# Evaluation of Functions Related to Tait's Mean Free Path*

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(Received January 24, 1942)

A new and extended table of values is given for the functions,

$$\Psi(x)=x\exp(-x^2)+(2x^2+1)\int_0^x \exp(-y^2)dy, \quad \text{and} \quad x^5/\Psi(x),$$

which occur frequently in kinetic theory. The integrals

$$\Theta_n=\int_0^\infty \frac{4x^n\exp(-x^2)}{\Psi(x)}dx$$

$n=3, 4, 5$, are evaluated by numerical quadrature. The new values disagree with the values found by Tait, but agree with the values found by Boltzmann. The value of $\Theta_4$, which determines Tait's mean free path for a single system, is 0.6774560. A table is given for the function

$$\Phi(\alpha)=\alpha^2\int_0^\infty \frac{x^5\exp(-\alpha x^2)}{\Psi(x)}dx,$$

which is useful in scattering experiments. $\Phi(\alpha)$ is evaluated by numerical quadrature for 27 selected values of $\alpha$.

THE function,

$$\Psi(x)=x\exp(-x^2)+(2x^2+1)\int_0^x \exp(-y^2)dy,$$

occurs frequently in kinetic theory, particularly in expressions for mean free paths, and in discussions of viscosity, diffusion, and thermal conduction.

Tait[1,2] published a table of $\Psi(x)$ which was limited to values of $x$ from 0.1 to 3.0 in intervals of 0.1. A table which is more extended than that of Tait has now been computed and is given in Table I.[3] The range of the argument $x$ is extended from 3.0 to 20.4, and the interval is reduced to 0.5 for values of $x$ below 1.5.

Table I gives also the values of $x^5/\Psi(x)$ which are necessary for computing functions such as $\Theta_n$ and $\Phi(\alpha)$, defined further below.

Three important expressions involving $\Psi(x)$ are the integrals

$$\Theta_n=\int_0^\infty \frac{4x^n\exp(-x^2)}{\Psi(x)}dx,$$

where $n=3, 4, 5$. $\Theta_4$ determines Tait's mean free path for a molecule in its own gas. These three integrals were evaluated numerically by Tait[1,4] and by Boltzmann,[5] but their results disagree after the third figure as shown in Table II.

In view of this discrepancy between the findings of Tait and of Boltzmann, it was desirable to perform new and independent numerical quadratures of $\Theta_n$. The results are given in the last line of Table II. In each case the new values of $\Theta_n$ disagree with the values of Tait, but agree rather well with the values of Boltzmann.

* Publication assisted by the Ernest Kempton Adams Fund for Physical Research of Columbia University.

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[1] P. G. Tait, Trans. Roy. Soc. Edinburgh **33**, 65 (1886); Tait, *Scientific Papers* (Cambridge University Press, 1900), Vol. 2, paper 77, p. 124.

[2] The values of $\Psi(x)$ in the references under footnote 1 are correct; but in all other published tables of $\Psi(x)$ two typographical errors appear repeatedly for $x=1.1$ and for $x=2.7$. The tables in which these errors appear are the following. J. H. Jeans, *The Dynamical Theory of Gases* (Cambridge University Press, 1904), first edition, p. 347; (1916), second edition, p. 430; (1921), third edition, p. 436; (1925), fourth edition, p. 438: J. H. Jeans, *An Introduction to the Kinetic Theory of Gases* (Cambridge University Press, 1940), p. 305: L. B. Loeb, *The Kinetic Theory of Gases* (McGraw-Hill Book Company, 1927), first edition, p. 531; (1934), second edition, p. 653. Tait's original table gave $\Psi(x)$ to 6 places past the decimal point; the tables in Jeans and in Loeb reproduced Tait's table to 5 places past the decimal point.

[3] Table I includes Tait's original values of $\Psi(x)$ except for $x=0.1, 0.2, 0.3, 0.4$, in which cases $\Psi(x)$ is recomputed in order to extend Tait's values to seven figures.

[4] P. G. Tait, Trans. Roy. Soc. Edinburgh **33**, 251 (1887); Tait, *Scientific Papers* (Cambridge University Press, 1900), Vol. 2, paper 78, p. 153.

[5] L. Boltzmann, Akad. d. Wissenschaften Wien Sitz. **96**, 891 (1887).

