Theory of the Continuous X-Ray Spectrum

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The theory of continuous x-radiation from thin targets—relativity and retardation of potential neglected—is completed by the incoherent integration of the single process transition probabilities over the unit sphere. This is carried out by means of series expansion. Comparison with the absolute intensity determination of Smick-Kirkpatrick gives a result too large by a factor of greater than 2.5. The theory compares satisfactorily with the absolute measurement of Clark-Kelly and the relative measurements of Harworth-Kirkpatrick.

I. INTRODUCTION

THE theory of the continuous x-ray spectrum —relativity and retardation of potential neglected—was worked out by Sommerfeld[1] in 1931. It is Sommerfeld's work upon which the present paper is directly based. One computes the probability that an electron moving toward an atom with velocity v_1 be deflected through a given angle θ with asymptotic velocity v_2, having experienced an energy loss $h\nu$. The dipole matrix elements of the single process have been rigorously computed by Sommerfeld, but the incoherent integration over all polar angles (θ, φ) of scattering of the incident electron has not in general been calculated in closed form.

Sauter[2] has given a non-relativistic treatment which makes use of the Born approximation, $Ze^2/\hbar v_{1,2} \ll 1$, for incident and emergent electron. Unfortunately, this condition is not met in a large number of experiments. When it is satisfied, on the other hand, the electron velocities are comparable with the velocity of light, so that the problem then requires a detailed relativistic treatment.

Elwert[3] has worked out an ingenious method for integration over all θ, φ of Sommerfeld's result, breaking off his series expansion under the assumption $(Ze^2/\hbar v_2) - (Ze^2/\hbar v_1) \ll 1$.

The present author employs a straightforward method whose accuracy is curtailed only by the physical limitations of the original Sommerfeld theory—in the main, the assumption of a pure Coulomb field as seen by the electron during the emission process. This is only strictly justified for an electron whose de Broglie wave-length is small compared with the radius of the K shell of the scattering atom. For $Z=28$ and $V=15$ kilovolts, for example, these lengths are of the same order of magnitude, so that at best the assumption of a pure Coulomb field is only approximate—the higher the voltage and the smaller Z, the better the approximation. The extreme difficulty associated with a more rigorous treatment of the field, plus the success in assuming a pure Coulomb field in numerous other problems (e.g., characteristic x-radiation, form of the Compton line from atomic electrons, etc.), justifies its assumption here. In addition, there is the question of what to assume for the value of Z, even once a Coulomb field is assumed. The procedure herein is to assign to Z the full atomic number of the scatterer. At worst, this may be an error of two parts in Z, since in general the electron wave-length is small compared with the L-shell radius of the scatterer. In the case of 15 kilovolt electrons incident upon nickel $(Z=28)$ and $\lambda=1.431$A, computations employing $Z=28$ and $Z=26$, the extreme values of effective nuclear charge, have yielded absolute intensity results differing by 15 percent. It turns out, however, that the discrepancy between experiment and theory in this single case is too great to be accounted for by merely an error in the choice of Z.

[1] A. Sommerfeld, Ann. d. Physik **11**, 257 (1931).
[2] F. Sauter, Ann. d. Physik **18**, 486 (1933).
[3] G. Elwert, Ann. d. Physik **34**, 178 (1939). The present work was undertaken before the author had complete knowledge of the significance or justification of Elwert's results. The latter's work is indeed elegant and provides a more rapidly convergent expansion in most cases; numerical work therefrom, however, requires computation of the Γ-function and its logarithmic derivative for imaginary argument. (There are tables of $\Gamma(z)$ for imaginary z for $|z| \leqslant 1$.) Also, Elwert computes merely $\mathfrak{M}_y^2 (= \mathfrak{M}_x^2)$, and uses the Sommerfeld-Maue result (Section II.D) for $\mathfrak{M}^2$ to determine $\mathfrak{M}_x^2 = M^2 - 2\mathfrak{M}_y^2$. He thereby loses the advantage of having the numerical check set forth in Section II.D.

