Electron Emission of Metals in Electric Fields

III. The Transition from Thermionic to Cold Emission*

EUGENE GUTH AND CHARLES J. MULLIN†
University of Notre Dame, Notre Dame, Indiana
(Received October 14, 1941)

The theory of Schottky emission is extended to include in the current those electrons which tunnel through the top of the potential barrier at the metal surface when rather strong electric fields are used. It is found that these tunnelling electrons contribute to the periodic deviations from the Schottky line, increasing the amplitude of the deviations. This increase leads to a better agreement with experiment, especially for large fields. This agreement requires that the Nottingham reflection coefficient be very small for fields greater than 10^4 volts per cm. Expressions are developed for the electron current emitted by a metal for various field intensities ranging from the small fields of thermionic work to the large fields used in cold emission. Results are obtained to indicate the temperature and field dependence of the electron current for all fields and temperatures of interest. Of particular interest is the expression developed for the current in the "transition region," i.e., $T\sim 500$–$1200°$K and $F\sim 10^7$–10^8 volts per cm. The modifications which must be made in the theory to take account of the polycrystalline nature of the emitting surface have no effect on the periodic deviations from the Schottky line, and very little effect on the fluctuations observed in field photo-currents.

I. INTRODUCTION

It is possible to obtain electron currents emitted from metals under widely varying experimental conditions. For example: the electrons may be ejected from a hot cathode and collected by fields with a rather wide range of intensities; or the electrons may be pulled from a cold cathode by the application of very intense electric fields. Despite the different experimental conditions employed in these cases, a single general expression which is valid (under the usual assumptions of the free electron theory of metals) for all electron energies, fields, and temperatures may be given:

$$i=\int_0^\infty N(W)D(W, F)dW, \tag{1}$$

where W is the energy which the electron has in the velocity component normal to the emitting surface, $N(W)$ is the number of electrons in the metal with this energy which strike unit area per second, and $D(W, F)$ is the probability that one of these electrons will escape through the metal surface under the influence of the applied field, F. Unfortunately, an evaluation of (1) valid for all energies cannot be given in an analytical form; however, some special cases of interest and importance can be fully treated. The familiar work on thermionic and Schottky emission has given a theory which holds for high temperatures and low fields [high values of W in Eq. (1)], and the well-known papers of Fowler and Nordheim[1] and of Nordheim[2] treat the case of very low temperatures and high fields [lower values of W in Eq. (1)]. In the present work an attempt is made to extend these theories and to bridge the gap between the two theories. In line with this,

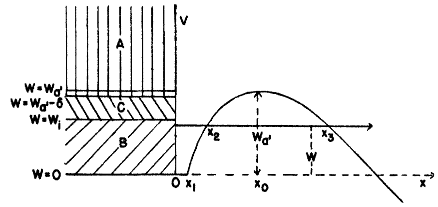

FIG. 1. Energy diagram for electrons in and out of the metal. W_a' is the total height of the barrier at the metal surface. Electrons from region A contribute to Schottky emission, those from region B to cold emission. x_2 and x_3 are the points at which $\phi=W-V=0$.

* This work is a section of a dissertation presented to the Faculty of the Graduate School of the University of Notre Dame, 1942, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
† Now at St. Louis University, St. Louis, Missouri.

[1] R. H. Fowler and L. W. Nordheim, Proc. Roy. Soc. **119**, 173 (1928).
[2] L. W. Nordheim, Proc. Roy. Soc. **121**, 626 (1928).

the recent theory[3] of the periodic deviations from the Schottky line is also extended to include the effect of the electrons which penetrate the top of the potential barrier at the metal surface (Fig. 1); this penetration effect is found to increase the amplitude of the deviations so as to improve somewhat the agreement between theory and experiment at the higher field intensities used. It is found that the improved agreement between the theoretical and experimental deviations requires that the reflection coefficient proposed by Nottingham[4] decrease with the applied field in such a way as to become nearly zero for fields of the intensities used to obtain the periodic deviations. An interesting result is also found for the current emitted when rather high fields and intermediate temperatures are used. In Part II of the present paper, the results which can be obtained from Eq. (1) for the case of an ideal surface with a single "effective" work function, or for a single crystal face, are discussed; in Part III the effects of the polycrystalline nature of the emitting surface on these results are considered; and in Part IV a brief derivation for the currents obtained under the various conditions discussed in II is given.

