Comparison Spaces in General Relativity

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A physical interpretation is given of the mathematical theorem that a Riemannian space can be defined by means of the system of local tangent flat spaces. This leads naturally to an elucidation of the status of Rosen's flat space in the general relativity theory. Comparison flat spaces can be chosen arbitrarily, and if taken as giving the metric of space time lead to different arbitrary values of ether drift. In particular, the flat-space tangent to the Riemannian space along the history of the observer (at the origin) is locally equivalent with the Riemannian space, and no ether drift would be involved in using it in place of the Riemannian space as a method of extrapolating measurements to great distances. It is further shown that the formal simplifications achieved by introducing the comparison metric do not depend on its flatness. A *de Sitter* type of isotropic empty space is introduced by means of which the distribution and laws of motion of matter can be expressed in terms of the differences between actual space containing matter and the empty comparison space. Comparison spaces in general are essentially ideal, and can be introduced to bring out the non-ideal characteristics of actual space.

INTRODUCTION

IT was shown by Rosen[1] that by arbitrarily introducing an Euclidean quadratic form in the coordinates side by side with the Riemannian metric of the Einstein theory, a formal simplification of important equations of general relativity could be secured.

Rosen further pointed out the possibility of abandoning the geometrical interpretation of the Riemannian metric in favor of the Euclidean form as the natural space-time metric of an observer in a gravitational field.[2] The field potential would be described in terms of the Riemannian fundamental tensor. The rays of light would be given by null geodesics in the Riemannian field, which would thus behave as a medium with a refractive index depending on the gravitational potential. An ether drag[3] caused by motion through the field is then to be expected, which circumstance was claimed to favor the flat-space point of view.

Since Rosen's work, Anderson[4] has published his measurements of the velocity of light, in which the greatest possible drag variation in six months was less than the mean daily variation. As his measurements are apparently the most precise to date, it would appear that the flat-space point of view has lost observational support in this respect.

It should perhaps be pointed out in passing that the argument would not be affected by the possible existence of secular changes in the velocity of light such as those discussed by Birge,[5] since small variations of gravitational potential in the local stellar cluster would suffice to account for these on either theory.

Certain fundamental theoretical difficulties involved in the flat-space point of view have been discussed elsewhere.[6] In the present paper we discuss the status of Rosen's flat space within the scheme of general relativity, and point out how the formal advantages of Rosen's method are independent of the flatness of the comparison space.

RIEMANNIAN SPACE AS A SYSTEM OF LOCAL SPACES

It is known[7] that a complete "curved" space can be built up from a system of local spaces by means of linear connections; also that the Riemannian space can be defined by the system of local tangent "flat" spaces.[8] From the point of view of physics, these mathematical theorems can be described as follows.

[1] N. Rosen, Phys. Rev. **57**, 147-150 (1940).
[2] N. Rosen, Phys. Rev. **57**, 150–154 (1940).
[3] N. Rosen, Phys. Rev. **57**, 154–155 (1940).
[4] W. C. Anderson, J. Opt. Soc. Am. **31**, 187–197 (1941).
[5] R. T. Birge, Nature **134**, 771 (1934).
[6] W. Band, Phys. Rev. **61**, 668 (1942).
[7] D. J. Struik, *Theory of Linear Connections* (Springer, 1934).
[8] O. Veblen, *Projective Relativitätstheorie* (Springer, 1933).

A freely falling observer finds the gravitational field locally absent, and his reference system locally flat. Let him employ unit reference vectors $\mathbf{i}^{\alpha}$, where[9]

$$\mathbf{i}^{\alpha}\cdot\mathbf{i}^{\beta}=\delta_{\alpha\beta}=0,\ \alpha\neq\beta;\quad =1,\ \alpha=\beta=0;$$
$$=-1,\ \alpha=\beta\neq 0. \quad (1)$$

Any attempts by an observer to make measurements outside of his own immediate locality are to be considered extrapolations; direct measurements are to be confined to his own neighborhood. The principle of connectivity, however, states that it is possible for any observer S to compare with his own, the unit reference vectors used nearly simultaneously by any other observer S' at a position not too far removed from S at the time of comparison.