II. ANALYTICAL TREATMENT

Sommerfeld finds for the elementary dipole moments (unnormalized) for a single electron incident in the x direction upon a bare nucleus of charge $+Ze$:

$$M_x = -A\{[1+n_1-(1+n_2)\cos\theta]F - 2\sin^2\tfrac{1}{2}\theta(1-\xi)F'\}, \tag{1}$$

$$\left.\begin{matrix} M_y \\ M_z \end{matrix}\right\} = A\sin\theta\left\{\begin{matrix} \cos\varphi \\ \sin\varphi \end{matrix}\right\}\{(1+n_2)F-(1-\xi)F'\}. \tag{2}$$

Here $F=F(1+n_1, 1+n_2, 1, \xi)$, the hypergeometric function of indicated parameters, where

$$F(a,b,c,\xi)=\sum_{\nu=0}^{\infty}\frac{\Gamma(a+\nu)\Gamma(b+\nu)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+\nu)\nu!}\xi^{\nu}, \quad \text{for } |\xi|<1; \tag{3}$$

$$\xi=-\frac{4n_1n_2}{(n_1-n_2)^2}\sin^2\frac{\theta}{2}; \quad n_{1,2}=\frac{Z}{ik_{1,2}a}=\frac{\alpha Z}{i\beta_{1,2}}; \quad \begin{cases} \hbar k_{1,2}=mv_{1,2}; & \beta_{1,2}=v_{1,2}/c \\ a=\hbar^2/me^2; & \alpha=e^2/\hbar c \end{cases}$$

$$A=-\frac{16\pi k_1k_2\exp -i\pi n_1}{(k_1+k_2)^2(k_1-k_2)^4}\left(\frac{k_1+k_2}{k_1-k_2}\right)^{n_1+n_2}.$$

The unnormalized eigenfunctions used in the calculation of the moments are

$$\psi_1=\exp(ik_1x)L_{n_1}(ik_1(r-x)), \quad \psi_2=\exp(ik_2x')L_{-n_2}(-ik_2(r+x')),$$

where

$$L_n(u)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\binom{n}{\nu}\frac{u^{\nu}}{\nu!},$$

and x' is the coordinate measured along the asymptotic direction of the emerging electron.

In the event that $\xi_0=-[4n_1n_2/(n_1-n_2)^2]<-1$, it is not permissible to employ the above (3) series development of F, which is valid only within the unit circle. We therefore use the analytic continuation:[5]

$$F(a,b,c,\xi)=(1-\xi)^{-a}F[a,c-b,c,\xi/(\xi-1)]. \tag{4}$$

Expansion of the second factor on the right in powers of $w=\xi/(\xi-1)$ according to (3) gives a series convergent for $-\infty<\xi<\frac{1}{2}$—in particular for all non-positive values of ξ.

A. Calculation of $\mathfrak{M}_x^2=\int_0^{2\pi}d\varphi\int_0^{\pi}|M_x|^2\sin\theta d\theta$

From the relation $\xi=\xi_0\sin^2\frac{1}{2}\theta$, (1) becomes

$$\begin{aligned} M_x &= -(A/\xi_0)\{\xi_0(n_1-n_2)F+2\xi[(1+n_2)F-(1-\xi)F']\} \\ &= (2n_1A/\xi_0)\{BF-F_1\}, \end{aligned}$$

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[4] A. Sommerfeld and A. W. Maue, Ann. d. Physik **23**, 589 (1935). In this paper, the authors introduce wave functions which differ slightly from those used in the 1931 work. The results (1) and (2) above, somewhat different from the expressions derived in 1931, are merely stated in the later paper. The physical results are of course unchanged, the reason for the altered procedure being one of better visualization.

[5] This is readily verified by substitution in the differential equation (Whittaker and Watson, *Modern Analysis*, fourth edition, p. 283) for $F(a,b,c,\xi)$, which the right-hand member of (4) is seen to satisfy. One then notes that the two members behave identically in the neighborhood of $\xi=0$. Equation (4) is actually a mere expression of the identity (p. 207, reference 5):

$$(1-\xi)^{a}P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & \xi \\ 1-c & b & c-a-b & \end{matrix}\right\}=P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & \xi/(\xi-1) \\ 1-c & b-a & c-b & \end{matrix}\right\},$$

so long as the indentical behavior about $\xi=0$ (or any other point) is established.

