The result is antisymmetric in the spins of the two nuclear particles. A deuteron in any stationary state has therefore no extra non-additive magnetic moment due to meson exchange in the pseudoscalar theory as well as the vector theory.

In contrast to the result in the vector theory, in the pseudoscalar theory a proton-neutron pair has no exchange electric dipole moment that reacts with an electrostatic field. There is also no electric quadripole moment that reacts with an electrostatic field.

In conclusion the writers would like to thank Professor W. Heitler for a comment at the early stage of this work.

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Tensor Forces and Heavy Nuclei

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It has been suggested that the interaction between a pair of nuclear particles contains, in addition to a central force term, a tensor term, and quantitative estimates of the relative magnitudes of these two terms have been given. In this paper a rough quantitative estimate of the relative contributions of these two terms to the binding energy of a nucleus is made on the basis of a highly simplified nuclear model. It is found that a tensor interaction between two nuclear particles of the form suggested by the neutral meson field theory of nuclear forces with constants determined to fit the experimental data on the two-particle neutron-proton system would lead to the existence of moderately light nuclei having binding energies, spins, and deviations from spherical shape much greater than those observed.

1. INTRODUCTION

THE discovery of the electric quadrupole moment of the deuteron[1] suggests the existence of a term lacking spherical symmetry in the interaction between a proton and a neutron. The current assumption of equality of forces between all pairs of nuclear particles then implies the existence of a similar term in the interaction between two like particles. General considerations of invariance[2] indicate that such a term should contain the operator

$$S_{12}=3(\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{r}/r^2)-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \qquad (1)$$

Field theories of nuclear forces have been proposed[3] which lend support to this view. They predict the existence of terms containing S_{12} as a factor, but little reliance can be placed on the detailed form of the interaction given by these theories because of divergence difficulties from which they suffer.

The influence of the S_{12} tensor term on the two-particle proton-neutron system has been investigated by Rarita and Schwinger[4] and by Bethe.[5] The former develop a purely phenomenological theory adopting simplified rectangular well potentials with constants chosen to fit the binding energy and quadrupole moment of the deuteron, and the scattering of slow neutrons in hydrogen. The range of the forces is taken equal to that deduced from proton-proton scattering. Bethe employs the form of the interaction predicted by the single-meson theory of nuclear forces cutting off at small distances to avoid divergence. The value of the "cut-off" distance and the strength of the interaction is determined to fit the deuteron binding energy

* The greater part of this work was completed during the spring term of 1940 while the author was in the Department of Physics at Princeton University, Princeton, New Jersey, on a Research Fellowship granted by the Royal Society of Canada.

[1] J. M. B. Kellogg, I. I. Rabi, N. F. Ramsey, Jr., and J. R. Zacharias, Phys. Rev. **55**, 318 (1939) and **57**, 677 (1940).

[2] E. Wigner, Phys. Rev. **51**, 106 (1937).

[3] H. A. Bethe, Phys. Rev. **57**, 260 (1940) (gives a list of references); R. E. Marshak, Phys. Rev. **57**, 1101 (1940).

[4] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941) and Phys. Rev. **59**, 556 (1941).

[5] H. A. Bethe, Phys. Rev. **57**, 390 (1940).

and the scattering cross section of slow neutrons in hydrogen. A discussion of H^3 and He^4 based on the use of tensor forces is given by Gerjuoy and Schwinger.[6]

It seems of interest to investigate the effect of the tensor interaction on the binding energy, spin, and shape of heavy nuclei. Wigner and Eisenbud[7] pointed out that forces of the general type of those used by Rarita and Schwinger and by Bethe will not have the property of saturation. Bethe[3] also mentioned the possibility of the tensor part of the interaction becoming predominant in heavy nuclei with the consequences not only of non-saturation, but also of large nuclear spin and markedly non-spherical shape of the nucleus. The object of the present paper is twofold: (a) On the basis of a highly simplified nuclear model, we obtain a rough estimate of the relative contributions of the spherically symmetrical and of the tensor interaction to the nuclear binding energy. (b) The application of these results to the "neutral" theory type interactions using the numerical constants as determined by Bethe[5] shows that the difficulties mentioned above will become pronounced well within the range of existing nuclei. The same holds for the values of the constants as determined by Schwinger and Rarita[4] if the spherically symmetrical part of the interaction is due to an ordinary potential. The case that it is an exchange interaction is not considered in the present paper and may well lead to a different conclusion. This makes it very probable that in the Schwinger and Rarita theory, the $J_0(r)$ of (2) has to be replaced by an exchange interaction.

2. THE FORM OF THE NUCLEAR INTERACTION

The general form of the interaction between two nuclear particles predicted by the neutral meson theory is given by

$$V = J_0(r) + J_1(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + J_2(r)S_{12}. \quad (2)$$

The symmetrical theory introduces an additional factor $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$, while the charged meson theory gives no interaction between like particles, and introduces a factor $(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ in the interaction between a proton and a neutron.