If one now imagines a system of free observers, one to every small four-dimensional region, one can at once set up the fundamental equations of differential geometry:

$$d\mathbf{i}^{\alpha}=\Gamma^{\mu}_{\alpha\nu}dy^{\nu}\mathbf{i}^{\mu}, \quad (2)$$

where dy^{ν} are the components of the position of S' against S:

$$\mathbf{SS}'=dy^{\alpha}\mathbf{i}^{\alpha}. \quad (3)$$

Since the members of this system are all free observers, the system obtained from (2) is everywhere flat, and the linear connections must satisfy the condition

$$B^{\epsilon}_{\mu\nu\sigma}=\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\epsilon}_{\alpha\nu}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{\epsilon}_{\alpha\sigma}+\frac{\partial}{\partial x^{\nu}}\Gamma^{\epsilon}_{\mu\sigma}-\frac{\partial}{\partial x^{\sigma}}\Gamma^{\epsilon}_{\mu\nu}=0. \quad (4)$$

It is obvious that the free observers cannot in fact all be permanent members of the system, for in general two free observers will be near neighbors only momentarily. This is the physical language for the mathematical statement that the system of coordinates set up will be non-affine. In practice we do not go on re-creating new free observers for each small four-dimensional element of space-time, but we set up, in principle, a system of observers which are supported in such a way as to be permanent members of the system whose origin is set permanently on the first-chosen free observer.

It is not necessary to specify exactly the system of observers beyond the single requirement that their relative motions shall be such as to maintain mutual neighborliness! Such a system, while impracticable, is conceivable, and hence acceptable as the basis of extrapolations from the origin. The principle of connectivity applied to such a system permits the relations

$$d\mathbf{e}^{\alpha}=\left\{{\mu \atop \alpha\nu}\right\}dx^{\nu}\mathbf{e}^{\mu}, \quad (5)$$

where $\mathbf{e}^{\mu}$ are the unit vectors for any supported observer, and dx are the components of the position of S against S':

$$\mathbf{SS}'=dx^{\alpha}\mathbf{e}^{\alpha}. \quad (6)$$

By the manner of its definition, the system is affine, so that

$$\left\{{\mu \atop \alpha\nu}\right\}=\left\{{\mu \atop \nu\alpha}\right\}, \quad (7)$$

but not in general flat because of the "supports" and local presence of gravitational fields.

It will now be conceivable for any one of the supported observers to compare his unit reference vectors with those of the appropriate member of the system of free observers. The relative acceleration between them will be equivalent to a linear transformation, and we shall expect the relation:

$$\mathbf{e}^{\mu}=\phi^{\alpha}{}_{\mu}\mathbf{i}^{\alpha}, \quad (8)$$

where the dyadic components $\phi^{\alpha}{}_{\mu}$ can be found in every small four-dimensional region. It will be assumed possible to choose the system of free observers in such a way that $\phi^{\alpha}{}_{\mu}$ vary continuously from region to region everywhere, and therefore are well-behaved functions of position in the affine x space of the supported observers.

Comparing (3) with (6), noting that the intervals $\mathbf{SS}'$ can be chosen identical in the two relations, we see that

$$dx^{\nu}\mathbf{e}^{\nu}=dy^{\nu}\mathbf{i}^{\nu} \quad (9)$$

and from (8)

$$dy^{\mu}=\phi^{\mu}{}_{\nu}dx^{\nu}. \quad (10)$$

This constitutes a linear transformation between the non-affine flat y space and the affine curved x space.