where $F_1=F(n_1, 1+n_2, 1, \xi)$, $B=(n_1+n_2)/(n_1-n_2)$, as one obtains directly through use of the easily verified relations:

$$F'(a, b, c, \xi)=(ab/c)F(a+1, b+1, c+1, \xi), \tag{5}$$

$$cF(a, b, c, \xi)-a(1-\xi)F(a+1, b+1, c+1, \xi)=(c-a)F(a, b+1, c+1, \xi), \tag{6}$$

$$(b\xi/c)F(a, b+1, c+1, \xi)=F(a, b, c, \xi)-F(a-1, b, c, \xi),$$

plus the fact that $\xi_0=-4n_1n_2/(n_1-n_2)^2$;

$$\therefore \quad |M_x|^2=\left|\frac{2n_1A}{\xi_0}\right|^2\{B^2|F|^2+|F_1|^2-2B\Re FF_1^*\}, \qquad \text{since } B \text{ is real.}$$

According to (4),

$$\left.\begin{aligned} F&=(1-\xi)^{-1-n_1}G_1; \quad G_1=F(1+n_1, -n_2, 1, w)\\ F_1&=(1-\xi)^{-n_1}G_2; \quad G_2=F(n_1, -n_2, 1, w)\end{aligned}\right\} w=\frac{\xi}{\xi-1}.$$

Since n_1 is pure imaginary and $(1-\xi)^{-1}=(1-w)$ is real,

$$|M_x|^2=\left|\frac{2n_1A}{\xi_0}\right|^2\{B^2(1-w)^2|G_1|^2+|G_2|^2-2B(1-w)\Re G_1G_2^*\}.$$

Also, since $\sin^2\frac{1}{2}\theta=\xi/\xi_0$,

$$2\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta d\theta=\sin\theta d\theta=\frac{2d\xi}{\xi_0}=-\frac{2}{\xi_0}\frac{dw}{(1-w)^2}.$$

$$\therefore \quad \mathfrak{M}_x^2=-\frac{4\pi}{\xi_0}\left|\frac{2n_1A}{\xi_0}\right|^2\left\{B^2\int_0^{w_0}|G_1|^2dw+\int_0^{w_0}\frac{|G_2|^2}{(1-w)^2}dw-2B\Re\int_0^{w_0}\frac{G_1G_2^*}{1-w}dw\right\}; \quad w_0=\frac{\xi_0}{\xi_0-1}.$$

Since $0<w_0<1$, it is permissible to expand G_1 and G_2 in powers of w. As a result, upon term by term integration:

$$\mathfrak{M}_x^2=-\frac{4\pi}{\xi_0}\left|\frac{2n_1A}{\xi_0}\right|^2\left\{B^2\sum_{r=0}^{\infty}\frac{s_rw_0^{r+1}}{r+1}-\xi_0+\sum_{r=1}^{\infty}t_r[(1-\xi_0)w_0^r-rI_{r-1}]-2B\sum_{r=0}^{\infty}I_r\Re u_r\right\}; \tag{7}$$

$$s_r=\sum_{\nu=0}^{r}g_{1\nu}g^*_{1,r-\nu}, \quad t_r=\sum_{\nu=0}^{r}g_{2\nu}g^*_{2,r-\nu}, \quad u_r=\sum_{\nu=0}^{r}g_{1\nu}g^*_{2,r-\nu},$$

where we have written $G_\sigma=\sum_{\nu=0}^{\infty}g_{\sigma\nu}w^\nu$, and where, according to (3):

$$g_{1\nu}=\frac{\Gamma(\nu+1+n_1)\Gamma(\nu-n_2)}{\Gamma(1+n_1)\Gamma(-n_2)\nu!\nu!}, \quad g_{2\nu}=\frac{\Gamma(\nu+n_1)\Gamma(\nu-n_2)}{\Gamma(n_1)\Gamma(-n_2)\nu!\nu!};$$

$$I_r=\log(1-\xi_0)-\sum_{k=1}^{r}\binom{r}{k}\frac{(w_0-1)^k-(-1)^k}{k}. \tag{8}$$