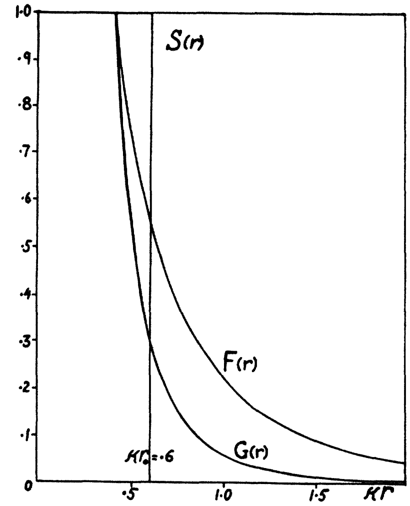

FIG. 1. The factor giving the dependence of the nuclear interaction on the distance (in units of $1/\kappa = 2.18\times10^{-13}$ cm) between the particles. $F(r)$ and $G(r)$ are Bethe's results on the neutral straight cut-off theory for the spherically symmetric and the tensor parts of the interaction. $S(r)$ is the step function used to replace both in the present calculations.

Bethe[5] arbitrarily assumed the spin independent term J_0 to be entirely absent, and took the other two terms to be of the form predicted by the meson theory of nuclear forces:

$$J_0(r)=0, \quad J_1(r)=\frac{2}{3}f^2\frac{e^{-\kappa r}}{r},$$
$$J_2(r) = -f^2e^{-\kappa r}\left\{\frac{1}{\kappa^2 r^2}+\frac{1}{\kappa r}+\frac{1}{3}\right\}, \quad (3)$$

where $1/\kappa = \hbar/\mu c = 2.18\times10^{-13}$ cm on the assumption that $\mu = 177$ m, and f^2 is a constant. To avoid divergence these interactions must be cut off at some value of $r=a$. Using straight cut-off ($J(r)\equiv J(a)$ for $r<a$) and the interaction (2) of the neutral theory Bethe obtained

$$f^2 = 0.0770\hbar c, \quad \kappa a = 0.405. \quad (4)$$

J_1 and J_2 may then be rewritten as

$$J_1(r) = AF(r), \quad J_2(r) = BG(r), \quad (5)$$

where $F(r)$ and $G(r)$ are unity for $r \leqq a$, and for $r>a$

$$F(r) = J_1(r)/J_1(a), \quad G(r) = J_2(r)/J_2(a). \quad (6)$$

[6] E. Gerjuoy and J. Schwinger, Phys. Rev. **61**, 138 (1942).
[7] E. Wigner and L. Eisenbud, Phys. Rev. **56**, 214 (1939).

TABLE I. Values of the function $W_1(k, \delta)$.

k	$\delta=0$	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
0.1	1.000		0.968		0.801		0.624	0.342	0.207	
0.2	1.000	1.000	1.000		0.946		0.805	0.483	0.300	
0.4	1.000	1.000	1.000	1.000	0.999		0.946	0.641	0.404	0.172
0.6	1.000	1.000	1.000	1.000	1.000		0.988	0.723	0.448	0.183
0.8	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.759	0.465	0.187
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.768	0.469	0.188
1.5	1.000	1.000	1.000	1.000		0.997	0.985	0.741	0.457	0.185
2.0	1.000	1.000	1.000	1.000		0.975	0.943	0.692	0.436	0.180
3.0	1.000	0.998		0.975		0.893	0.843	0.598	0.394	0.170
4.0	1.000	0.994		0.923		0.810	0.755	0.524	0.357	0.161
8.0	1.000	0.879		0.717		0.588	0.537	0.361	0.259	0.134
20.0	1.000	0.606		0.437		0.346	0.311	0.208	0.152	0.091

TABLE II. Values of the function $W_2(k, \delta)$.

k	$\delta=0$	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
0.1	−0.939		−0.908		−0.742		−0.566	−0.290	−0.162	
0.2	−0.833	−0.833	−0.833		−0.779		−0.641	−0.336	−0.177	
0.4	−0.595	−0.595	−0.595	−0.595	−0.594		−0.543	−0.283	−0.121	−0.029
0.6	−0.370	−0.370	−0.370	−0.370	−0.370		−0.359	−0.175	−0.069	−0.016
0.8	−0.172	−0.172	−0.172	−0.172	−0.172	−0.172	−0.171	−0.076	−0.029	−0.006
1.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1.5	0.338	0.338	0.338	0.338		0.333	0.310	0.124	0.046	0.010
2.0	0.582	0.582	0.582	0.582		0.535	0.483	0.194	0.073	0.016
3.0	0.906	0.906		0.858		0.708	0.621	0.263	0.103	0.023
4.0	1.111	1.101		0.962		0.754	0.654	0.290	0.119	0.028
8.0	1.496	1.252		0.937		0.693	0.595	0.290	0.140	0.036
20.0	1.778	0.953		0.662		0.484	0.419	0.223	0.128	0.043

A and B are constants given by

$$A = J_1(a) = 7.7 \text{ Mev}, \quad B = J_2(a) = -102 \text{ Mev}. \quad (7)$$

$F(r)$ and $G(r)$ are plotted in Fig. 1 together with the step function $S(r)$ of range r_0 defined by

$$\begin{aligned} S(r) &= 1 \quad 0 \leqq r \leqq r_0 \\ &= 0 \quad\quad r > r_0. \end{aligned} \quad (8)$$

Bethe also found that the factor $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ introduced by the symetmric theory alters the values of a and f^2 and leads to several serious difficulties.

