Electromagnetic Properties of Nuclei in the Meson Theory

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The mathematical treatment of the interaction of a meson field with an electromagnetic field is simplified by using Pauli's method. A discussion of calculation of exchange multipole moments of nuclei by the method of canonical transformation and the results for dipole moments in the pseudoscalar theory are given.

1. INTRODUCTION

THE theory of the interaction of a meson field and an electromagnetic field has been developed by several authors.[1] More details have been worked out by Bhabha. The latter has given complete expressions for the electromagnetic interaction of a meson field in the presence of nuclear particles and also equations of continuity for the meson current and the current of nuclear particles separately. Both the mathematical treatment and the final result can be simplified if we adopt a method used by Pauli.[2] Pauli's treatment is based on the general properties of the Lagrangian function instead of the explicit form of Proca's equations which are derivable from the Lagrangian function as variational equations. The method is therefore quite general and is applicable not only to the vector meson theory, but also to the scalar, pseudovector, and pseudoscalar theories as well. Pauli treated only the electromagnetic interaction of a free meson. We give below a generalization of his method to include nuclear particles.

Owing to the exchange nature of the meson theory, it becomes now necessary, in treating the interaction of a nucleus with an external electromagnetic field, to take into account not only the direct influence of the electromagnetic field on the nuclear particles, but also the interaction of the external field with the exchange meson field. Investigations in this direction have recently been made by several authors.[3] It has been shown that the electromagnetic interaction of a nuclear system can be expressed in terms of interactions between the external field and the electric and magnetic multipole moments of the nuclear system. The electric and magnetic dipole moments and the electric quadripole moment have been calculated on the vector theory by means of the ordinary perturbation method in quantum theory. It might appear objectionable to use a perturbation method for this purpose, because perturbations of as high as the third order are involved in such calculations. It can be shown, however, that all the previous results can be obtained, in a simpler manner, as the first approximation in the canonical transformation carried out by Møller and Rosenfeld.[4] We give below a discussion of the calculation of the exchange multipole moments by the method of canonical transformation and also give results in the pseudoscalar theory.

2. GENERAL THEORY OF THE INTERACTION OF MESON FIELD WITH ELECTROMAGNETIC FIELD

We give below a mathematical treatment of the interaction of a meson field with an electromagnetic field in the vector meson theory on the basis of Pauli's method. Pauli's method may be easily extended to other forms of meson theory. We give at the end of this section the results for the pseudoscalar theory, which can be similarly derived.

The vector meson field is described by a four vector U_ρ, with $U_4=\epsilon U_0$, ϵ being the square root of -1 introduced by Proca.[5] Let the complex conjugates of the field variables U_ρ be denoted

[1] Yukawa, Sakata, Kobayasi, and Taketani, Proc. Phys. Math. Soc. Japan **20**, 319, 720 (1938); Fröhlich, Heitler, and Kemmer, Proc. Roy. Soc. **A166**, 154 (1938); Kemmer, Proc. Roy. Soc. **A166**, 127 (1938); Bhabha, Proc. Roy. Soc. **A166**, 501 (1938).

[2] W. Pauli, Report of the Solvay Conference (1939).

[3] A. J. F. Siegert, Phys. Rev. **52**, 787 (1937); W. E. Lamb, Jr. and L. I. Schiff, Phys. Rev. **53**, 651 (1938); Fröhlich, Heitler, and Hahn, Proc. Roy. Soc. **A174**, 85 (1940); S. T. Ma, Proc. Camb. Phil. Soc. **36**, 351, 438 (1940).

[4] Møller and Rosenfeld, K. Danske Vidensk. Selskab. **17**, 8 (1940).

[5] Proca, J. de phys. et rad. **7**, 347 (1936).

by U_ρ^*. We introduce the operators

$$\pi_\nu = \nabla_\nu - i\Phi_\nu, \quad \pi_\nu^* = \nabla_\nu + i\Phi_\nu, \tag{1}$$

where $\nabla_\nu = \partial/\partial x_\nu$, $(\Phi_1, \Phi_2, \Phi_3) = (e/\hbar c)(A_1, A_2, A_3)$ and $\Phi_0 = \Phi_4/\epsilon = (e/\hbar c)A_0$, $\mathbf{A}$ and A_0 being the electromagnetic potentials from which the antisymmetric tensor $f_{\rho\nu}$ representing the field strengths can be derived. We also adopt the notations

$$\begin{aligned} U_{\rho\nu} &= \pi_\nu U_\rho, \quad & U^*_{\rho\nu} &= \pi_\nu^* U_\rho^*, \\ U_{\rho\nu\lambda} &= \pi_\lambda \pi_\nu U_\rho, \quad & U^*_{\rho\nu\lambda} &= \pi_\lambda^* \pi_\nu^* U_\rho^*. \end{aligned} \tag{2}$$

From these we define the new field variables

$$G_{\rho\nu} = U_{\nu\rho} - U_{\rho\nu}, \quad G^*_{\rho\nu} = U^*_{\nu\rho} - U^*_{\rho\nu}. \tag{3}$$

It follows from the relations

$$\pi_\nu \pi_\lambda - \pi_\lambda \pi_\nu = -\frac{ie}{\hbar c} f_{\nu\lambda}, \quad \pi_\nu^* \pi_\lambda^* - \pi_\lambda^* \pi_\nu^* = \frac{ie}{\hbar c} f_{\nu\lambda} \tag{4}$$

that

$$\begin{aligned} U_{\rho\nu\lambda} - U_{\rho\lambda\nu} &= +\frac{ie}{\hbar c} f_{\nu\lambda} U_\rho, \\ U^*_{\rho\nu\lambda} - U^*_{\rho\lambda\nu} &= -\frac{ie}{\hbar c} f_{\nu\lambda} U_\rho^*. \end{aligned} \tag{5}$$