The metrics of the two spaces are given by their unit reference vectors. Thus the quadratic intervals are, respectively,

$$ds^2=\delta_{\mu\nu}dy^{\mu}dy^{\nu},\ \text{where}\ \mathbf{ds}=dy^{\nu}\mathbf{i}^{\nu}, \quad (11)$$

and

$$ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu},\ \text{where}\ \mathbf{ds}=dx^{\mu}\mathbf{e}^{\mu}, \quad (12)$$

and

$$g_{\mu\nu}=\mathbf{e}^{\mu}\cdot\mathbf{e}^{\nu}=\delta_{\alpha\beta}\phi^{\alpha}{}_{\mu}\phi^{\beta}{}_{\nu}. \quad (13)$$

[9] W. Band, Am. J. Phys. 8, 162–164 (1940).

It should be remarked that the above transformation relations between flat and curved metrics are possible only because the flat metric is non-affine; the Eqs. (10), for example, are not integrable. It is of interest also to note that the relations (13) signify that whereas the metric of the curved space is determined by the transformation (8), the converse is not true; there are many possible systems of free observers which would lead to the same curved metric with a given single free observer as permanent origin.

COMPARISON FLAT SPACES

If Eqs. (10) are used in the linear connection (2) we obtain

$$d\mathbf{i}^{\alpha}=\Lambda^{\mu}_{\alpha\nu}dx^{\nu}\mathbf{i}^{\mu}, \tag{14}$$

where

$$\Lambda^{\mu}_{\alpha\nu}=\phi^{\sigma}_{\nu}\Gamma^{\mu}_{\alpha\sigma}. \tag{15}$$

Now the relations (14) constitute an alternative linear connection in x space for comparison with (5); since x space is already affine the coefficients of (14) must satisfy

$$\Lambda^{\mu}_{\alpha\nu}=\Lambda^{\mu}_{\nu\alpha}.$$

Moreover, since the reference system $\mathbf{i}^{\alpha}$ is everywhere flat

$$\Lambda^{\alpha}_{\mu\sigma}\Lambda^{\epsilon}_{\alpha\nu}-\Lambda^{\alpha}_{\mu\nu}\Lambda^{\epsilon}_{\alpha\sigma}+\frac{\partial}{\partial x^{\nu}}\Lambda^{\epsilon}_{\mu\sigma}-\frac{\partial}{\partial x^{\sigma}}\Lambda^{\epsilon}_{\mu\nu}=0. \tag{16}$$

These last two equations are not inconsistent with (4) nor with the non-affine condition in y space, namely

$$\Gamma^{\mu}_{\alpha\nu}\neq\Gamma^{\mu}_{\nu\alpha}. \tag{17}$$

The differences between the linear connection coefficients in the two spaces can be expressed in terms of the components ϕ^{α}_{μ}. Thus, let $\phi^{*\mu}_{\alpha}$ be the components of the reciprocal conjugate dyadic, so that (8) solves in the form:

$$\mathbf{i}^{\alpha}=\phi^{*\mu}_{\alpha}\mathbf{e}^{\mu}. \tag{18}$$

Then it is easily shown that

$$d\mathbf{i}^{\alpha}=(\phi^{*\mu}_{\alpha;\gamma}+\{{}^{\sigma}_{\alpha\gamma}\}\phi^{*\mu}_{\sigma})\mathbf{e}^{\mu}dx^{\gamma}, \tag{19}$$

where $\phi^{*\mu}_{\alpha;\gamma}$ is the set of intrinsic { }-derivatives of $\phi^{*\mu}_{\alpha}$. By comparison of (19) with (14) and by use of (8) on the right of (19) we obtain

$$\Delta^{\alpha}_{\mu\nu}\equiv\{{}^{\alpha}_{\mu\nu}\}-\Lambda^{\alpha}_{\mu\nu}=\phi^{*\sigma}_{\mu;\nu}\phi^{\alpha}_{\sigma}. \tag{20}$$

The analogous reasoning starting directly from (8) instead of (18) leads to the alternative expression

$$\Delta^{\alpha}_{\mu\nu}=\phi^{\sigma}_{\mu,\nu}\phi^{*\alpha}_{\sigma}, \tag{21}$$

where $\phi^{\sigma}_{\mu,\nu}$ are the intrinsic Λ derivatives of ϕ^{σ}_{μ}.