Rarita and Schwinger[4] assumed rectangular well potentials of equal range, but different depths:

$$\begin{aligned} J_0(r) &= -(1-\tfrac{1}{2}g)V_0S(r) = -13.39S(r), \\ J_1(r) &= -\tfrac{1}{2}gV_0S(r) \qquad = -\ \ 0.50S(r), \\ J_2(r) &= -\gamma V_0S(r) \qquad\ \ = -10.75S(r). \end{aligned} \quad (9)$$

$S(r)$ is the step function defined by (8), and the constants g, γ, V_0, and r_0 were evaluated to be:

$$g = 0.0715, \quad \gamma = 0.775, \quad V_0 = 13.89 \text{ Mev}, \quad r_0 = 2.80 \times 10^{-13} \text{ cm}. \quad (10)$$

The known properties of the ground state of the deuteron and low energy collision phenomena are insufficient to discriminate on the basis of a purely phenomenological theory between the interaction (2) and those obtained from it by the inclusion of the factors $-\frac{1}{3}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ or $-\frac{1}{2}(1+\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$, which both have the value unity for states antisymmetrical in the charge.

3. THE SIMPLIFIED NUCLEAR MODEL

The following model is used in the present calculations:

(a) The interaction V between two particles is assumed to be of the form (2) corresponding to the neutral meson theory. The particles are taken to have all their spins parallel. V then becomes an ordinary potential depending on the distance r between the two particles, and the angle θ which r makes with the direction of their spins:

$$V = [J_0(r) + J_1(r)] + (3\cos^2\theta - 1)J_2(r). \quad (11)$$

(b) The functions $J(r)$ are replaced by rectangular potential wells of appropriate range r_0. The ranges need not all be the same, but will be taken the same for convenience. Then:

$$V = AS(r) + BS(r)(3\cos^2\theta - 1). \quad (12)$$

(c) The nucleus is taken to have the shape of an ellipsoid of revolution with its axis of symmetry parallel to the spins of the particles. The shape of the ellipsoid is specified by the ratio k of its length along the axis of symmetry to the diameter of its greatest circular cross section ($k>1$ corresponds to prolate, and $k<1$ to oblate ellipsoids). The volume Ω of the ellipsoid is specified by δ which is defined by $\Omega = (\pi/6)(\delta r_0)^3$; i.e., δ is the ratio of the diameter of the sphere having the same volume as the ellipsoid to the range r_0 of the step functions used.

(d) The density ρ of the particles is assumed to be uniform over the volume of the nucleus.

(e) The Coulomb energy, and the exchange integrals of the purely nuclear interaction are neglected, the total energy being calculated as the sum of the direct nuclear interaction and the kinetic energy of all the particles.

If the Coulomb and the exchange integrals are not neglected, and if the surface effect[8] is properly taken into account then calculations based on the above crude model must *necessarily* give an overestimate of the total energy and hence an underestimate of the binding energy, so that any conclusions as to lack of saturation for such a model would *a fortiori* hold for a better one. The additional approximation made in neglecting the Coulomb, the exchange, and the surface energy makes this argument somewhat less conclusive, but the results obtained below are still suggestive enough to make such a model of interest.

4. EVALUATION OF THE POTENTIAL ENERGY

On the above model the potential energy is given by

$$\text{P.E.} = \tfrac{1}{2}\int\int \rho(\mathbf{x})\,V(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d\mathbf{x}d\mathbf{x}' = \tfrac{1}{2}N^2[A\,W_1(k,\delta) + B\,W_2(k,\delta)] \equiv W(k,\delta)N^2. \quad (13)$$

Here N is the number of particles in the nucleus and

$$W_1(k,\delta) \equiv \frac{1}{\Omega^2}\int\int S(|\mathbf{x}-\mathbf{x}'|)d\mathbf{x}d\mathbf{x}',$$
$$W_2(k,\delta) \equiv \frac{1}{\Omega^2}\int\int S(|\mathbf{x}-\mathbf{x}'|)(3\cos^2\theta - 1)d\mathbf{x}d\mathbf{x}', \quad (14)$$

both integrations being extended twice over the volume of the nucleus. The integrals in (14) may be carried out analytically, and the results are stated in the appendix. From these results, Tables I and II, giving numerical values of $W_1(k,\delta)$ and $W_2(k,\delta)$ as functions of k and δ for several selected values of these parameters, were constructed.