II. DISCUSSION OF THE RESULTS FOR AN IDEAL "SINGLE WORK FUNCTION SURFACE" OR FOR A SINGLE CRYSTAL

In order to evaluate (1) it is convenient to split the integral up into parts corresponding to various ranges of the energy W. If only values of W greater than W_a' (Fig. 1) are considered, that is, if the field applied is not strong enough to cause an appreciable number of electrons with energies below W_a' to be emitted, Eq. (1) leads to an expression for thermionic or Schottky emission; if only values of W less than W_i are considered, that is, if the filament temperature is so low that only a negligible number of electrons have energies greater than W_i, Eq. (1) leads to an expression for cold emission. For values of W ranging from W_i up to W_a', the current gradually takes on a stronger temperature dependence and weaker field dependence, passing from cold emission to thermionic emission; however, an analytical expression for the current valid in the entire region $W_i \leq W \leq W_a'$ has not yet been given. In the following work some limiting cases are treated.

(a) Low Fields, High Temperatures

In this case only electrons with energies above a certain value $W_a'-\delta$ contribute to the current, the others being reflected back into the metal when they strike the surface; the contributing electrons are those from region A in Fig. 1. The current from this region is

$$i=B(kT)^2e^{-\chi'/kT}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)\mu}{n(n+1)\mu^2+\mu-1}\right.$$
$$-\frac{W_a^{\frac{1}{2}}}{2kT}\left(1-\frac{1}{4x_0^2}W_a^2\right)\left[\frac{\cos v_a}{\{[\pi\alpha+(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}\right.$$
$$\left.\left.+\frac{\cos v_b}{\{[2\pi\alpha-(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}\right]\right\}, \quad (2)$$

where atomic units of length ($a=\hbar^2/me^2=0.527$A) and energy (unit$=e^2/2a=13.67$ ev) are used, and $\alpha=(\frac{1}{2}x_0^3)^{\frac{1}{2}}$; $\mu=2\pi\alpha kT$; $x_0=$char. Schottky distance (Fig. 1): $x_0=[(300e)^{\frac{1}{2}}/(2\times0.527)](10^8/F^{\frac{1}{2}})$ $F=$volts/cm; $\chi=$work function for zero field χ' for field F: $\chi'=\chi-(1/x_0)$; $W_a'=W_i+\chi'$ $\beta=\gamma+2\ln 2$; $\gamma=$Euler's const.$=0.5772$ $B=160\pi^6m^3e^9/ch^7$

$$v_a=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}-\tan^{-1}\frac{\beta\alpha}{\pi\alpha+(1/kT)},$$
$$v_b=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}+\tan^{-1}\frac{\beta\alpha}{\pi\alpha-(1/kT)}.$$

If the relatively small periodic term is neglected

$$i=B(kT)^2e^{-\chi'/kT}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)\mu}{n(n+1)\mu^2+\mu-1}. \quad (3)$$

If only the first term of the summation is taken this result reduces to one obtained by Bethe[5]

$$i=B\frac{\mu}{\mu-1}(kT)^2e^{-\chi'/kT}. \quad (4)$$

If $\mu\gg1$, which means that high temperatures and

[3] E. Guth and C. J. Mullin, Phys. Rev. **59**, 575 (1941).
[4] W. B. Nottingham, Phys. Rev. **49**, 78 (1936).
[5] A. Sommerfeld and H. Bethe, *Handbuch der Physik* (1933), Vol. 24/2, p. 442.