Furthermore, for two complex scalar quantities A and B, we have

$$(\pi_\rho^* A^*) B + A^*(\pi_\rho B) = \nabla_\rho (A^* B). \tag{6}$$

Let the wave functions of the nuclear particles be denoted by Ψ which consists of the relativistic wave functions of the proton and the neutron. On introducing the notations

$$\Psi^\dagger = \epsilon \Psi^* \gamma^4, \quad \gamma^\rho = -i\beta\alpha^\rho, \quad \alpha^4 = \epsilon, \tag{7}$$

where α^1, α^2, α^3, β are Dirac's matrices, we obtain the following four vectors and six vectors:

$$\begin{aligned} \Psi_P^\dagger \Pi \gamma^\rho \Psi_N &= \Gamma^\rho, \quad & \Psi_N^\dagger \Pi^* \gamma^\rho \Psi_P &= \Gamma^{\rho *}, \\ \Psi_P^\dagger \Pi \gamma^\rho \gamma^\nu \Psi_N &= \Gamma^{\rho\nu}, \quad & \Psi_N^\dagger \Pi^* \gamma^\rho \gamma^\nu \Psi_P &= \Gamma^{\rho\nu *}. \end{aligned} \tag{8}$$

The complete Lagrangian function for the system of nuclear particles, the meson field, their mutual interaction, and their interaction with the external electromagnetic field may be written as follows:

$$L = L^P + L^N + L^M + L^{PN} + L^{NP}, \tag{9}$$

$$\begin{aligned} L^P &= -i\Psi_P^\dagger (c\hbar \gamma^\rho \pi_\rho + M_P c^2) \Psi_P, \\ L_N &= -i\Psi_N^\dagger (c\hbar \gamma^\rho \nabla_\rho + M_N c^2) \Psi_N, \end{aligned} \tag{10}$$

$$L^M = \kappa^2 U_\rho^* U_\rho + \tfrac{1}{2} G^*_{\rho\nu} G_{\rho\nu}, \tag{11}$$

$$\begin{aligned} L^{PN} &= -g_1 \Gamma^\rho U_\rho - \frac{g_2}{2\kappa} \Gamma^{\rho\nu} G_{\rho\nu}, \\ L^{NP} &= -g_1 \Gamma^{\rho *} U_\rho^* - \frac{g_2}{2\kappa} \Gamma^{\rho\nu *} G^*_{\rho\nu}. \end{aligned} \tag{12}$$

It follows from the variational principle

$$\delta \int\int\int\int L dx_1 dx_2 dx_3 dx_4 = 0 \tag{13}$$

that

$$\pi_\nu^* \frac{\partial L}{\partial U_{\rho\nu}} = \frac{\partial L}{\partial U_\rho}, \quad \pi_\nu \frac{\partial L}{\partial U^*_{\rho\nu}} = \frac{\partial L}{\partial U_\rho^*}, \tag{14}$$

$$\pi_\nu^* \frac{\partial L}{\partial \Psi_{P\nu}} = \frac{\partial L}{\partial \Psi_P}, \quad \pi_\nu \frac{\partial L}{\partial \Psi^*_{P\nu}} = \frac{\partial L}{\partial \Psi_P^*}, \tag{15}$$

$$\nabla_\nu \frac{\partial L}{\partial \Psi_{N\nu}} = \frac{\partial L}{\partial \Psi_N}, \quad \nabla_\nu \frac{\partial L}{\partial \Psi^*_{N\nu}} = \frac{\partial L}{\partial \Psi_N}. \tag{16}$$

From these equations we obtain

$$\begin{aligned} \nabla_\nu \left(\frac{\partial L}{\partial U_{\lambda\nu}} U_\lambda \right) &= \frac{\partial L}{\partial U_\lambda} U_\lambda + \frac{\partial L}{\partial U_{\lambda\nu}} U_{\lambda\nu}, \\ \nabla_\nu \left(U_\lambda^* \frac{\partial L}{\partial U^*_{\lambda\nu}} \right) &= U_\lambda^* \frac{\partial L}{\partial U_\lambda^*} + U^*_{\lambda\nu} \frac{\partial L}{\partial U^*_{\lambda\nu}}, \\ \nabla_\nu \left(\frac{\partial L}{\partial U_{\lambda\nu}} U_{\lambda\rho} \right) &= \frac{\partial L}{\partial U_\lambda} U_{\lambda\rho} + \frac{\partial L}{\partial U_{\lambda\nu}} U_{\lambda\rho\nu}, \\ \nabla_\nu \left(U^*_{\lambda\rho} \frac{\partial L}{\partial U^*_{\lambda\nu}} \right) &= U^*_{\lambda\rho} \frac{\partial L}{\partial U_\lambda^*} + U^*_{\lambda\rho\nu} \frac{\partial L}{\partial U^*_{\lambda\nu}}. \end{aligned} \tag{17}$$

The Lagrangian function defined above, besides being relativistically invariant, is invariant under a gauge transformation. A gauge transformation for the wave functions of the protons, neutrons, and mesons is

$$\begin{aligned} \Psi_P &\to \Psi_P e^{i\alpha}, \quad & \Psi_P^* &\to \Psi_P^* e^{-i\alpha}, \\ \Psi_N &\to \Psi_N, \quad & \Psi_N^* &\to \Psi_N^*, \\ U_\rho &\to U_\rho e^{i\alpha}, \quad & U_\rho^* &\to U_\rho^* e^{-i\alpha}, \end{aligned} \tag{18}$$

where α is an arbitrary function of x_1, x_2, x_3, and x_4. The corresponding gauge transformation for the electromagnetic fields is

$$\Phi_\nu \to \Phi_\nu + \nabla_\nu \alpha. \tag{19}$$

From (18) and (19) it follows that

$$\pi_\nu \to \pi_\nu - i\nabla_\nu \alpha, \quad \pi_\nu^* \to \pi_\nu^* + i\nabla_\nu \alpha, \tag{20}$$