TABLE I. Values of $\Psi(x)$ and $x^5/\Psi(x)$.

| $x$ | $\Psi(x)$ | $x^5/\Psi(x)$ | $x$ | $\Psi(x)$ | $x^5/\Psi(x)$ | $x$ | $\Psi(x)$ | $x^5/\Psi(x)$ |
|---|---|---|---|---|---|---|---|---|
| 0.05 | 0.1000833 | 0.000003122399 | 4.1 | 30.68118 | 37.76133 | 9.7 | 167.6564 | 512.1988 |
| .1 | .2006660 | .00004983405 | 4.2 | 32.15231 | 40.64754 | 9.8 | 171.1127 | 528.2605 |
| .15 | .3022450 | .0002512449 | 4.3 | 33.65890 | 43.67595 | 9.9 | 174.6044 | 544.6541 |
| .2 | .4053121 | .0007895150 | 4.4 | 35.20093 | 46.84996 | 10.0 | 178.1316 | 561.3827 |
| .25 | .5103522 | .001913507 | 4.5 | 36.77842 | 50.17294 | 10.2 | 185.2923 | 595.8591 |
| .3 | .6178401 | .003933057 | 4.6 | 38.39135 | 53.64828 | 10.4 | 192.5948 | 631.7164 |
| .35 | .7282393 | .007212173 | 4.7 | 40.03973 | 57.27936 | 10.6 | 200.0391 | 668.9822 |
| .4 | .8419993 | .01216153 | 4.8 | 41.72356 | 61.06956 | 10.8 | 207.6252 | 707.6829 |
| .45 | .9595544 | .01923060 | 4.9 | 43.44284 | 65.02228 | 11.0 | 215.3531 | 747.8462 |
| .5 | 1.081321 | .02889984 | 5.0 | 45.19757 | 69.14088 | 11.2 | 223.2228 | 789.4991 |
| .55 | 1.207701 | .04167293 | 5.1 | 46.98775 | 73.42877 | 11.4 | 231.2343 | 832.6684 |
| .6 | 1.339068 | .05807024 | 5.2 | 48.81338 | 77.88931 | 11.6 | 239.3876 | 877.3813 |
| .65 | 1.475789 | .07862174 | 5.3 | 50.67445 | 82.52590 | 11.8 | 247.6827 | 923.6648 |
| .7 | 1.618194 | .1038627 | 5.4 | 52.57098 | 87.34192 | 12.0 | 256.1196 | 971.5461 |
| .75 | 1.766608 | .1343279 | 5.5 | 54.50295 | 92.34075 | 12.2 | 264.6983 | 1021.052 |
| .8 | 1.921318 | .1705496 | 5.6 | 56.47038 | 97.52578 | 12.4 | 273.4187 | 1072.211 |
| .85 | 2.082609 | .2130526 | 5.7 | 58.47325 | 102.9004 | 12.6 | 282.2810 | 1125.048 |
| .9 | 2.250723 | .2623557 | 5.8 | 60.51157 | 108.4680 | 12.8 | 291.2851 | 1179.591 |
| .95 | 2.425902 | .3189663 | 5.9 | 62.58534 | 114.2319 | 13.0 | 300.4309 | 1235.868 |
| 1.0 | 2.608351 | .3833840 | 6.0 | 64.69456 | 120.1956 | 13.2 | 309.7186 | 1293.905 |
| 1.05 | 2.798269 | .4560970 | 6.1 | 66.83923 | 126.3624 | 13.4 | 319.1480 | 1353.729 |
| 1.1 | 2.995825 | .5375848 | 6.2 | 69.01935 | 132.7356 | 13.6 | 328.7193 | 1415.368 |
| 1.15 | 3.201185 | .6283164 | 6.3 | 71.23492 | 139.3188 | 13.8 | 338.4323 | 1478.848 |
| 1.2 | 3.414479 | .7287554 | 6.4 | 73.48593 | 146.1153 | 14.0 | 348.2872 | 1544.197 |
| 1.25 | 3.635847 | .8393527 | 6.5 | 75.77240 | 153.1284 | 14.2 | 358.2838 | 1611.443 |
| 1.3 | 3.865384 | .9605592 | 6.6 | 78.09431 | 160.3616 | 14.4 | 368.4222 | 1680.609 |
| 1.35 | 4.103208 | 1.092812 | 6.7 | 80.45168 | 167.8181 | 14.6 | 378.7025 | 1751.726 |