The invariant interval associated with (14) is evidently

$$d\boldsymbol{\sigma}=dx^{\mu}\mathbf{i}^{\mu}, \tag{22}$$

which will be called the comparison interval with respect to the true interval ds given by (12). To permit transformations to different flat systems a more general notation than (11) is desirable for the comparison metric, which will, therefore, be written in the form

$$d\sigma^2=\gamma_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{23}$$

Here

$$\gamma_{\mu\nu}=\mathbf{i}^{\mu}\cdot\mathbf{i}^{\nu}, \tag{24}$$

and (13) is replaced by

$$g_{\mu\nu}=\gamma_{\alpha\beta}\phi^{\alpha}{}_{\mu}\phi^{\beta}{}_{\nu}. \tag{25}$$

Transformations of coordinates will leave invariant both $d\sigma$ and ds, while $\phi^{\alpha}{}_{\mu}$ will transform as tensor components.

Given a particular Riemannian metric, the choice of a flat comparison metric is still undetermined. For example, we may choose $\gamma_{\mu\nu}$ equal to $g_{\mu\nu}$ at the origin, so that the flat space is tangent to the Riemannian space at the origin. This would require, at the origin

$$\phi^{\alpha}{}_{\beta}=\delta^{\alpha}{}_{\beta}, \tag{26}$$

where $\delta^{\alpha}{}_{\beta}$ is the ordinary δ symbol [not the same components as those introduced in (1) and used in (13)]. With this tangent comparison space we derive from (20) and (21)

$$\Delta^{\alpha}_{\mu\nu}=\phi^{*\alpha}_{\mu;\nu}=\phi^{\alpha}_{\mu,\nu}. \tag{27}$$

There will be no distinction between the two spaces in the neighborhood of the observer at the origin, and they will merely represent two different possible methods of extrapolating measurements at great distances.

If on the other hand the Riemannian space tends towards flatness at great distances, it is equally possible to set up a flat comparison space which is tangent to the Riemannian space at infinity. This would give the flat space suggested by Rosen,[1] and there will be a definite distinction between the two spaces in the neighborhood of the observer at the origin. There is no necessity to make any particular choice, so far as formal convenience is concerned. In fact Rosen's choice would become impossible were the Riemannian space not tending towards flatness at infinity, and we should have to be satisfied with tangency at any arbitrarily chosen point.

Since the Riemannian space and the comparison space are based on the same coordinate system, transformations of coordinates take place in both systems simultaneously and their common point of tangency is invariant. The set $\phi^{\alpha}{}_{\mu}$ transforms as a tensor, and hence, through (21) the differences between the two linear connection coefficients also transform as a tensor. The formal simplifications obtained by the use of the Rosen flat space are consequences of any of the choices of flat comparison spaces here discussed, and do not depend upon the point of tangency between the two spaces.

EQUATIONS OF MOTION

Since the gravitational field is absent in the local flat space, the observer's own track will be a straight line in terms of the local space:

$$0=d^2\mathbf{s}=d(dy^{\mu}\mathbf{i}^{\mu})=\mathbf{i}^{\alpha}(d^2y^{\alpha}+\Gamma^{\alpha}_{\mu\nu}dy^{\mu}dy^{\nu}). \quad (28)$$

In terms of the comparison space, the equation of a straight line would be rather

$$0=d^2\boldsymbol{\sigma}=d(dx^{\mu}\mathbf{i}^{\mu})=\mathbf{i}^{\alpha}(d^2x^{\alpha}+\Lambda^{\alpha}_{\mu\nu}dx^{\mu}dx^{\nu}). \quad (29)$$