and therefore

$$U_{\rho\nu}\rightarrow U_{\rho\nu}e^{i\alpha},\quad U^*_{\rho\nu}\rightarrow U^*_{\rho\nu}e^{-i\alpha},$$
$$U_{\rho\nu\lambda}\rightarrow U_{\rho\nu\lambda}e^{i\alpha},\quad U^*_{\rho\nu\lambda}\rightarrow U^*_{\rho\nu\lambda}e^{-i\alpha},$$
$$\Psi_{P\nu}\rightarrow \Psi_{P\nu}e^{i\alpha},\quad \Psi^*_{P\nu}\rightarrow \Psi^*_{P\nu}e^{-i\alpha},$$
$$\Psi_{N\nu}\rightarrow \Psi_{N\nu},\quad \Psi^*_{N\nu}\rightarrow \Psi^*_{N\nu}, \tag{21}$$

and also that

$$\Gamma^{\nu}\rightarrow\Gamma^{\nu}e^{-i\alpha},\quad \Gamma^{\nu *}\rightarrow\Gamma^{\nu *}e^{i\alpha},$$
$$\Gamma^{\rho\nu}\rightarrow\Gamma^{\rho\nu}e^{-i\alpha},\quad \Gamma^{\rho\nu *}\rightarrow\Gamma^{\rho\nu *}e^{i\alpha}. \tag{22}$$

Since L is invariant under a gauge transformation, we have

$$\partial L/\partial\alpha=0.$$

Putting $\alpha=0$ in this equation we obtain

$$\frac{\partial L}{\partial U_\nu}U_\nu+\frac{\partial L}{\partial U_{\nu\rho}}U_{\nu\rho}+\frac{\partial L}{\partial \Psi_P}\Psi_P+\frac{\partial L}{\partial \Psi_{P\nu}}\Psi_{P\nu}=U_\nu{}^*\frac{\partial L}{\partial U_\nu{}^*}+U^*_{\nu\rho}\frac{\partial L}{\partial U^*_{\nu\rho}}+\Psi_P{}^*\frac{\partial L}{\partial \Psi_P{}^*}+\Psi^*_{P\nu}\frac{\partial L}{\partial \Psi^*_{P\nu}}. \tag{23}$$

With the help of this equation we obtain

$$\nabla_\rho L=U^*_{\lambda\rho}\frac{\partial L}{\partial U_\lambda{}^*}+\frac{\partial L}{\partial U_\lambda}U_{\lambda\rho}+U^*_{\lambda\nu\rho}\frac{\partial L}{\partial U^*_{\lambda\nu}}+\frac{\partial L}{\partial U_{\lambda\nu}}U_{\lambda\nu\rho}+\left(\Psi_\rho{}^*\frac{\partial L}{\partial \Psi^*}+\frac{\partial L}{\partial \Psi}\Psi_\rho+\Psi^*_{\nu\rho}\frac{\partial L}{\partial \Psi_\nu{}^*}+\frac{\partial L}{\partial \Psi_\nu}\Psi_{\nu\rho}\right)_{P,N}. \tag{24}$$

The energy and momentum tensor may be defined as

$$T_{\rho\nu}=U^*_{\lambda\rho}\frac{\partial L}{\partial U^*_{\lambda\nu}}+\frac{\partial L}{\partial U_{\lambda\nu}}U_{\lambda\rho}+\left(\Psi_\rho{}^*\frac{\partial L}{\partial \Psi_\nu{}^*}+\frac{\partial L}{\partial \Psi_\nu}\Psi_\rho\right)_{P,N}-L\delta_{\rho\nu}. \tag{25}$$

By (17),

$$\nabla_\nu T_{\rho\nu}=U^*_{\lambda\rho\nu}\frac{\partial L}{\partial U^*_{\lambda\nu}}+\frac{\partial L}{\partial U_{\lambda\nu}}U_{\lambda\rho\nu}+U^*_{\lambda\rho}\frac{\partial L}{\partial U_\lambda{}^*}+\frac{\partial L}{\partial U_\lambda}U_{\lambda\rho}+\left(\Psi^*_{\rho\nu}\frac{\partial L}{\partial \Psi_\nu{}^*}+\frac{\partial L}{\partial \Psi_\nu}\Psi_{\rho\nu}+\Psi_\rho{}^*\frac{\partial L}{\partial \Psi^*}+\frac{\partial L}{\partial \Psi}\Psi_\rho\right)_{P,N}-\nabla_\rho L. \tag{26}$$

Substituting (24) into this expression we obtain the result

$$\nabla_\nu T_{\rho\nu}=f_{\rho\nu}s_\nu, \tag{27}$$

where

$$s_\nu=-\frac{ie}{\hbar c}\left(U_\lambda{}^*\frac{\partial L}{\partial U^*_{\lambda\nu}}-\frac{\partial L}{\partial U_{\lambda\nu}}U_\lambda+\Psi_P{}^*\frac{\partial L}{\partial \Psi^*_{P\nu}}-\frac{\partial L}{\partial \Psi_{P\nu}}\Psi_P\right) \tag{28}$$

may be interpreted as the four vector expressing the current and charge densities of the nuclear system including the nuclear particles and the meson field. From (28) we have

$$\nabla_\nu s_\nu=-\frac{ie}{\hbar c}\left(U_\lambda{}^*\frac{\partial L}{\partial U_\lambda{}^*}-\frac{\partial L}{\partial U_\lambda}U_\lambda+U^*_{\lambda\nu}\frac{\partial L}{\partial U^*_{\lambda\nu}}-\frac{\partial L}{\partial U_{\lambda\nu}}U_{\lambda\nu}+\Psi_P{}^*\frac{\partial L}{\partial \Psi_P{}^*}-\frac{\partial L}{\partial \Psi_P}\Psi_P+\Psi^*_{P\nu}\frac{\partial L}{\partial \Psi^*_{P\nu}}-\frac{\partial L}{\partial \Psi_{P\nu}}\Psi_{P\nu}\right). \tag{29}$$

The equation of continuity for the total current

$$\nabla_\nu s_\nu=0 \tag{30}$$

follows from (29) and (23).

