' can be computed. It should be reminded that $r^2 J_{ar}$ depends on r as r^{x+1}. This gives

$$V' = V'(+n) + V'(-n),$$

$$V'(+n) = -\int_0^b E_r'(+n)dr = -\frac{8\pi^2 i b^3 J_{ar}(+n)}{c\lambda(x_+ +2)[(x_+ +2)^2-n^2]}. \tag{82}$$

This must be calculated for the middle of the first anode, i.e., for $r=b$, $\theta=\pi/2n$. Formula (72) may be written

$$I_{ar} = \epsilon 2b[J_{ar}(+n) + J_{ar}(-n)], \tag{83}$$

for $r=b$, $\theta=\pi/2n$, while

$$\epsilon V' = -\epsilon\frac{8\pi^2 i}{c\lambda}b^3\left[\frac{J_{ar}(+n)}{(x_+ +2)[(x_+ +2)^2-n^2]} + \frac{J_{ar}(-n)}{(x_- +2)[(x_- +2)^2-n^2]}\right], \quad r=b, \; \theta=\pi/2n. \tag{84}$$

This shows that the additional voltage V' is out of phase with the current (i factor) and very small except for the cases

$$x = \pm n - 2. \tag{85}$$

An order of magnitude can be obtained, if one remembers that the x values are of the order of n. This gives

$$\epsilon V' \approx -\frac{i\pi^2 b^2}{c\lambda(n+2)(n+1)} I_{ar}. \tag{86}$$

The addition of a voltage $\epsilon V'$ to the voltage $\epsilon V_a(b)$ already assumed on the anodes is equivalent to adding a series impedance $Z' = -\epsilon V'/I_{ar}$, the order of magnitude of which would thus be

$$Z' = i\frac{\pi^2 b^2}{c\lambda(n+2)(n+1)}. \tag{87}$$

Let us take, for instance, $b = \frac{1}{10}\lambda$ and $\lambda = 1$ cm which will be a very unfavorable situation.

$$Z' = i\frac{0.3\pi^2}{(n+2)(n+1)},$$

which is of the order of magnitude of a fraction of an ohm, a very small quantity when compared to the thousands of ohms which electron tubes yield as internal resistance.

Let us now discuss the conditions (85) for which the additional electric field takes a really important part. A glance at Fig. 3 shows that this condition can be fulfilled only for $x(+n)$, between $y=n-\sqrt{2}$ and n (where x_+ is slightly above 1) or above $y=n+\sqrt{2}$. The first case is for $n=3$, $x=1$, $y=3-\sqrt{2}$, 3, $3+\sqrt{2}$; then $n=4$, etc. These points are plotted on Fig. 6. They all lie in the region where magnetrons have been proved to yield pure imaginary internal impedance and to be unfit for sustaining oscillations.

It thus appears legitimate to neglect the role played by the additional field E' induced by alternating internal magnetic fields.

8. CONCLUSIONS

The results of this study are summarized in Fig. 6 which shows the type of solutions obtained in different regions of the y, n plane. These results will perhaps be easier to understand for practical applications if we notice that Larmor's angular velocity ω_H is proportional to the magnetic field, which means

$$\omega_H = 0.884\ 10^7 H, \quad H \text{ in Gauss}$$
$$y = \omega/\omega_H = 2\pi c/\lambda\omega_H = 21310/\lambda H. \quad \lambda \text{ in cm} \quad (88)$$

This enables a comparison with experimental results, which are usually given in terms of λH corresponding to different types of oscillations.

On the map drawn in Fig. 6, the different regions corresponding to the cases I to V defined in Section 6 [Eq. (62)] have been represented. The physical meaning of the various cases seems to be:

I and IV. Large oscillations.

$$y_5 < y < n - \sqrt{2}, \quad n < y < n + \sqrt{2}. \quad (64)$$

II. No oscillations, the magnetron being equivalent to a pure imaginary impedance.

III and V. The solution still contains an arbitrary constant; probably no oscillations. The different regions corresponding to these cases I to V are neatly divided for magnetrons corresponding to $n=2, 3, 4 \cdots$ (number of anodes $2n=4, 6, 8 \cdots$); one always finds two different regions, defined by conditions (64), where oscillations could possibly take place.

For the usual split-anode magnetron ($n=1$; 2 anodes) it is hard to foresee the conditions of oscillations, as the whole diagram changes just on the line $n=1$. These very typical circumstances should make this magnetron very sensitive to all sorts of perturbations such as increase in the diameter of the filament, effect of large oscillations, etc. Experimental values on the usual split-anode magnetron are found in a paper by G. R. Kilgore,[3] with an attempt at theoretical explanation, which unfortunately does not take account of space charge effects, and therefore remains far away from actual conditions in magnetrons. Figure 7 is a reproduction of Kilgore's Fig. 3, showing the static characteristic for a split-anode magnetron. This means $\omega=0$, $y=0$. The magnetic field applied was 1.5 times the critical field. The average anode potential was 500 volts, and variations as large as ±400 volts (giving $E_A - E_B = 800$ volts) were applied. This means very large perturbations, for which our theory would certainly be only a rough first approximation. It appears from the curves that the negative resistance is zero for small ΔV and reaches a maximum of about −1500 ohms for $\Delta V = 400$ volts. Efficiencies of magnetrons operated under different conditions are given by Kilgore on page 372 and can be summarized as follows ($H = 1.5H_c$, H_c critical field):

Efficiency	10%	20%	30%	40%	50%	
$\lambda H 10^{-4}$	5.45	6.7	8.1	9.8	12	(82)
$y = \omega/\omega_H$.	0.39	0.32	0.265	0.218	0.178.	

If these y values are taken with a negative sign, they lie just between the horizontal n axis and the line $y = n - \sqrt{2}$. Another type of oscillation has been found on split-anode magnetrons with λH values around 12,000, which gives

$$y \approx 1.8.$$

This value of y seems to correspond to the upper band of our diagram, as it lies between 1 and $1+\sqrt{2}$.

A further prediction of the theory concerns the possibility for a magnetron of type n to work as magnetrons of types $3n$, $5n$, $\cdots$ but with lower efficiency.

Magnetrons $n=3$ (6 anodes) should behave differently, as the perturbation by the oscillating magnetic field is especially large in that case.

[3] G. R. Kilgore, Proc. I. R. E. (August, 1936). Reprinted by RCA in *Radio at Ultra-High Frequencies* (1940), p. 360.