The actual equations of motion in terms of the x coordinates are obtained by going over from (28) to the Riemannian space:

$$0=d^2\mathbf{s}=d(dx^{\mu}\mathbf{e}^{\mu})=\mathbf{e}^{\alpha}(d^2x^{\alpha}+\{^{\alpha}_{\mu\nu}\}dx^{\mu}dx^{\nu}), \quad (30)$$

which, by (20), can be written as

$$d^2x^{\alpha}+\Lambda^{\alpha}_{\mu\nu}dx^{\mu}dx^{\nu}=-\Delta^{\alpha}_{\mu\nu}dx^{\mu}dx^{\nu}. \quad (31)$$

Comparison between (29) and (31) shows that, if the observer makes use of the comparison flat space, the true equations of motion will appear as if a gravitational field were present disturbing the motion. The force depends only upon the connection differences; but since the actual choice of comparison space is not prescribed, the force is actually arbitrary. If the comparison space is chosen tangent to the Riemannian space along the history of the observer at the origin, then the connection differences, and hence the apparent force, will permanently vanish in the immediate neighborhood of the origin. If the comparison space is tangent to the Riemannian space at infinity, the apparent force will approximate the force of Newtonian theory. But none of these choices is compulsory, and no preference should be given to any special value of the force in a claim to "reality."

NON-EXISTENCE OF AN ETHER DRAG

Consider two momentarily coincident planetary observers with uniform relative velocity. Since their local flat spaces are tangent with their respective curved systems in their own (common) neighborhood, the transformation between their curved spaces will be the Lorentz transformation between their local flat spaces. The coefficients of the quadratic forms are functions only of the gravitational potentials and these are the same for the two observers. The Lorentz transformation thus leaves unchanged the coefficients of the quadratic form.[10] If we take

$$ds^2=c^2(1+2\phi/c^2)dt^2-(1-2\phi/c^2)dr^2, \quad (32)$$

the transformation will be approximately

$$dx=k'(dx'+vdt'), \quad dt=k'(dt'+vdx'/c'^2), \quad (33)$$

where

$$k'=(1-v^2/c'^2)^{-\frac{1}{2}}, \quad \text{and} \quad c'=c(1+2\phi/c^2). \quad (34)$$

In terms of x', t', we have, of course,

$$ds^2=c^2(1+2\phi/c^2)dt'^2-(1-2\phi/c^2)dr'^2. \quad (35)$$

The velocity of light on the two systems is the same, and no "ether drag" can be expected.

If we use the Rosen comparison space we may write

$$d\sigma^2=c^2dt^2-dr^2, \quad (36)$$

where c is the velocity of light in regions remote from matter, or at infinity in the Riemannian space. If we arbitrarily assume that the correct transformation between the two systems leaves

[10] The writer wishes to thank Dr. Rosen for a discussion of this point.

the coefficients of (36) unchanged, then we shall have to use the ordinary Lorentz transformations like (33) and (34) with the unprimed symbols. The application of this to (32) leads to

$$ds^2=c^2dt'^2(1+2\phi'/c^2)+8\phi k^2 v dx'dt' -dx'^2(1-2\phi'/c^2), \quad (37)$$

where

$$\phi'=\phi(1+v^2/c^2)k^2. \quad (38)$$

The velocity of light, given by $ds=0$ in (37) is easily shown to be, when terms in $(v/c)^2$ are neglected,

$$c(1+2\phi/c^2)(1+4\phi v/c^3). \quad (39)$$

This gives the ether drag coefficient $-4\phi v/c^3$ as in Rosen's work. The result is the same as if we regarded the gravitational field relative to the metric of (36) as a medium of refractive index $(1+2\phi/c^2)$.

If instead of Rosen's particular comparison space we were to choose some other flat space, and arbitrarily assume again that the coefficients of the quadratic form shall remain unchanged by transformations between the two observers, we should of course derive different expressions for the gravitational potential and the ether drag coefficient. Since a change in the velocity of light is a question of fact rather than of mere convenience, we cannot claim that the different assumptions are equivalent. From the general relativity point of view which we are here developing, the error is in assuming that the correct transformations will leave unchanged the coefficients in the flat quadratic forms. There is only one correct transformation, namely (33) and (34), whatever comparison space we choose; and it is the coefficients of the flat quadratic forms which must change, not those of (32).