The current s_ν can be separated into two parts: the proton and the meson current defined, respectively, by

$$s_\nu{}^P=-\frac{ie}{\hbar c}\left(\Psi_P{}^*\frac{\partial L}{\partial \Psi^*_{P\nu}}-\frac{\partial L}{\partial \Psi_{P\nu}}\Psi_P\right), \tag{31}$$

$$s_\nu{}^M=-\frac{ie}{\hbar c}\left(U_\lambda{}^*\frac{\partial L}{\partial U^*_{\lambda\nu}}-\frac{\partial L}{\partial U_{\lambda\nu}}U_\lambda\right). \tag{32}$$

The two parts do not satisfy (30) separately because of the exchange of charge. We have

$$\nabla_\nu s_\nu{}^P = -\frac{ie}{\hbar c}\left(\Psi_P{}^*\frac{\partial L}{\partial \Psi_P{}^*}+\Psi^*_{P\lambda}\frac{\partial L}{\partial \Psi^*_{P\lambda}}-\frac{\partial L}{\partial \Psi_P}\Psi_P-\frac{\partial L}{\partial \Psi_{P\lambda}}\Psi_{P\lambda}\right).$$

On account of the relations

$$\Psi_P{}^*\frac{\partial L}{\partial \Psi_P{}^*}=L^P+L^{PN},\quad \frac{\partial L}{\partial \Psi^*_{P\lambda}}=0,\quad \frac{\partial L}{\partial \Psi_{P\lambda}}\Psi_{P\lambda}+\frac{\partial L}{\partial \Psi_P}\Psi_P=L^P+L^{NP},$$

we obtain the result

$$\nabla_\nu s_\nu{}^P = -\nabla_\nu s_\nu{}^M = -\frac{ie}{\hbar c}(L^{PN}-L^{NP}). \tag{33}$$

This equation expresses the rate of flow of the exchange current into or out of the nuclear particles. It has been given by Bhabha[1] but it is here given in a much simpler form.

The explicit expressions for the variational equations and the current densities are

$$\pi_\rho{}^* G'^*_{\rho\nu} = \kappa^2 U_\nu{}^* - g_1\Gamma^\nu,\quad \pi_\rho G'_{\rho\nu} = \kappa^2 U_\nu - g_1\Gamma^{\nu *}, \tag{34}$$

$$s_\nu{}^P = e\Psi_P{}^\dagger\gamma^\nu\Psi_P,\quad s_\nu{}^M = -\frac{ie}{\hbar c}(U_\lambda{}^* G'_{\nu\lambda} - G'^*_{\nu\lambda}U_\lambda), \tag{35}$$

where

$$G'_{\rho\nu} = G_{\rho\nu} - \frac{g_2}{\kappa}\Gamma^{\rho\nu *},\quad G'^*_{\rho\nu} = G^*_{\rho\nu} - \frac{g_2}{\kappa}\Gamma^{\rho\nu}. \tag{36}$$

Let us now write the four and six vectors in terms of ordinary vectors and scalars as follows:

$$\begin{aligned} &U_\rho = (\mathbf{U}, V),\quad G'_{\rho\nu} = (\mathbf{G}, \mathbf{F}),\quad (G'_{12}=G_3,\quad G'_{01}=F_1,\ \text{etc.}),\\ &g_1\Gamma^\rho = (\mathbf{M}, N) = (g_1\Psi_P{}^*\Pi\boldsymbol{\alpha}\Psi_N,\ g_1\Psi_P{}^*\Pi\Psi_N),\\ &-\frac{g_2}{\kappa}\Gamma^{\rho\nu} = (\mathbf{S}, \mathbf{T}) = \left(\frac{g_2}{\kappa}\Psi_P{}^*\Pi\beta\boldsymbol{\sigma}\Psi_N,\ \frac{ig_2}{\kappa}\Psi_P{}^*\Pi\beta\boldsymbol{\alpha}\Psi_N\right).\end{aligned} \tag{37}$$

Equations (34) and (36) become then

$$\begin{aligned} \left(\frac{1}{c}\frac{\partial}{\partial t}+i\Phi_0\right)\mathbf{F} &= (\boldsymbol{\nabla}-i\boldsymbol{\Phi})\times\mathbf{G}+\kappa^2\mathbf{U}-\mathbf{M}^*,\\ \kappa^2 V &= -(\boldsymbol{\nabla}-i\boldsymbol{\Phi})\cdot\mathbf{F}+\mathbf{N}^*,\\ \mathbf{G} &= (\boldsymbol{\nabla}-i\boldsymbol{\Phi})\times\mathbf{U}+\mathbf{S}^*,\\ \left(\frac{1}{c}\frac{\partial}{\partial t}+i\Phi_0\right)\mathbf{U} &= -\mathbf{F}-(\boldsymbol{\nabla}-i\boldsymbol{\Phi})V+\mathbf{T}^*,\end{aligned} \tag{38}$$

together with the complex conjugates of these equations. The meson current and charge densities are, in the present notations,

$$\mathbf{j} = c(s_1, s_2, s_3) = -\frac{ie}{\hbar}(\mathbf{U}^*\times\mathbf{G}-\mathbf{U}\times\mathbf{G}^*+V^*\mathbf{F}-V\mathbf{F}^*), \tag{39}$$

$$\rho = s_4/\epsilon = -\frac{ie}{c\hbar}(\mathbf{U}^*\cdot\mathbf{F}-\mathbf{F}^*\cdot\mathbf{U}). \tag{40}$$