B. Calculation of $\mathfrak{M}_y^2=\mathfrak{M}_z^2=\int_0^{2\pi}d\varphi\int_0^{\pi}|M_y|^2\sin\theta d\theta$

By application of (5), (6), and (4) to (2) above, we obtain

$$M_y=-An_1(1+n_2)\cos\varphi\sin\theta(1-w)^{1+n_1}F(1+n_1, -n_2, 2, w),$$

* *Note added in proof:* P. C. Rosenbloom has pointed out that $I_r=\int_0^{w_0}\frac{w^r}{1-w}dw$ may be evaluated more simply in the form $I_r=\log(1-\xi_0)-\sum_{\nu=1}^{r}\frac{w_0^\nu}{\nu}$.

whence

$$|M_y|^2=(4/\xi_0^2)\,|An_1(1+n_2)|^2\cos^2\varphi[\xi_0 w(w-1)-w^2]|G_3|^2,$$

where

$$G_3=F(1+n_1,\ -n_2,\ 2,\ w).$$

$$\therefore\quad \mathfrak{M}_y^2=\mathfrak{M}_z^2=\frac{8\pi}{\xi_0^3}|An_1(1+n_2)|^2\int_0^{w_0}\left\{\frac{\xi_0 w}{1-w}+\frac{w^2}{(1-w)^2}\right\}|G_3|^2dw$$

$$=\frac{8\pi}{\xi_0^3}|An_1(1+n_2)|^2\sum_{r=0}^{\infty}p_r\{(\xi_0-r-2)I_{r+1}+(1-\xi_0)w_0{}^{r+2}\},\tag{9}$$

as one finds on expansion of G_3 and term by term integration. Here,

$$p_r=\sum_{\nu=0}^{r}g_{3\nu}g^*_{3,r-\nu},\quad\text{and}\quad g_{3\nu}=\frac{\Gamma(\nu+1+n_1)\Gamma(\nu-n_2)}{\Gamma(1+n_1)\Gamma(-n_2)\nu!(\nu+1)!},$$

while I_r is the quantity defined in (8) above.

C. Normalization

The question of normalization is treated in detail by Sommerfeld.[1] We may accomplish his result by means of a slightly different viewpoint, however. Namely, we employ the usual artifice of limiting the system to a finite volume Ω, sufficiently large that only the first term in the asymptotic expansions of the wave functions is needed in computing the integrals over Ω of the absolute squares. The continuous range of energy levels is approximated by a discrete set whose separation goes to zero as Ω increases without limit. This is accomplished in the usual manner of requiring periodicity with Ω as the fundamental interval of volume. As usual, Ω drops out, and the result is that which we should obtain as Ω increases to infinity. In short, $\mathfrak{M}_x^2$, $\mathfrak{M}_y^2$, $\mathfrak{M}_z^2$ must be multiplied by the following factors in order to bring the result to the units, "ergs per unit solid angle per unit frequency range per bombarding electron per atom-per-square-centimeter of target area":

(i) Normalization of the final state eigenfunction:

$$\frac{1}{\Omega}\frac{2\pi i n_2}{1-\exp(-2\pi i n_2)}.$$

(ii) Reduction of bombarding current to one electron per second per square centimeter incident upon one atom per square centimeter:

$$\frac{2\pi i n_1}{1-\exp(-2\pi i n_1)}\frac{m}{\hbar k_1}.$$

(iii) Number of states in volume Ω in unit range of k_2:

$$\frac{\Omega}{(2\pi)^3}k_2^2.$$

(iiii) Conversion from "per unit k_2 interval" to "per unit frequency range":

$$2\pi m/\hbar k_2.$$

(iiiii) Factors arising in formula for dipole radiation:

$$\frac{e^2}{2\pi c^3}\frac{\hbar^4}{(2m)^4}(k_1^2-k_2^2)^4.$$

If we call the product of the above five factors (i to iiiii) N^2, we find the result:

$$N^2|A|^2=\frac{1}{(1-\exp(-2\pi i n_2))(\exp(2\pi i n_1)-1)}\left(\frac{Z}{a}\right)^2\frac{e^2}{8\pi c^3}\left(\frac{h}{m}\right)^2\frac{\xi_0^2}{k_1^2}.\quad\left(a=\frac{\hbar^2}{me^2}\right).$$