Stated rather differently, the fundamental point at issue as between the present point of view and the flat point of view suggested by Rosen, is this; whether we accept as absolute constant the actual velocity of light at the point of interest, or whether only its value at some arbitrarily assigned point—say at infinity—is to remain constant during transformations at the point of interest; whether the actual transformation between two observers at a given point is determined by conditions at the point or by conditions at infinity, or some other arbitrarily chosen point.

ISOTROPIC COMPARISON SPACES

Let us introduce a comparison metric

$$d\sigma^2=\gamma_{\mu\nu}dx^\mu dx^\nu \quad (40)$$

in the same coordinates as the Riemannian space, where now however the tensor $\gamma_{\mu\nu}$ does not satisfy the conditions for flatness. Also write $\Lambda^{\alpha}_{\mu\nu}$ for the linear connection or Christoffel symbols in the γ's; but do not require them to satisfy (16). Retain the notation

$$\Delta^{\alpha}_{\mu\nu}=\{^{\alpha}_{\mu\nu}\}-\Lambda^{\alpha}_{\mu\nu} \quad (41)$$

for the differences between the Christoffel symbols. Then it is easy to prove that when $H_{\mu\nu}$ is any tensor

$$H_{\mu\nu;\sigma}=H_{\mu\nu,\sigma}-\Delta^{\alpha}_{\mu\sigma}H_{\alpha\nu}-\Delta^{\alpha}_{\nu\sigma}H_{\mu\alpha}, \quad (42)$$

where again ; σ means intrinsic g differentiation, and , σ means intrinsic γ differentiation. In particular

$$\Delta^{\sigma}_{\mu\nu}=\tfrac{1}{2}g^{\alpha\sigma}(g_{\mu\alpha,\nu}+g_{\nu\alpha,\mu}-g_{\mu\nu,\alpha}) \quad (43)$$

and in general, as in Rosen's work, γ differentiation can be substituted for ordinary differentiation, and $\Delta^{\alpha}_{\mu\nu}$ for $\{^{\alpha}_{\mu\nu}\}$ in any first-order differential tensor equation originally expressed in terms of the Riemannian space. The flatness of the comparison metric is not a condition for this result. Proceeding to second-order differential equations, we find that the difference between the two expressions is given by the above substitution. Thus if $R_{\mu\nu}$ and $P_{\mu\nu}$ are the contracted Riemann-Christoffel tensors in the original and comparison spaces, respectively, their difference is given simply by

$$R_{\mu\nu}-P_{\mu\nu}=\Delta^{\alpha}_{\alpha\mu,\nu}-\Delta^{\alpha}_{\mu\nu,\alpha}+\Delta^{\alpha}_{\beta\mu}\Delta^{\beta}_{\alpha\nu}-\Delta^{\alpha}_{\alpha\beta}\Delta^{\beta}_{\mu\nu}. \quad (44)$$

This is a direct corollary of a theorem given by Levi-Civita.[11]

Referring now to Rosen's first paper,[1] we may write in place of his Eq. (3) the corresponding relation in a *de Sitter* type of world[12]

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(R-2\lambda). \quad (45)$$

If for the comparison metric we adopt the *de Sitter* metric for an empty world, the corresponding expression in this space is zero. Hence, $G_{\mu\nu}$ is

[11] T. Levi-Civita, *Absolute Differential Calculus* (Blackie, 1927), Chapter 8, §3.

[12] A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge, 1924), Chapter 4, §54 (54–71).

also the difference between the corresponding quantities, and the substitutions used by Rosen in his §3 Eq. (11) can be carried over into the present argument. The introduction of the *de Sitter* comparison space instead of a flat space makes no essential difference to the argument.