low fields are employed, Eqs. (3) and (4) reduce to the familiar equation for Schottky emission, in which the mean transmission coefficient $\bar{D}\sim 1$,

$$i=B(kT)^2e^{-\chi'/kT}. \qquad (5)$$

It is seen that (5) and (3) differ in that (5) effectively considers only those electrons which have energies greater than W_a', i.e., which surmount the potential barrier of Fig. 1, while (3) also takes account of those electrons which tunnel through the barrier in the region $W_a'-\delta\leq W\leq W_a'$. This tunnelling occurs even for rather weak fields because even then the transmission coefficient is not zero at the top of the barrier of Fig. 1. A comparison of Eq. (2) with the equation obtained for the current by neglecting the tunnel effect shows that the second periodic term ($\cos v_b$) of (2) arises from the electrons penetrating the potential barrier at the metal surface. An interesting fact about the periodic term in (2) is that it becomes small both for low and high fields; for sufficiently high fields $x_0\rightarrow x_1\sim 1/2W_a$, so that the coefficient $(1-1/4x_0^2W_a^2)$ becomes smaller as the field becomes very high. The contribution of the second periodic term due to the tunnel effect gives a better agreement between the theoretical and experimental[6–8] amplitudes of the deviations at the high end of the field range used.

With all electrons accounted for in Eq. (2), the periodic deviations have an amplitude in good agreement with observation. This agreement affords a good opportunity to observe whether or not the electrons reaching the metal surface are suffering, at these rather high field intensities, a reflection of the type suggested by Nottingham.[4] In order to explain the non-Maxwellian distribution of the slow electrons in thermionic emission, Nottingham proposed that the electrons suffered a reflection

$$R_N=e^{-(W-W_a')/\omega}, \quad \omega\sim 0.2 \text{ ev}. \qquad (6)$$

However, as Dr. Nottingham pointed out to the authors, the velocity distribution experiments require only that $\omega\sim 0.2$ ev for the very low fields used in those experiments. A transmission coefficient which meets the conditions imposed by R_N and at the same time introduces some of the influence of the barrier of Fig. 1 is given by

$$D=D_c(1-R_N), \qquad (7)$$

where D_c is the transmission coefficient for the barrier of Fig. 1. With this transmission, the current from the region A becomes

$$i=B(kT)^2e^{-\chi'/kT}\left\{\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n\mu+1}-\frac{1}{n\mu+1+(kT/\omega)}\right)\right.$$
$$\left.-\frac{W_a^{\frac{1}{2}}}{2kT}\left(1-\frac{1}{4x_0^2W_a^2}\right)\left[\frac{\cos u_1}{\{[\pi\alpha+(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}-\frac{\cos u_2}{\{[2\pi\alpha+(1/kT)+(1/\omega)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}\right]\right\}, \qquad (8)$$

where

$$u_1=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}-\tan^{-1}\frac{\beta\alpha}{\pi\alpha+(1/kT)},$$

$$u_2=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}-\tan^{-1}\frac{\beta\alpha}{\pi\alpha+(1/kT)+(1/\omega)}.$$

To keep the amplitude of the deviations appreciable would require that

$\pi\alpha\sim 1/\omega$, or $\omega\sim 1/2000$ atomic units, $\omega\sim 1/200$ ev.

Thus, ω must depend upon the field in such a way that $\omega\sim 0.2$ ev when $F\sim 0$, and $\omega\rightarrow 0$ as F takes on the rather large values needed to obtain the periodic deviations. Practically, Eq. (8) shows that the Nottingham reflection does not operate at these relatively high fields. A discussion of the Nottingham reflection *versus* the theory of emission from "patches" is given by Nichols.[9]

[6] R. L. E. Seifert and T. E. Phipps, Phys. Rev. **56**, 652 (1939).
[7] D. Turnbull and T. E. Phipps, Phys. Rev. **56**, 663 (1939).
[8] W. B. Nottingham, Phys. Rev. **57**, 935 (1940).
[9] M. H. Nichols, to be submitted for publication.