In order to link up the present notations with that of reference 3, we divide $\mathbf{U}$ and $\mathbf{F}$ into longitudinal and transverse parts and put

$$\mathbf{U}_l=\frac{1}{(4\pi)^{\frac{1}{2}}}\phi,\quad \mathbf{F}_e=-\frac{\kappa}{(4\pi)^{\frac{1}{2}}}\psi.$$

The longitudinal part of $\mathbf{U}$ and the transverse part of $\mathbf{F}$ may then be expressed in terms of ϕ and ψ with the help of (38) with Φ_0, $\mathbf{\Phi}$, $\mathbf{M}$, N, $\mathbf{S}$, and $\mathbf{T}$ omitted. We have

$$\mathbf{U}=\frac{1}{(4\pi)^{\frac{1}{2}}}\left(\phi-\frac{1}{c\kappa}\dot{\psi}\right),\quad \mathbf{F}=-\frac{\kappa}{(4\pi)^{\frac{1}{2}}}\left(\psi+\frac{1}{c\kappa}\dot{\phi}\right). \tag{41}$$

The Hamiltonian is equal to the 44 component of the energy-momentum tensor

$$T_{\rho\nu}=-L\delta_{\rho\nu}+\overset{*}{U}_{\lambda\rho}G'_{\nu\lambda}+G'^{*}_{\nu\lambda}U_{\lambda\rho}-ic\hbar\Psi^{\dagger}\gamma^{\nu}(\nabla_{\rho}-i\Phi_{\rho}\tau_P)\Psi \tag{42}$$

multiplied by -1. Since

$$\overset{*}{U}_{\lambda\rho}G'_{\nu\lambda}=\overset{*}{G}_{\rho\lambda}G_{\nu\lambda}-\frac{g_2}{\kappa}\overset{*}{G}_{\rho\lambda}\Gamma^{\nu\lambda *}+\overset{*}{U}_{\rho\lambda}G'_{\nu\lambda},$$

and

$$\begin{aligned}\nabla_{\nu}(\overset{*}{U}_{\rho\lambda}G'_{\nu\lambda})&=\nabla_{\nu}\nabla_{\lambda}(U_{\rho}{}^{*}G'_{\nu\lambda})+\nabla_{\nu}[U_{\rho}{}^{*}(\kappa^2U_{\nu}-g_1\Gamma^{\nu *})]\\&=\nabla_{\nu}[U_{\rho}{}^{*}(\kappa^2U_{\nu}-g_1\Gamma^{\nu *})],\end{aligned}$$

we may write

$$T_{\rho\nu}=-L\delta_{\rho\nu}-ic\hbar\Psi^{\dagger}\gamma^{\nu}(\nabla_{\rho}-i\Phi_{\rho}\tau_P)\Psi+\overset{*}{G}_{\rho\lambda}G_{\nu\lambda}+\kappa^2U_{\rho}{}^{*}U_{\nu}-g_1\Gamma^{\nu}U_{\rho}-\frac{g_2}{\kappa}\Gamma^{\nu\lambda}G_{\rho\lambda}+\text{the complex conjugate.} \tag{43}$$

Hence

$$\begin{aligned}H=-T_{44}&=L+ic\hbar\Psi^{\dagger}\gamma^{4}(\nabla_4-i\Phi_4\tau_P)\Psi-2(\overset{*}{G}_{4\lambda}G_{4\lambda}+\kappa^2U_4{}^{*}U_4)\\&\qquad+g_1\Gamma^4U_4+\frac{g_2}{\kappa}\Gamma^{4\lambda}G_{4\lambda}+\text{the complex conjugate}\\&=-i\Psi^{\dagger}[c\hbar\boldsymbol{\gamma}\cdot(\boldsymbol{\nabla}-i\mathbf{\Phi}\tau_P)+(M_P\tau_P+M_N\tau_N)c^2]\Psi+\kappa^2\mathbf{U}^{*}\cdot\mathbf{U}+\tfrac{1}{2}\sum_{i,j=1}^{3}\overset{*}{G}_{ij}G_{ij}\\&\qquad+\frac{1}{\kappa^2}|\boldsymbol{\pi}\cdot\mathbf{F}-N^{*}|^2+|\mathbf{F}-\mathbf{T}^{*}|^2-\mathbf{M}\cdot\mathbf{U}+\tfrac{1}{2}\sum_{i,j=1}^{3}S^{ij}G_{ij}+\text{the complex conjugate.}\end{aligned} \tag{44}$$

This result can be separated into a term independent of the electromagnetic field and a term describing the electromagnetic interaction, namely,

$$\begin{aligned}H_0=&-i\Psi^{\dagger}[c\hbar(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})+(M_P\tau_P+M_N\tau_N)c^2]\Psi+\kappa^2\mathbf{U}^{*}\cdot\mathbf{U}+\mathbf{F}^{*}\cdot\mathbf{F}\\&+\frac{1}{\kappa^2}|\boldsymbol{\nabla}\cdot\mathbf{F}|^2+|\boldsymbol{\nabla}\times\mathbf{U}|^2-M\cdot U-\frac{1}{\kappa^2}N(\boldsymbol{\nabla}\cdot\mathbf{F})+\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{U})-\mathbf{F}\cdot\mathbf{T}\\&+\text{the complex conjugate}+\text{terms containing } g_1{}^2,\ g_2{}^2,\end{aligned} \tag{45}$$

and

$$H_I=-\frac{1}{c}\mathbf{j}\cdot\mathbf{A}. \tag{46}$$

In the second equation terms proportional to the square of the A's have been omitted. In order to obtain the interaction with an external electrostatic field A_0 we have further to introduce into the Hamiltonian the zero term

$$A_0\left(\rho-\frac{1}{4\pi}\boldsymbol{\nabla}\cdot\mathbf{E}\right), \tag{47}$$

where ρ is the charge density given by (40) and $\mathbf{E}$ is the electric field strength.[6]

[6] Heisenberg and W. Pauli, Zeits. f. Physik **59**, 168 (1930).