That is, the "components" of radiation associated with the three moments $\mathfrak{M}_x$, $\mathfrak{M}_y$, $\mathfrak{M}_z$ are:

$$I_\nu^{(x,y,z)}=\frac{1}{(1-\exp(-2\pi i n_2))(\exp(2\pi i n_1)-1)}\left(\frac{Z}{a}\right)^2\frac{e^2}{8\pi c^3}\left(\frac{h}{m}\right)^2\frac{\xi_0^2}{k_1^2}\frac{\mathfrak{M}^2_{x,y,z}}{|A|^2},$$

where, for the direction of observation making an angle Θ with the (x-) direction of bombardment, the absolute intensity is given by:[6]

$$I_\nu^{(\Theta)}=I_\nu^{(x)}\sin^2\Theta+I_\nu^{(z)}\cos^2\Theta+I_\nu^{(y)}$$

ergs per unit solid angle per unit frequency range per bombarding electron per atom-per-square-centimeter of target area.

In particular, for 90° (z direction) observation,

$$I_\nu^{(90°)}=\frac{1}{(1-\exp(-2\pi i n_2))(\exp(2\pi i n_1)-1)}\left(\frac{Z}{a}\right)^2\frac{e^2}{8\pi c^3}\left(\frac{h}{m}\right)^2\frac{\xi_0^2}{k_1^2}\frac{(\mathfrak{M}_x^2+\mathfrak{M}_y^2)}{|A|^2}.$$

D. Numerical Check

Since, in the numerical calculation, the series expansions (7) and (9) are broken off after a finite number of terms, it is desirable to have a check which permits an estimation of the error thus involved. Such a check is available through the work of Sommerfeld and Maue,[4] who have been able, by ingenious application of the differential equation for the hypergeometric function, to express $\mathfrak{M}^2=\mathfrak{M}_x^2+\mathfrak{M}_y^2+\mathfrak{M}_z^2$ in closed form. According to their work,

$$\mathfrak{M}^2=\frac{8\pi}{\xi_0}|A|^2\frac{d}{d\xi_0}|\mathfrak{F}|^2,$$

where $\mathfrak{F}=F(-n_1,\ -n_2,\ 1,\ \xi_0)$.

For $\xi_0<-1$, we use the analytic continuation of the hypergeometric function:[7]

$$F(-n_1,\ -n_2,\ 1,\ \xi_0)=\frac{\Gamma(n_1-n_2)}{n_1\Gamma(n_1)\Gamma(-n_2)}(-\xi_0)^{n_1}G_4+\frac{\Gamma(n_2-n_1)}{n_2\Gamma(n_2)\Gamma(-n_1)}(-\xi_0)^{n_2}G_5, \tag{10}$$

where

$$G_4=F(-n_1,\ -n_1,\ 1+n_2-n_1,\ \xi_0^{-1}),\quad G_5=F(-n_2,\ -n_2,\ 1+n_1-n_2,\ \xi_0^{-1}).$$

Whence, on expanding in powers of ξ_0^{-1} and performing the differentiation, we obtain:

$$\mathfrak{M}^2=-\frac{8\pi}{\xi_0}|A|^2\left\{\left|\frac{\Delta}{n_1}\right|^2\sum_{r=1}^{\infty}rq_r\xi_0^{-r-1}+\left|\frac{\Delta}{n_2}\right|^2\sum_{r=1}^{\infty}rl_r\xi_0^{-r-1}\right.$$
$$\left.-2\Re\left\{\frac{\Delta^2}{n_1n_2}(-\xi_0)^{n_1-n_2}\left[\sum_{r=1}^{\infty}rj_r\xi_0^{-r-1}-(n_1-n_2)\sum_{r=0}^{\infty}j_r\xi_0^{-r-1}\right]\right\}\right\},$$

[6] Here the y direction is taken perpendicular to both the direction of incident electron and direction of observation.