| 1.4 | 4.349386 | 1.236552 | 6.8 | 82.84449 | 175.5016 | 14.8 | 389.1245 | 1824.820 |
| 1.45 | 4.604011 | 1.392207 | 6.9 | 85.27275 | 183.4151 | 15.0 | 399.6883 | 1899.918 |
| 1.5 | 4.867132 | 1.560210 | 7.0 | 87.73646 | 191.5623 | 15.2 | 410.3940 | 1977.047 |
| 1.6 | 5.419114 | 1.934958 | 7.1 | 90.23562 | 199.9464 | 15.4 | 421.2414 | 2056.234 |
| 1.7 | 6.005696 | 2.364184 | 7.2 | 92.77023 | 208.5710 | 15.6 | 432.2306 | 2137.507 |
| 1.8 | 6.627149 | 2.851253 | 7.3 | 95.34029 | 217.4392 | 15.8 | 443.3616 | 2220.891 |
| 1.9 | 7.283658 | 3.399527 | 7.4 | 97.94580 | 226.5546 | 16.0 | 454.6344 | 2306.416 |
| 2.0 | 7.975359 | 4.012359 | 7.5 | 100.5868 | 235.9203 | 16.2 | 466.0490 | 2394.107 |
| 2.1 | 8.702340 | 4.693107 | 7.6 | 103.2632 | 245.5400 | 16.4 | 477.6054 | 2483.990 |
| 2.2 | 9.464667 | 5.445128 | 7.7 | 105.9750 | 255.4172 | 16.6 | 489.3036 | 2576.096 |
| 2.3 | 10.26236 | 6.271796 | 7.8 | 108.7223 | 265.5549 | 16.8 | 501.1436 | 2670.448 |
| 2.4 | 11.09547 | 7.176461 | 7.9 | 111.5051 | 275.9565 | 17.0 | 513.1254 | 2767.076 |
| 2.5 | 11.96402 | 8.162497 | 8.0 | 114.3233 | 286.6257 | 17.2 | 525.2490 | 2866.005 |
| 2.6 | 12.86798 | 9.233288 | 8.1 | 117.1769 | 297.5658 | 17.4 | 537.5143 | 2967.264 |
| 2.7 | 13.80739 | 10.39220 | 8.2 | 120.0660 | 308.7800 | 17.6 | 549.9215 | 3070.878 |
| 2.8 | 14.78225 | 11.64259 | 8.3 | 122.9906 | 320.2717 | 17.8 | 562.4705 | 3176.876 |
| 2.9 | 15.79255 | 12.98786 | 8.4 | 125.9506 | 332.0444 | 18.0 | 575.1613 | 3285.284 |
| 3.0 | 16.83830 | 14.43138 | 8.5 | 128.9460 | 344.1016 | 18.2 | 587.9938 | 3396.129 |
| 3.1 | 17.91951 | 15.97653 | 8.6 | 131.9769 | 356.4465 | 18.4 | 600.9682 | 3509.439 |
| 3.2 | 19.03615 | 17.62669 | 8.7 | 135.0433 | 369.0823 | 18.6 | 614.0843 | 3625.240 |
| 3.3 | 20.18825 | 19.38523 | 8.8 | 138.1450 | 382.0130 | 18.8 | 627.3423 | 3743.559 |
| 3.4 | 21.37579 | 21.25555 | 8.9 | 141.2823 | 395.2412 | 19.0 | 640.7420 | 3864.424 |
| 3.5 | 22.59879 | 23.24101 | 9.0 | 144.4550 | 408.7709 | 19.2 | 654.2836 | 3987.862 |
| 3.6 | 23.85723 | 25.34502 | 9.1 | 147.6631 | 422.6053 | 19.4 | 667.9669 | 4113.900 |
| 3.7 | 25.15112 | 27.57092 | 9.2 | 150.9067 | 436.7477 | 19.6 | 681.7921 | 4242.565 |
| 3.8 | 26.48046 | 29.92213 | 9.3 | 154.1858 | 451.2013 | 19.8 | 695.7590 | 4373.882 |
| 3.9 | 27.84525 | 32.40201 | 9.4 | 157.5002 | 465.9702 | 20.0 | 709.8677 | 4507.882 |
| 4.0 | 29.24549 | 35.01395 | 9.5 | 160.8502 | 481.0568 | 20.2 | 724.0828 | 4644.817 |
| | | | 9.6 | 164.2356 | 496.4653 | 20.4 | 738.5106 | 4784.033 |