Equation (44) can also be reduced to the form

$$H=-i\Psi^\dagger[c\hbar\boldsymbol{\gamma}\cdot(\nabla-i\boldsymbol{\Phi}\tau_P)+(M_P\tau_P+M_N\tau_N)c^2]\Psi+\kappa^2\mathbf{U}^*\cdot\mathbf{U}+\mathbf{G}^*\cdot\mathbf{G}+\kappa^2V^*V$$
$$+\mathbf{F}^*\cdot\mathbf{F}-\mathbf{F}\cdot\mathbf{T}-\mathbf{F}^*\cdot\mathbf{T}^*-\mathbf{M}\cdot\mathbf{U}-\mathbf{M}^*\cdot\mathbf{U}^*, \quad (48)$$

if we add the relativistically invariant term

$$\frac{1}{2}\frac{g_2^2}{\kappa^2}\Gamma^{\rho\nu *}\Gamma^{\rho\nu}=\mathbf{S}^*\cdot\mathbf{S}-\mathbf{T}^*\cdot\mathbf{T}$$

to the Lagrangian function (9).

The treatment of other forms of meson theory is similar to the above. We state briefly the results in the pseudoscalar theory. Let ψ be a pseudoscalar describing the meson field and $\psi_\rho=\pi_\rho\psi$ be the pseudovector derived from ψ. For the nuclear particles we have the following pseudovector and pseudoscalar:

$$\begin{gathered}\Gamma^{\rho\nu\lambda}=i\epsilon\Psi_P^\dagger\Pi\gamma^\rho\gamma^\nu\gamma^\lambda\Psi_N,\quad \Gamma^{\rho\nu\lambda *}=i\epsilon\Psi_N^\dagger\Pi^*\gamma^\rho\gamma^\nu\gamma^\lambda\Psi_P,\\ (\Gamma^{234},\Gamma^{314},\Gamma^{124},\Gamma^{321})=(\Gamma^{1\prime},\Gamma^{2\prime},\Gamma^{3\prime},\Gamma^{4\prime}),\\ (\Gamma^{234*},\Gamma^{314*},\Gamma^{124*},\Gamma^{321*})=(\Gamma^{1\prime *},\Gamma^{2\prime *},\Gamma^{3\prime *},\Gamma^{4\prime *}),\\ \Gamma^{1234}=-i\epsilon\Psi_P^\dagger\Pi\gamma^1\gamma^2\gamma^3\gamma^4\Psi_N=\Gamma',\quad \Gamma^{1234*}=-i\epsilon\Psi_N^\dagger\Pi^*\gamma^1\gamma^2\gamma^3\gamma^4\Psi_P=\Gamma'^*.\end{gathered}\tag{49}$$

While the Lagrangian function for the nuclear particles is still given by (10), for the meson field and its interactions we have

$$L^M=\kappa^2\psi^*\psi+\psi_\rho^*\psi_\rho,\quad L^{PN}=-f_1\Gamma'\psi-\frac{f_2}{\kappa}\Gamma'^\rho\psi_\rho,\quad L^{NP}=-f_1\Gamma'^*\psi^*-\frac{f_2}{\kappa}\Gamma'^{\rho *}\psi_\rho^*.\tag{50}$$

The variational equations for the pseudoscalar meson field are

$$\pi_\nu^*\partial L/\partial\psi_\nu=\partial L/\partial\psi,\quad \pi_\nu\partial L/\partial\psi_\nu^*=\partial L/\partial\psi^*,\tag{51}$$

or

$$\pi_\rho\varphi_\rho=-\kappa^2\psi+f_1\Gamma'^*,\quad \pi_\rho^*\varphi_\rho^*=-\kappa^2\psi^*+f_1\Gamma',\tag{52}$$

where

$$\varphi_\rho=-\psi_\rho+\frac{f_2}{\kappa}\Gamma^{\rho\prime *},\quad \varphi_\rho^*=-\psi_\rho^*+\frac{f_2}{\kappa}\Gamma^{\rho\prime}.\tag{53}$$

The present scheme is also gauge invariant and therefore (23) still holds. From the energy-momentum tensor

$$T_{\rho\nu}=\psi_\rho^*\frac{\partial L}{\partial\psi_\nu^*}+\frac{\partial L}{\partial\psi_\nu}\psi_\rho+\left(\Psi_\rho^*\frac{\partial L}{\partial\Psi_\nu^*}+\frac{\partial L}{\partial\Psi_\nu}\Psi_\rho\right)_{P,N}-L\delta_{\rho\nu}\tag{54}$$

it follows that the four vector of the current and charge density is given by

$$s_\nu=s_\nu^P+s_\nu^M,$$

where

$$s_\nu^M=\frac{ie}{\hbar c}(\psi^*\varphi_\nu-\varphi_\nu^*\psi).\tag{55}$$

The proof of the equations of continuity (30) and (33) is the same as in the vector theory.