[7] Whittaker and Watson, reference 5, p. 289. Since the text suffers a typographical error, we restate the result:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,\ b,\ c,\ z)=\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}(-z)^{-a}F(a,\ 1-c+a,\ 1-b+a,\ z^{-1})+\frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)}(-z)^{-b}F(b,\ 1-c+b,\ 1-a+b,\ z^{-1}),$$

where $|\arg(-z)|<\pi$.

where

$$\Delta=\frac{\Gamma(n_1-n_2)}{\Gamma(n_1)\Gamma(-n_2)};\quad q_r=\sum_{\nu=0}^{r} g_{4\nu}g^*_{4,r-\nu};\quad l_r=\sum_{\nu=0}^{r} g_{5\nu}g^*_{5,r-\nu};\quad j_r=\sum_{\nu=0}^{r} g_{4\nu}g^*_{5,r-\nu},$$

and the $g_{\sigma\nu}$ are the expansion coefficients of the G_σ:

$$g_{4\nu}=\frac{\Gamma(\nu-n_1)\Gamma(\nu-n_1)\Gamma(1+n_2-n_1)}{\Gamma(-n_1)\Gamma(-n_1)\Gamma(\nu+1+n_2-n_1)\nu!};\quad g_{5\nu}=\frac{\Gamma(\nu-n_2)\Gamma(\nu-n_2)\Gamma(1+n_1-n_2)}{\Gamma(-n_2)\Gamma(-n_2)\Gamma(\nu+1+n_1-n_2)\nu!}.$$

For $\xi_0\ll-1$, this expansion converges quite rapidly and thereby provides a splendid check for the numerical work by comparison with the sum of the individually computed quantities $\mathfrak{M}_x^2$, $\mathfrak{M}_y^2$, $\mathfrak{M}_z^2$.

III. NUMERICAL EVALUATION

In all cases it was found sufficient to neglect all but the first nine or ten terms in each of the infinite series in order to obtain the accuracy stated below. The procedure lends itself well to compactness, so that with the aid of computing machine the time needed for each complete calculation is reasonably short. It is possible to obtain more rapid convergence if one uses for part of the interval in ξ the expansion in w (4), and for the remainder the expansion (10) in ξ^{-1} employed in the Sommerfeld-Maue check calculation of $\mathfrak{M}^2$. In this case, however, the evil of employing two different representations is by far greater than that of the somewhat slower convergence. Originally, in the $Z=28$, $\lambda=1.431$A, $V=15$ kilovolts computation of $\mathfrak{M}_y^2$, this procedure was followed; the result merely served later as a check in the calculation of $\mathfrak{M}_y^2$ as set forth in Section IIB.

For all computations, the hypergeometric coefficients were computed with the aid of the following recursion formulae:

$$g_{2,\nu+1}=\frac{\nu^2-n_1n_2+\nu(n_1-n_2)}{(\nu+1)^2}g_{2\nu};\quad (g_{\sigma 0}=1)$$

$$g_{1\nu}=\left(1+\frac{\nu}{n_1}\right)g_{2\nu};\quad g_{3\nu}=\frac{1}{\nu+1}g_{1\nu}.$$

Since only two or three terms per series were required in the calculation of $\mathfrak{M}^2$ as check, the coefficients $g_{4\nu}$ and $g_{5\nu}$ were calculated directly.

The following detailed remarks concern the computations for $Z=13$, $\lambda=0.474$A, $V=31.7$ kilovolts. The various parameters were found to be: $in_1=0.269$, $in_2=0.647$, $\xi_0=-4.87$, $w_0=0.830$. Although $(1-w_0)^2\ll 1$, it was found inadvisable to neglect $(1-w_0)^k$ for $k>1$ in the computation of $I_r(8)$; wholesale cancellations in the alternating series for I_r render these terms significant. A table of the values of I_r so obtained follows:

$r=$	0	1	2	3	4	5	6	7	8	9
$I_r=$	1.769	0.939	0.595	0.404	0.285	0.206	0.152	0.113	0.085	0.064

The results for the unnormalized moments are: $\mathfrak{M}_x^2=0.691|A|^2$, $\mathfrak{M}_y^2=\mathfrak{M}_z^2=0.059|A|^2$, and $\mathfrak{M}_x^2+\mathfrak{M}_y^2+\mathfrak{M}_z^2=0.809|A|^2$. The latter is within about two percent of the Sommerfeld-Maue check result of $\mathfrak{M}^2=0.826|A|^2$. An assumed total arithmetic error of at most four percent is, therefore, by no means an unreasonable claim of accuracy.