Recent experiments[6–8] on the scattering of molecular beams have involved a modification of $\Theta_5$, namely, the function

$$\Phi(\alpha)=\alpha^2\int_0^\infty \frac{x^5 \exp(-\alpha x^2)}{\Psi(x)}dx.$$

[6] W. H. Mais, Phys. Rev. **45**, 773 (1934).

[7] S. Rosin and I. I. Rabi, Phys. Rev. **48**, 373 (1935). Rosin and Rabi give values of $\Psi(x)$ to three significant figures which disagree with the results of the present calculations in the second and third significant figures. Rosin and Rabi's value of $\Theta_5$ is 0.844 which disagrees with the results of Tait, of Boltzmann, and of Rosenberg, in the second significant figure (cf. Table II). (In the notation used by Rosin and Rabi $\Theta_5=4I(\alpha)$ for $\alpha=1$.)

[8] P. Rosenberg, Phys. Rev. **55**, 1267 (1939); **57**, 561 (1940).

$\Phi(\alpha)$ is used to determine the average mean free path of a molecular beam particle traversing a scattering gas. $\alpha$ is the square of the ratio of the most probable speed of the particles of the scattering gas to the most probable speed of the particles of gas in the source of the beam.

$\Phi(\alpha)$ is now evaluated by numerical quadrature for 27 selected values of $\alpha$. The results are given in Table III.

The following convergence test is used to determine the upper limits in the numerical quadratures of $\Phi(\alpha)$. If the quadrature is carried out from $x=0$ to a value $x=L$, the remainder $R$ is given by

$$R=\alpha^2\int_L^\infty \frac{x^5 \exp(-\alpha x^2)}{\Psi(x)}dx < \alpha^2\int_L^\infty \frac{x^3 \exp(-\alpha x^2)}{2P}dx = S,$$

where $$P=\int_0^L \exp(-y^2)dy.$$

$S$ can be evaluated analytically, yielding

$$S=[(\alpha L^2+1)/4P]\exp(-\alpha L^2).$$

In each quadrature of $\Phi(\alpha)$, $L$ is chosen large enough so that $S$ is negligibly small compared to the eighth significant figure in the value of $\Phi(\alpha)$.

TABLE II. Values of $\Theta_n$.

| | $\Theta_3$ | $\Theta_4$ | $\Theta_5$ |
|---|---|---|---|
| Tait | 0.650404 | 0.677072 | 0.838098 |
| Boltzmann | 0.650511 | 0.677464 | 0.838264 |
| This paper | 0.6505144 | 0.6774560 | 0.8382656 |

TABLE III. Values of $\Phi(\alpha)$.

| $\alpha$ | $\Phi(\alpha)$ | $\alpha$ | $\Phi(\alpha)$ |
|---|---|---|---|
| 0. | 0.2820948 $(=1/2\pi^{\frac{1}{2}})$ | 8. | 0.1071319 |
| .05 | .2756271 | 9. | .1019461 |
| .1 | .2698879 | 10. | .09744985 |
| .5 | .2364264 | 11. | .09350219 |
| .6 | .2301268 | 12. | .08999987 |
| .8 | .2190537 | 15. | .08146816 |
| 1. | .2095664 | 17. | .07696826 |
| 1.2 | .2012941 | 20. | .07143003 |
| 2. | .1762257 | 21. | .06983411 |
| 3. | .1553010 | 22. | .06834015 |
| 4. | .1405107 | 25. | .06437612 |
| 5. | .1293167 | 30. | .05907038 |
| 6. | .1204532 | 35. | .05489229 |
| 7. | .1132050 | 40. | .05149165 |

A similar convergence test is used in the quadratures of $\Theta_n$.

The quadratures of $\Theta_n$ and $\Phi(\alpha)$ are performed by Simpson's $\frac{1}{3}$ rule. In all but a few of the quadratures, the interval in $x$ is so small that the sum of the odd terms is equal to the sum of the even terms to the first six or more figures. The equality of these two sums is regarded as an internal check on the accuracy of the numerical computations.

The values of $\Psi(x)$ and $x^5/\Psi(x)$ are exact to only six of the seven figures given in Table I. The values of $\Theta_n$ and $\Phi(\alpha)$ are computed to eight figures and are given correct to seven figures in Table II and Table III.

Values of the probability integral are taken from the table of Burgess.[9] Values of the descending exponential are obtained from the table of Newman.[10]

Mr. David S. Cohen assisted in the numerical computations.

[9] J. Burgess, Trans. Roy. Soc. Edinburgh **39**, 257 (1898).
[10] F. W. Newman, Trans. Camb. Phil. Soc. **13**, 145 (1883).