We can also prove that, instead of (43), we could write

$$\overset{\sigma}{\Delta}_{\mu\nu}=\tfrac{1}{2}\gamma^{\alpha\sigma}(\gamma_{\mu\alpha;\nu}+\gamma_{\nu\alpha;\mu}-\gamma_{\mu\nu;\alpha}) \tag{46}$$

and in consequence obtain in place of (44) the alternative form

$$R_{\mu\nu}-P_{\mu\nu}=\overset{\alpha}{\Delta}_{\alpha\mu;\nu}-\overset{\alpha}{\Delta}_{\mu\nu;\alpha}-\overset{\alpha}{\Delta}_{\beta\mu}\overset{\beta}{\Delta}_{\alpha\nu}+\overset{\alpha}{\Delta}_{\alpha\beta}\overset{\beta}{\Delta}_{\mu\nu}. \tag{47}$$

Adding (44) and (47) we obtain

$$R_{\mu\nu}-P_{\mu\nu}=\tfrac{1}{2}(\overset{\alpha}{\Delta}_{\alpha\mu;\nu}+\overset{\alpha}{\Delta}_{\alpha\mu,\nu}-\overset{\alpha}{\Delta}_{\mu\nu;\alpha}-\overset{\alpha}{\Delta}_{\mu\nu,\alpha}). \tag{48}$$

This can be expressed approximately in terms of the differences between the two metric tensors. Thus if we write

$$h_{\mu\nu}=g_{\mu\nu}-\gamma_{\mu\nu}, \tag{49}$$

we can express (43) and (46) in the forms

$$\overset{\sigma}{\Delta}_{\mu\nu}=\tfrac{1}{2}g^{\alpha\sigma}(h_{\mu\alpha,\nu}+h_{\nu\alpha,\mu}-h_{\mu\nu,\alpha})$$
$$=\tfrac{1}{2}\gamma^{\alpha\sigma}(h_{\mu\alpha;\nu}+h_{\nu\alpha;\mu}-h_{\mu\nu;\alpha}). \tag{50}$$

Using the first of these in the g derivatives and the second in the γ derivatives in (48), we are led to an expression which, on neglect of products of h terms, becomes approximately

$$R_{\mu\nu}-P_{\mu\nu}$$
$$=\tfrac{1}{2}\gamma^{\alpha\beta}\left(\frac{\partial^2 h_{\alpha\beta}}{\partial x^\mu\partial x^\nu}+\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha\partial x^\beta}-\frac{\partial^2 h_{\mu\beta}}{\partial x^\nu\partial x^\alpha}-\frac{\partial^2 h_{\nu\alpha}}{\partial x^\mu\partial x^\beta}\right). \tag{51}$$

At this point we connect with the reasoning given by Eddington, reference (11) §46. Our Eq. (51) is essentially the analogue of Eddington's (46–3) which refers to a space which is flat when empty. Our $h_{\mu\nu}$ are thus equivalent with his $g_{\mu\nu}$ and the differences $R_{\mu\nu}-P_{\mu\nu}$ take the place of the contracted $R-C$ tensor in Eddington's equation. Following his argument, we therefore find that the differences are given by the density of a continuous distribution of matter.

In general we may summarize this reasoning by asserting that we may choose any kind of space whatever for comparison with the actual space. In particular, we have chosen to compare actual space regarded as containing a static distribution of matter with an ideally empty isotropic space of the *de Sitter* variety; we have found that the distribution and laws of motion of matter can be expressed in terms of the differences between the two spaces. In a similar manner, we might choose to compare an actual space containing an admittedly non-static distribution of matter with an ideal space containing the same mean density of matter in a static distribution. Different comparisons may be convenient for different purposes, and the tensor analysis is capable of handling any of them. Merely because in this way we achieve some kind of formal simplification of the analysis is not a valid reason for regarding any one comparison space as in any sense "actual"; they are essentially ideal, and are introduced merely to bring out the non-ideal characteristics of the actual space.