5. KINETIC ENERGY

In the statistical model the wave function of each individual particle is taken to be that of a free particle in a box of a size and shape corresponding to the particular nuclear model under investigation. The number of particles in each level is restricted by the Pauli exclusion principle. Usually two particles are allowed in each level corresponding to the two possible spin orientations. In our case, where the spins are all taken

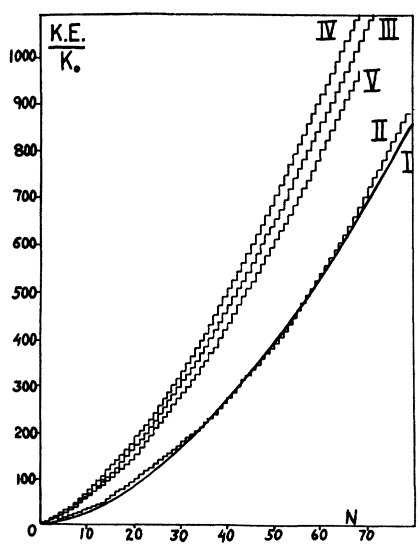

FIG. 2. The total kinetic energy (in units of $K_0 = \pi^2\hbar^2/2M\Omega^{\frac{2}{3}}$) of N particles obeying the exclusion principle confined to a box of volume Ω. I—Approximate calculation. II—Cubical box, periodic boundary conditions. III—Cubical box, zero boundary conditions. IV—Rectangular box, ratio of sides 1 : 1 : 3, zero boundary conditions. V—Spherical box, zero boundary conditions.

[8] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936); E. Feenberg, Phys. Rev. **60**, 204 (1941).

TABLE III. Values of the function $W_B(k, \delta) = \frac{1}{2}[7.7W_1(k, \delta) - 102W_2(k, \delta)]$ in Mev.

k	$\delta=0.50$	0.70	0.90	1.0	1.5	2.0	3.0
1.0	3.8	3.8	3.8	3.8	3.0	1.8	0.7
1.5	−13.4	−13.4	−13.1	−12.0	− 3.5	−0.6	0.2
2.0	−25.8	−25.8	−23.5	−21.0	− 7.2	−2.1	−0.1
3.0	−42.4	−40.0	−32.7	−28.4	−11.1	−3.8	−0.5
4.0	−52.3	−45.5	−35.3	−30.4	−12.8	−4.7	−0.8
8.0	−60.5	−45.0	−33.1	−28.2	−13.4	−6.2	−1.3
20.0	−46.3	−32.1	−23.3	−20.2	−10.6	−5.9	−1.8

parallel, there are still two particles in each level—a proton and a neutron.

The familiar formula for the kinetic energy of a large number N of particles obeying the exclusion principle and restricted to a volume Ω, obtained by assigning two particles to each cell of size h^3 in phase space, is

$$\text{K.E.} = \frac{3}{5}\left(\frac{3}{\pi}\right)^{2/3} \frac{\pi^2\hbar^2}{2M\Omega^{2/3}} N^{5/3} = 0.582 K_0 N^{5/3}, \tag{15}$$

where

$$K_0 \equiv \pi^2\hbar^2/2M\Omega^{2/3}. \tag{16}$$

This is seen to depend on the mass M of each particle, and on the volume Ω of the box, but not on its shape. However, a more detailed calculation for a small number of particles shows that the result does depend on the shape of the volume Ω, and also on the type of boundary conditions used. Results are given below for a few simple cases.

The simplest case is that of a rectangular box of sides a, k_2a, k_3a and volume $\Omega = k_2k_3a^3$. Two types of boundary conditions are possible: periodic or zero boundary conditions. The wave functions are complex exponentials in the former case, and sines in the latter. The corresponding expressions for the kinetic energy are:

$$(\text{K.E.})_{\exp} = 4(k_2k_3)^{\frac{2}{3}} K_0 \times \sum [n_1^2 + n_2^2/k_2^2 + n_3^2/k_3^2]; \tag{17}$$

$$(\text{K.E.})_{\sin} = (k_2k_3)^{\frac{2}{3}} K_0 \times \sum [n_1^2 + n_2^2/k_2^2 + n_3^2/k_3^2]. \tag{18}$$

K_0 is defined by (16). The summation in (17) extends over zero, positive, and negative integral values of n_1, n_2, n_3, while in (18) it extends only over the positive integral values.