Let us rewrite the above four-dimensional pseudovectors and pseudoscalars in terms of three-dimensional pseudovectors and pseudoscalars as follows:

$$\varphi_\rho=(\boldsymbol{\Gamma},\varphi),\tag{56}$$

$$f_1\Gamma'=R,\quad \frac{f_2}{\kappa}(\Gamma^{1\prime},\Gamma^{2\prime},\Gamma^{3\prime})=\mathbf{P},\quad \frac{f_2}{\kappa}\Gamma^{0\prime}=Q.\tag{57}$$

Then (52) and (53) become

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+i\Phi_0\right)\psi=\varphi-Q^*,\quad -\left(\frac{1}{c}\frac{\partial}{\partial t}+i\Phi_0\right)\varphi=\kappa^2\psi+(\nabla-i\mathbf{\Phi})\cdot\mathbf{\Gamma}-R^*,\quad \mathbf{\Gamma}=-(\nabla-i\mathbf{\Phi})\psi+P^*, \tag{58}$$

together with the complex conjugates of these equations, and (55) takes the form

$$\mathbf{j}^M=(ie/\hbar)(\psi^*\mathbf{\Gamma}=\mathbf{\Gamma}^*\psi), \tag{59}$$

$$\rho^M=(ie/\hbar c)(\psi^*\varphi-\varphi^*\psi). \tag{60}$$

The Hamiltonian is given by

$$H=-i\Psi^\dagger\{c\hbar\boldsymbol{\gamma}\cdot(\nabla-i\mathbf{\Phi}\tau_P)+(M_P\tau_P+M_N\tau_N)c^2\}\Psi+\kappa^2\psi^*\psi+|\pi\psi|^2+\varphi^*\varphi-f_1\Gamma'\psi$$
$$-\frac{f_2}{\kappa}(\Gamma^{0\prime}\varphi+\sum_{k=1}^{3}\Gamma^{k\prime}\pi_k\psi)+\text{the complex conjugate}+\frac{f_2^2}{\kappa^2}\Gamma^{0\prime *}\Gamma^{0\prime}, \tag{61}$$

in which the term describing the electromagnetic interaction is of the form (46), $\mathbf{j}^M$ being now given by (59). Equation (61) may be rewritten in the form

$$H=-i\Psi^\dagger\{c\hbar\boldsymbol{\gamma}\cdot(\nabla-i\mathbf{\Phi}\tau_P)+(M_P\tau_P+M_N\tau_N)c^2\}\Psi+\kappa^2\psi^*\psi+\varphi^*\varphi+\mathbf{\Gamma}^*\cdot\mathbf{\Gamma}-R\psi$$
$$-Q\varphi-R^*\psi^*-Q^*\varphi^*+|Q|^2-|\mathbf{P}|^2, \tag{62}$$

which is the expression given in reference 4.

3. CALCULATION OF EXCHANGE MULTIPOLE MOMENTS

As in the case of the nuclear forces the multipole moments of a nuclear system due to exchange processes may be calculated either by using the ordinary method of quantum theory or by adopting the method of canonical transformation in reference 4. Calculations of the multipole moments by the former method have been given in reference 3. We discuss below the method of canonical transformation and also calculate the multipole moments in the pseudoscalar theory.

Let us take as an example the exchange magnetic moment of the deuteron in the vector theory. The static parts of the field variables are given by

$$\begin{aligned}
V^0(\mathbf{r}')&=\int N^*(\mathbf{r})\varphi(\mathbf{r}-\mathbf{r}')d\mathbf{r},\\
\mathbf{U}^0(\mathbf{r}')&=-\int\nabla\times\mathbf{S}^*(\mathbf{r})\varphi(\mathbf{r}-\mathbf{r}')d\mathbf{r},\\
\mathbf{F}^0(\mathbf{r}')&=-\int\nabla N^*(\mathbf{r})\varphi(\mathbf{r}-\mathbf{r}')d\mathbf{r},\\
\mathbf{G}^0(\mathbf{r}')&=\kappa^2\int\mathbf{S}^*(\mathbf{r})\varphi(\mathbf{r}-\mathbf{r}')d\mathbf{r}-\int\nabla(\nabla\cdot\mathbf{S}^*(\mathbf{r}))\varphi(\mathbf{r}-\mathbf{r}')d\mathbf{r},
\end{aligned} \tag{63}$$

together with the complex conjugates of these equations, in which

$$\varphi(\mathbf{r}-\mathbf{r}')=\frac{1}{4\pi}\frac{\exp(-\kappa|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \tag{64}$$

The exchange magnetic moment is given by

$$\boldsymbol{\mu}=\frac{1}{2c}\int \mathbf{r}\times\mathbf{j}^0 d\mathbf{r}=\frac{ie}{2c\hbar}\int\{\mathbf{r}\times\mathbf{F}^{0*}V^0-\mathbf{r}\times\mathbf{F}^0V^{0*}-\mathbf{r}\times(\mathbf{G}^{0*}\times\mathbf{U}^0)+\mathbf{r}\times(\mathbf{G}^0\times\mathbf{U}^{0*})\}d\mathbf{r}. \quad (65)$$

Take any point O as the origin. Let the vectors OP and ON from O to the proton and the neutron, respectively, be denoted by $\mathbf{R}_1$ and $\mathbf{R}_2$. We write $\mathbf{R}_1-\mathbf{R}_2=\mathbf{R}$, $\mathbf{R}_1+\mathbf{R}_2=\boldsymbol{\varrho}$, $\mathbf{r}-\mathbf{R}_1=\mathbf{r}_1$, $\mathbf{r}-\mathbf{R}_2=\mathbf{r}_2$. Then the first term in the integrand of (65) gives a contribution

$$-\int\int\int \mathbf{r}\times(\nabla_2 N_2)N_1^*\varphi(r_1)\varphi(r_2)d\mathbf{r}d\mathbf{R}_1d\mathbf{R}_2.$$

With the help of the relation

$$\int\varphi(r_1)\varphi(r_2)d\mathbf{r}=\frac{1}{8\pi\kappa}e^{-\kappa R}, \quad (66)$$

we obtain

$$-\frac{1}{16\pi\kappa}\int\int\boldsymbol{\varrho}\times(\nabla_2N_2)N_1^*e^{-\kappa R}d\mathbf{R}_1d\mathbf{R}_2=\frac{1}{16\pi}\int\int N_1^*N_2(\boldsymbol{\varrho}\times\mathbf{R})\frac{e^{-\kappa R}}{R}d\mathbf{R}_1d\mathbf{R}_2.$$