IV. COMPARISON WITH EXPERIMENT

In order that the foregoing theory be applied to the conditions of any given experiment, the x-radiation upon which measurement is made must originate from a thin target. The requirement is necessary for the reason that we have assumed both initial direction and energy of the electron to be known. For a target to be acceptable as "thin," therefore, the occurrence of all processes altering the direction or energy of the electron before it radiates must be negligibly frequent. A number of experiments have been performed recently with targets sufficiently thin for comparison with the theory to be valid.

Smick and Kirkpatrick,[8] bombarding a nickel target (thickness ~500A) with 15 kilovolt electrons, observed a narrow frequency range of the continuous spectrum centered about 1.431A at an angle negligibly less than 90° (~88°).

[8] E. Smick and P. Kirkpatrick, Phys. Rev. **60**, 162 (1941).

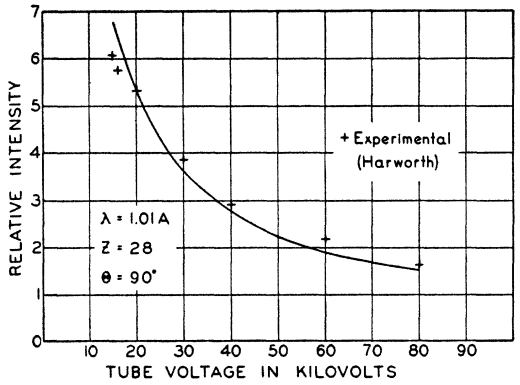

FIG. 1.

These investigators obtained the absolute intensity measurement of 2.2×10^{-50} erg per unit solid angle per unit frequency range per bombarding electron per atom-per-square-centimeter of target area. Using the full atomic number, 28, of nickel, we have, by the method given above, arrived at a much too high value of 6.09×10^{-50} in the same units. In the extreme case of assumed complete screening by the nickel K electrons ($Z=26$), the result is somewhat better, 5.27×10^{-50} erg, etc., yet still too large by a factor of greater than two. Although it does not seem likely that deviation from the Coulomb field should cause so large a discrepancy, it has been suggested the experiment be repeated with somewhat harder electrons and/or target of lower atomic number, whereupon this difficulty may drop out. At present, such a repetition is in prospect at Stanford.

The only other investigation of absolute intensity from a thin target known to the author fulfills this requirement, and here the agreement between theory and experiment is quite satisfactory. Employing 31.7-kilovolt electrons and an aluminum target, Clark and Kelly[9] have observed a narrow band of the continuous spectrum centered about 0.474A at an angle (Θ) of 60°. Their result, in the proper units, is 6.17×10^{-51} erg, etc., with a stated error of about ±33 percent. Our computations give 4.23×10^{-51} erg, etc., which does fall within the experimental error of the Clark-Kelly result. It may be significant that the agreement occurs in a region where the theory should rigorously hold, and the disagreement in a region in which the uncertainties due to deviation from the Coulomb field render its strict application somewhat hazardous.

The final comparison with experiment, and by far exhibiting the best agreement to date, is with the relative intensity measurements of Harworth and Kirkpatrick.[10] The results are given in Fig. 1. Since the measurements are merely relative, normalization is necessary; this is done by fitting the results at 20 kilovolts. This not only gives the best fit, but also finds justification in that below 20 kilovolts screening has its greatest effect; above, the relativistic effects are greater. As in the case of the Smick-Kirkpatrick measurement, the computation has been made assuming an observation angle of 90°, whereas the actual mean angle was 93.5°. The error thus involved is of course negligible. The good agreement may be in part attributed to the thinness of the target (nickel) used. It was about 200A thick.

V. ACKNOWLEDGMENT

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Finally, his thanks go to the National Youth Administration and to their patient, untiring Miss Dorothy Pearson, whose careful computation forms the largest part of the numerical results obtained.

[9] J. Clark and H. Kelly, Phys. Rev. **59**, 220 (1941). Here the result is reported as being a factor of 20 larger than predicted by the Sauter (note 2) theory. The greatness of the discrepancy was due merely to an oversight in that the experimental result was not reduced to the proper units for comparison. This fact has been discovered and corrected by Clark and Kelly themselves in a personal communication.

[10] K. Harworth and P. Kirkpatrick, Phys. Rev. **60**, 163 (1941). Among others, the results represented in the figure are to be published in a forthcoming paper by Harworth and Kirkpatrick.