TABLE IV. Values of $N_0^{\frac{1}{3}} = -107/\delta^2 W_B(k, \delta)$.

k	$\delta=0.50$	0.70	0.90	1.0	1.5	2.0	3.0
1.0							
1.5	32.0	16.3	10.1	8.9	13.7	44.7	
2.0	16.6	8.4	5.6	5.1	6.6	13.0	91.7
3.0	10.1	5.5	4.0	3.8	4.3	7.1	22.4
4.0	8.2	4.8	3.7	3.5	3.7	5.7	15.2
8.0	7.1	4.8	4.0	3.8	3.5	4.3	9.1
20.0	9.3	6.8	5.7	5.3	4.5	4.5	6.5

TABLE V. Average energy per particle in Mev (from Bethe's interaction) for a nucleus containing 45 particles.

k	$\delta=0.9$	1.0	1.5	2.0
3.0	202	76	101	169
4.0	82	−16	25	126
8.0	183	82	−2	61
20.0	621	445	125	72

TABLE VI. Average energy per particle in Mev (from Bethe's interaction) for a nucleus containing 216 particles.

k	$\delta=0.7$	0.9	1.0
3.0	− 778	−2298	−2283
4.0	−1965	−2871	−2723
8.0	−1864	−2388	−1349

Another simple case is that of a spherical box with zero boundary conditions for which

$$\text{K.E.} = (4/3\pi^2)^{\frac{2}{3}} K_0 \sum_{n,i} Z^2_{n,i}, \tag{19}$$

where K_0 is given by (16), and $Z_{n,i}$ $(i = 1, 2, 3, \cdots)$ are zeros of Bessel's functions of order $n = 1/2, 3/2, 5/2, \cdots$, the sum being taken over a sufficient number of the lower lying zeros.

Figure 2 gives the ratio K.E./K_0 as a function of the total number of particles for five different cases:

I. Calculated from (15).
II. Calculated from (17) with $k_2 = k_3 = 1$.
III. Calculated from (18) with $k_2 = k_3 = 1$.
IV. Calculated from (18) with $k_2 = 1$, $k_3 = 3$.
V. Calculated from (19).

It is of interest to note that although for a given volume of the box the K.E. does not depend strongly on its shape (cubical, oblong, and spherical, respectively, in III, IV, V), it does depend markedly on the type of boundary conditions used (periodic in II, zero in III). For a very large number of particles the ratio of the K.E. obtained from any of the formulae above to that given by (15) must necessarily approach unity. But for small N it is seen that zero boundary conditions, which lead to a greater concentration of particles towards the center of the box, give a considerably higher energy than the periodic boundary conditions, which for all values of N give results very close to those obtained from (15).

Since in our simple model of the nucleus the density of particles is assumed to be constant over the nucleus, it is consistent to use the periodic rather than the zero boundary conditions on the wave functions in calculating the kinetic energy, the former giving a more uniform particle density. Since the dependence of the kinetic energy on the shape of the nucleus is slight, for simplicity expression (15) is used for all values of k and δ. Rewritten in terms of δ, (15) becomes:

$$\text{K.E.} = KN^{5/3}\delta^{-2}, \tag{20}$$

TABLE VII. Values of the function
$W_{RS}(k, \delta) = \frac{1}{2}[-13.89W_1(k, \delta) - 10.75W_2(k, \delta)]$
in Mev.

k	$\delta=0.5$	0.7	0.9	1.0	1.5	2.0	3.0
1	−6.94	−6.94	−6.94	−6.94	−5.33	−3.26	−1.30
1.5	−8.76	−8.76	−8.72	−8.51	−5.81	−3.42	−1.34
2.0	−10.07	−10.07	−9.65	−9.15	−5.85	−3.42	−1.34
3.0	−11.81	−11.39	−10.01	−9.19	−5.57	−3.29	−1.31
4.0	−12.83	−11.58	−9.71	−8.76	−5.20	−3.12	−1.27
8.0	−12.84	−10.02	−7.81	−6.93	−3.94	−2.55	−1.12
20.0	−9.33	−6.60	−5.01	−4.42	−2.64	−1.74	−0.86

TABLE VIII. Values of $N_0^{\frac{1}{3}} = -23.3/\delta^2 W_{RS}(k, \delta)$.

k	$\delta=0.5$	0.7	0.9	1.0	1.5	2.0	3.0
1	13.41	6.84	4.14	3.35	1.92	1.79	1.98
1.5	10.63	5.42	3.30	2.74	1.77	1.70	1.93
2.0	9.24	4.72	2.98	2.54	1.75	1.70	1.93
3.0	7.88	4.17	2.87	2.53	1.84	1.77	1.98
4.0	7.26	4.10	2.96	2.66	1.97	1.87	2.04
8.0	7.25	4.74	3.68	3.36	2.60	2.28	2.31
20.0	9.98	7.20	5.74	5.27	3.88	3.34	2.99

TABLE IX. Average energy per particle in Mev (from Rarita and Schwinger's interaction) for a nucleus containing 9 particles.

k	$\delta=1.0$	1.5	2.0
1	38.2	−3.6	−4.1
1.5	24.1	−7.9	−5.6
2.0	18.4	−8.3	−5.6
3.0	18.0	−5.7	−4.4

where

$$K = \frac{9\pi}{5}\left(\frac{3}{2\pi}\right)^{\frac{1}{3}} \frac{\hbar^2}{Mr_0^2} = \frac{38.4}{(\kappa r_0)^2} \text{ Mev}, \tag{21}$$

with $1/\kappa = 2.18 \times 10^{-13}$ cm.