The first two terms in the integrand of (65) give therefore a contribution

$$\frac{ie}{16\pi c\hbar}\int\int N_1^*N_2(\boldsymbol{\varrho}\times\mathbf{R})\frac{e^{-\kappa R}}{R}d\mathbf{R}_1d\mathbf{R}_2=\frac{1}{2c}\frac{ie}{\hbar}\int\!\cdot\!\int\left(\frac{\boldsymbol{\varrho}}{2}\times\mathbf{R}\right)N_1^*N_2\varphi(R)d\mathbf{R}_1d\mathbf{R}_2. \quad (67)$$

It is known that (see reference 3) the total exchange current between the proton and the neutron is equal to $(ie/\hbar)J$, J being the nuclear potential between the proton and the neutron due to exchange processes. Equation (67) is therefore just one part of the magnetic moment due to the exchange current with respect to the point O. If we choose the origin at the midpoint of the two particles then $\boldsymbol{\varrho}=0$ and (67) vanishes. Henceforth we shall omit all terms containing $\boldsymbol{\varrho}$ in the final result.

The contribution of the third and fourth terms in the integrand of (65) can be similarly calculated with the help of the general formula for partial integration

$$\int u\mathbf{v}\cdot(\nabla\times\mathbf{w})d\mathbf{r}=\int(\mathbf{v}\times\mathbf{w})\cdot\nabla u d\mathbf{r}+\int u\mathbf{w}\cdot(\nabla\times\mathbf{v})d\mathbf{r}. \quad (68)$$

The final result for the exchange magnetic moment with respect to the midpoint of P and N is

$$\boldsymbol{\mu}=\frac{ie}{16\pi\mu}(\mathbf{S}_1^*\times\mathbf{S}_2)(\nabla^2+\kappa^2)e^{-\kappa R}=\frac{ieg_2^2}{8\pi\mu\kappa^2}\Pi_1^*\Pi_2(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\left(\kappa^2-\frac{\kappa}{R}\right)e^{-\kappa R}, \quad (69)$$

in agreement with the result obtained in reference 3.

In the pseudoscalar theory the exchange magnetic dipole moment is given by

$$\boldsymbol{\mu}=\frac{ie}{2\hbar c}\int\mathbf{r}\times(\psi^{0*}\boldsymbol{\Gamma}^0-\boldsymbol{\Gamma}^{0*}\psi^0)d\mathbf{r}$$

$$=\frac{ie}{2\hbar c}\int\int\int\mathbf{r}\times[(\kappa^2\mathbf{P}_2+\nabla_2\times\nabla_2\times\mathbf{P}_2)(\nabla_1\cdot\mathbf{P}_1^*)-(\kappa^2\mathbf{P}_1^*+\nabla_1\times\nabla_1\times\mathbf{P}_1^*)(\nabla_2\cdot\mathbf{P}_2)]\varphi(r_1)\varphi(r_2)d\mathbf{r}d\mathbf{R}_1d\mathbf{R}_2.$$

On evaluating the integral as above, we find

$$\boldsymbol{\mu}=\frac{ief_2^2}{8\pi\mu}\Pi_1^*\Pi_2\left[\frac{(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\mathbf{R}}{R}\frac{\mathbf{R}}{R}\left(1+\frac{1}{\kappa R}\right)-(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\right]e^{-\kappa R}. \quad (70)$$

The result is antisymmetric in the spins of the two nuclear particles. A deuteron in any stationary state has therefore no extra non-additive magnetic moment due to meson exchange in the pseudoscalar theory as well as the vector theory.

In contrast to the result in the vector theory, in the pseudoscalar theory a proton-neutron pair has no exchange electric dipole moment that reacts with an electrostatic field. There is also no electric quadripole moment that reacts with an electrostatic field.

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Tensor Forces and Heavy Nuclei

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It has been suggested that the interaction between a pair of nuclear particles contains, in addition to a central force term, a tensor term, and quantitative estimates of the relative magnitudes of these two terms have been given. In this paper a rough quantitative estimate of the relative contributions of these two terms to the binding energy of a nucleus is made on the basis of a highly simplified nuclear model. It is found that a tensor interaction between two nuclear particles of the form suggested by the neutral meson field theory of nuclear forces with constants determined to fit the experimental data on the two-particle neutron-proton system would lead to the existence of moderately light nuclei having binding energies, spins, and deviations from spherical shape much greater than those observed.

1. INTRODUCTION

THE discovery of the electric quadrupole moment of the deuteron[1] suggests the existence of a term lacking spherical symmetry in the interaction between a proton and a neutron. The current assumption of equality of forces between all pairs of nuclear particles then implies the existence of a similar term in the interaction between two like particles. General considerations of invariance[2] indicate that such a term should contain the operator

$$S_{12}=3(\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{r}/r^2)-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \quad (1)$$

Field theories of nuclear forces have been proposed[3] which lend support to this view. They predict the existence of terms containing S_{12} as a factor, but little reliance can be placed on the detailed form of the interaction given by these theories because of divergence difficulties from which they suffer.

The influence of the S_{12} tensor term on the two-particle proton-neutron system has been investigated by Rarita and Schwinger[4] and by Bethe.[5] The former develop a purely phenomenological theory adopting simplified rectangular well potentials with constants chosen to fit the binding energy and quadrupole moment of the deuteron, and the scattering of slow neutrons in hydrogen. The range of the forces is taken equal to that deduced from proton-proton scattering. Bethe employs the form of the interaction predicted by the single-meson theory of nuclear forces cutting off at small distances to avoid divergence. The value of the "cut-off" distance and the strength of the interaction is determined to fit the deuteron binding energy

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