6. CALCULATION OF TOTAL ENERGY FROM BETHE'S CONSTANTS

Our approximation to the total energy of a heavy nucleus is obtained by adding (13) and (20):

$$E = W(k, \delta)N^2 + KN^{5/3}\delta^{-2}. \tag{22}$$

In order to apply this to Bethe's[5] neutral theory interaction with straight cut-off the functions $F(r)$ and $G(r)$ of Fig. 1 are replaced by the step function $S(r)$ whose range r_0 is so chosen that the area under $G(r)$ is the same as that under $S(r)$. $F(r)$ could be replaced by a step function of a different range, but A is numerically so much smaller than B in (7) as to make the contribution of $W_1(k, \delta)$ to the total energy relatively unimportant, so that for convenience $F(r)$ and $G(r)$ are replaced by the same step function $S(r)$ of range determined by

$$\kappa r_0 = 0.6. \tag{23}$$

The value of K_B is then obtained from (21):

$$K_B = 107 \text{ Mev}, \tag{24}$$

and $W_B(k, \delta)$ calculated from

$$W_B(k, \delta) = \tfrac{1}{2}[7.7W_1(k, \delta) - 102W_2(k, \delta)] \text{ Mev} \tag{25}$$

is given for selected values of k and δ in Table III. The value of N for which $E=0$ is given by

$$N_0^{\frac{1}{3}} = -\frac{K_B}{\delta^2 W_B(k, \delta)} = -\frac{107}{\delta^2 W_B(k, \delta)}. \tag{26}$$

For values of N greater than N_0, E rapidly becomes more and more negative, very soon numerically exceeding the observed value of

TABLE X. Average energy per particle in Mev (from Rarita and Schwinger's interaction) for a nucleus containing 216 particles.

k	$\delta=0.9$	1.0	1.5
2.0	−1048	−1137	−891
3.0	−1126	−1146	−830
4.0	−1062	−1053	−750

roughly $-8N$ Mev. Table IV gives the values of $N_0^{\frac{1}{3}}$ from (26) for selected values of k and δ. In this table the lowest value of $N_0=(3.5)^3=43$ is seen to occur for $k=4$, $\delta=1$. Table V gives the values of the average energy per particle E/N in Mev when $N=45$, and Table VI gives the average energy per particle when $N=216$. If the range r_0 of the step function $S(r)$ replacing $F(r)$ and $G(r)$ in the above calculations is increased so as to satisfy $\kappa r_0=0.7$ instead of $\kappa r_0=0.6$, the lowest value of N_0 is further reduced to $N_0=17$.

The dimensions of the ellipsoid specified by $k=4$, $\delta=1$, $\kappa r_0=0.6$ are: major axis$=3.3\times 10^{-13}$ cm, minor axis$=0.83\times 10^{-13}$ cm. The ellipsoid is of the order of magnitude of nuclear dimensions, but is excessively elongated.

7. CALCULATION OF TOTAL ENERGY FROM RARITA AND SCHWINGER'S CONSTANTS

Rarita and Schwinger[4] use the step function $S(r)$ of range $r_0=2.80\times 10^{-13}$ cm, i.e.,

$$\kappa r_0=1.285. \tag{27}$$

The corresponding value of K_{RS} from (21) is

$$K_{RS}=23.3 \text{ Mev}, \tag{28}$$

and $W_{RS}(k, \delta)$ calculated from

$$W_{RS}(k,\delta)=\tfrac{1}{2}[-13.89W_1(k,\delta) -10.75W_2(k,\delta)] \text{ Mev} \tag{29}$$

is given for selected values of k and δ in Table VII. The value of N for which $E=0$ is given by

$$N_0^{\frac{1}{3}}=-\frac{K_{RS}}{\delta^2 W_{RS}(k,\delta)}=-\frac{23.3}{\delta^2 W_{RS}(k,\delta)}. \tag{30}$$

Table VIII gives the values of $N_0^{\frac{1}{3}}$ from (30) for selected values of k and δ. In this table the lowest value of $N_0=(1.7)^3=5$ is seen to occur for $k=2$, $\delta=2$. Table IX gives the values of the average energy per particle E/N in Mev when $N=9$, and Table X gives the average energy per particle when $N=216$.

The dimensions of the ellipsoid specified by $k=2$, $\delta=1.5$, $\kappa r_0=1.285$ are: major axis$=6.7\times 10^{-13}$ cm, minor axis$=3.4\times 10^{-13}$ cm.

8. DISCUSSION

Although the model used in the above calculations has little meaning when applied to nuclei containing as few particles as the values of N_0 obtained above, nevertheless the inference may be drawn from these calculations that interactions of the type suggested by the neutral meson field theory of nuclear forces, although useful for describing the lighter nuclei, will allow stable nuclei having large spins, excessive binding energies, and deviating appreciably from spherical shape to exist well within the range of the atomic weights of the periodic table, contrary to experimental evidence.

The small values of N_0 given by (26) are due to the large negative values of $W_B(k, \delta)$ obtained by using Bethe's interaction. The feature of the interaction responsible for this is the excessive depth of the attractive tensor term as compared with the magnitude of the repulsive spherically symmetric term. This disparity in the relative strengths of the two parts of the interaction between a proton and a neutron is, however, required to give the observed binding energy of the deuteron, if the wave function for the ground state of the deuteron is not to deviate appreciably from spherical symmetry.

The values of $W_{RS}(k, \delta)$ obtained by using Rarita and Schwinger's interaction are numerically much smaller than those of $W_B(k, \delta)$ because in their interaction the tensor and the spherically symmetric terms are both attractive, and consequently the observed binding energy of the deuteron can be obtained with relatively shallow potential wells. In spite of this the values of N_0 given by (30) are even smaller than those given by (26) because K_{RS} in (30) is much smaller than K_B in (26). This is because the width of the potential well used by Rarita and Schwinger is twice the width of the square well introduced above to replace Bethe's potential. The greater is the width of the square well used, the more important is the potential energy relative to the kinetic energy, since the particles can occupy a larger volume, and hence have less kinetic energy, for the same value of the potential energy.

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APPENDIX

The analytical expressions for the functions $W_1(k, \delta)$ and $W_2(k, \delta)$ defined by (14) are given below.

$k \geqq 1$ (Prolate Ellipsoids)

$$\begin{aligned} W_1(k, \delta) &= 1 && \text{for } \delta \leqq k^{-2/3}, \\ &= 8\delta^{-3} - B(k)\delta^{-4} + C(k)\delta^{-6} \\ &\quad - (k^2-1)^{-1/2} D(\delta k^{-1/3}) && \text{for } k^{-2/3} \leqq \delta \leqq k^{1/3}, \\ &= 8\delta^{-3} - B(k)\delta^{-4} + C(k)\delta^{-6} && \text{for } k^{1/3} \leqq \delta, \end{aligned}$$

where

$$\begin{aligned} B(k) &\equiv \tfrac{9}{2} k^{-4/3} [(k^2-1)^{-1/2} \cos^{-1}(k^{-1}) + k^{-2}], \\ C(k) &\equiv \tfrac{1}{4} [3 + 2k^{-2} + 3k^2(k^2-1)^{-1/2} \cos^{-1}(k^{-1})], \\ D(x) &\equiv \tfrac{3}{4}(x^{-2}-1)^{5/2} + \tfrac{11}{2}(x^{-2}-1)^{3/2} \\ &\quad + \tfrac{15}{4}(x^{-2}-1)^{1/2} - \tfrac{9}{2}(x^{-4} - \tfrac{1}{6}x^{-6}) \cos^{-1} x, \end{aligned}$$

and

$$\begin{aligned} W_2(k, \delta) &= E(k), && \text{for } \delta \leqq k^{-2/3}, \\ &= F(k)\delta^{-4} - G(k)\delta^{-6} + H(k, \delta) && \text{for } k^{-2/3} \leqq \delta \leqq k^{1/3}, \\ &= F(k)\delta^{-4} - G(k)\delta^{-6} && \text{for } k^{1/3} \leqq \delta, \end{aligned}$$

where

$$\begin{aligned} E(k) &\equiv (k^2-1)^{-1}(2k^2+1) - 3k^2(k^2-1)^{-3/2} \cos^{-1}(k^{-1}), \\ F(k) &\equiv \tfrac{9}{8} k^{4/3} [(k^2-4)(k^2-1)^{-3/2} \cos^{-1}(k^{-1}) \\ &\quad + (1+2k^{-2})(k^2-1)^{-1}], \\ G(k) &\equiv \tfrac{1}{8} [3k^2(k^2-2)(k^2-1)^{-3/2} \cos^{-1}(k^{-1}) \\ &\quad + (3k^2 - 4 + 4k^{-2})(k^2-1)^{-1}], \\ H(k, \delta) &\equiv \tfrac{3}{8} k^2 (k^2-1)^{-3/2} [(k^2-2)\delta^{-6} - 3(k^2-4)k^{-2/3}\delta^{-4} - 8] \\ &\quad \times \cos^{-1}(\delta k^{-1/3}) + \tfrac{15}{4}(k^2-1)^{-1/2}(k^{2/3}\delta^{-2}-1)^{1/2} \\ &\quad - \tfrac{1}{8}(k^2+44)(k^2-1)^{-3/2}(k^{2/3}\delta^{-2}-1)^{3/2} \\ &\quad + \tfrac{3}{8}(k^2-2)(k^2-1)^{-3/2}(k^{2/3}\delta^{-2}-1)^{5/2}. \end{aligned}$$

$k \leqq 1$ (Oblate Ellipsoids)[9]

$$\begin{aligned} W_1(k, \delta) &= 1 && \text{for } \delta \leqq k^{1/3}, \\ &= 1 + (1-k^2)^{-1/2} D'(\delta k^{-1/3}) && \text{for } k^{1/3} \leqq \delta \leqq k^{-2/3}, \\ &= 8\delta^{-3} - B'(k)\delta^{-4} + C'(k)\delta^{-6} && \text{for } k^{-2/3} \leqq \delta, \end{aligned}$$

where

$$\begin{aligned} B'(k) &\equiv \tfrac{9}{2} k^{4/3} [(1-k^2)^{-1/2} \cosh^{-1}(k^{-1}) + k^{-2}], \\ C'(k) &\equiv \tfrac{1}{4} [3 + 2k^{-2} + 3k^2(1-k^2)^{-1/2} \cosh^{-1}(k^{-1})], \\ D'(x) &\equiv \tfrac{3}{4}(1-x^{-2})^{5/2} - \tfrac{11}{2}(1-x^{-2})^{3/2} \\ &\quad + \tfrac{15}{4}(1-x^{-2})^{1/2} - \tfrac{9}{2}(x^{-4} - \tfrac{1}{6}x^{-6}) \cosh^{-1} x, \end{aligned}$$

and

$$\begin{aligned} W_2(k, \delta) &= -E'(k) && \text{for } \delta \leqq k^{1/3}, \\ &= -E'(k) + H'(k, \delta) && \text{for } k^{1/3} \leqq \delta \leqq k^{-2/3}, \\ &= -F'(k)\delta^{-4} + G'(k)\delta^{-6} && \text{for } k^{-2/3} \leqq \delta, \end{aligned}$$

where

$$\begin{aligned} E'(k) &\equiv (1-k^2)^{-1}(2k^2+1) - 3k^2(1-k^2)^{-3/2} \cosh^{-1}(k^{-1}), \\ F'(k) &\equiv \tfrac{9}{8} k^{4/3} [(k^2-4)(1-k^2)^{-3/2} \cosh^{-1}(k^{-1}) \\ &\quad + (1+2k^{-2})(1-k^2)^{-1}], \\ G'(k) &\equiv \tfrac{1}{8} [3k^2(k^2-2)(1-k^2)^{-3/2} \cosh^{-1}(k^{-1}) \\ &\quad + (3k^2 - 4 + 4k^{-2})(1-k^2)^{-1}], \\ H'(k, \delta) &\equiv \tfrac{3}{8} k^2 (1-k^2)^{-3/2} [(k^2-2)\delta^{-6} \\ &\quad - 3(k^2-4)k^{-2/3}\delta^{-4} - 8] \cosh^{-1}(\delta k^{-1/3}) \\ &\quad - \tfrac{15}{4}(1-k^2)^{-1/2}(1-k^{2/3}\delta^{-2})^{1/2} \\ &\quad + \tfrac{1}{8}(k^2+44)(1-k^2)^{-3/2}(1-k^{2/3}\delta^{-2})^{3/2} \\ &\quad + \tfrac{3}{8}(k^2-2)(1-k^2)^{-3/2}(1-k^{2/3}\delta^{-2})^{5/2}. \end{aligned}$$

The three separate ranges of validity of all the above results correspond, respectively, to the range r_0 of the step function $S(r)$ being greater than the maximum dimension of the ellipsoid, intermediate between the maximum and minimum dimensions, and less than the minimum dimension.

As $\delta \to \infty$ the values of $W_1(k, \delta)$, $W_2(k, \delta)$, and $W_2(k, \delta)/W_1(k, \delta)$ all approach zero for all values of k.

As $k \to \infty$ for any finite value of δ the values of $W_1(k, \delta)$ and $W_2(k, \delta)$ both approach zero, while their ratio $W_2(k, \delta)/W_1(k, \delta)$ approaches 2.

As $k \to 0$ for any finite value of δ the values of $W_1(k, \delta)$ and $W_2(k, \delta)$ both approach zero, while their ratio $W_2(k, \delta)/W_1(k, \delta)$ approaches -1.

$W_1(k, \delta)$ is always positive, while $W_2(k, \delta)$ is positive for prolate ellipsoids ($k > 1$), negative for oblate ellipsoids ($k < 1$), and zero for a sphere ($k = 1$).

The values of $W_1(k, \delta)$ and $W_2(k, \delta)$ calculated from the above formulae for selected values of k and δ are given in Tables I and II.

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[9] The formulae for the oblate case were also obtained independently by W. Rarita and J. Schwinger. I am indebted to them for communicating to me their results and the following numerical values of $W_1(k, \delta)$ and $W_2(k, \delta)$ some of which supplement those already listed in Tables I and II:

δ	k	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
1	$W_1(k, \delta)$	1.000	1.000	0.999	0.996	0.988	0.973	0.945	0.896	0.805	0.624
	$W_2(k, \delta)$	0.000	−0.082	−0.172	−0.266	−0.360	−0.454	−0.542	−0.612	−0.642	−0.566
0	$W_1(k, \delta)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$W_2(k, \delta)$	0.000	−0.082	−0.172	−0.268	−0.370	−0.480	−0.594	−0.714	−0.832	−0.940