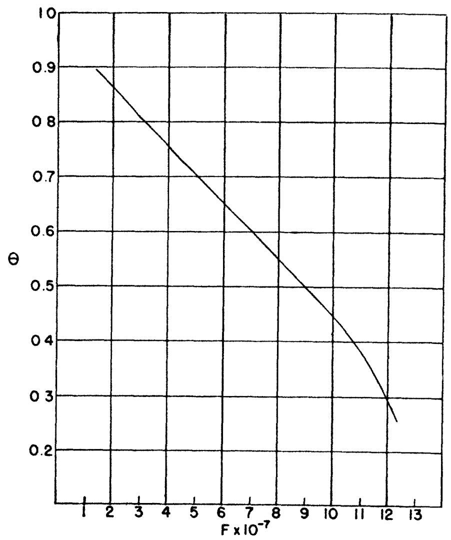

FIG. 2. The Nordheim function θ plotted against the accelerating field F.

It may be noted that the value $\omega=0.2$ ev leads to a mean transmission coefficient $\bar{D}=\frac{1}{3}$, while the value $\omega\sim 1/200$ ev leads to a $\bar{D}\sim 0.95$.

In the case of photoelectric emission under the influence of fields high enough to cause penetration of the topmost part of the barrier, the penetrating current plays a much less important role than in the above cited case of Schottky emission; the reason for this is immediately clear: In the photoelectric case the transmission decreases exponentially as the electron energy decreases from W_a', but the number of electrons available remains almost constant (since these electrons have energies less than W_i); hence, the current, which again depends upon the product $D(W, F)N(W)$, decreases very rapidly as W becomes progressively less than W_a'; this quick decrease makes the contribution to both non-periodic and periodic parts of the current unimportant.

(b) High Fields, Low Temperatures

At low temperatures there are very few electrons with energy greater than W_i. However, with sufficiently high values of the applied field, electrons with energies less than W_i are pulled from the metal. The electrons contributing to this cold emission are from the region B of Fig. 1, most of them coming from the uppermost part of region B, since the transmission coefficient for this region decreases exponentially with decreasing energy. The current from this region is then,

$$i=B\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{\frac{3}{2}}\right]\left\{\frac{1}{16\Theta^2x_0^4\chi}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{(kT)^2}{(n+4x_0^2\Theta\chi^{\frac{1}{2}}kT}\right\} \quad (9)$$

or, if we express χ in electron volts and F in volts per cm,

$$i=\left\{1.55\times 10^{-6}\frac{F^2}{\Theta^2\chi}+120.5T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times\frac{1}{n+8.81\times 10^{+3}\Theta\chi^{\frac{1}{2}}T/F}\right\}\cdot\exp\left(-6.838\times 10^7\Theta\chi^{\frac{3}{2}}/F\right)\ \text{amp./cm}^2 \quad (10)$$

where Θ is a slowly varying function[10] of F which varies from 1 to 0 as F varies from 0 to ∞. Θ is plotted as a function of F in Fig. 2. If the summation term is neglected completely (This corresponds to taking only the first term in the expansion of the Fermi distribution function, a valid approximation for low temperatures), Eq. (10) reduces to

$$i=1.55\times 10^{-6}\frac{F^2}{\Theta^2\chi}\exp\left(-6.838\times 10^7\Theta\chi^{\frac{3}{2}}/F\right)\ \text{amp cm}^2. \quad (11)$$

This formula has been recently checked experimentally by Haefer.[11] By taking the sum into consideration one obtains the temperature dependence of the current from region B of Fig. (1);

[10] This function was introduced by L. W. Nordheim, reference 1, to determine the effect of the image force on cold emission.

[11] R. Haefer, Zeits. f. Physik **116**, 604 (1940). This author determined the actual field strength acting on the point of the field emitting metal by obtaining the curvature of the point from electron microscope investigations. He found that field emission sets in at about 3×10^7 volts per cm as required by Eq. (11). Thus no "sensitive spots" need be introduced in order to explain the difference between the empirical and theoretical thresholds. Haefer accurately verified the dependence of the current on $\chi^{\frac{3}{2}}$ as given by (11).

if T is small so that $8.81\times 10^3\chi^{1/2}VT/F\ll 1$, Eq. (10) gives for the temperature dependence

$$\frac{i(T)}{i(0)}=1+7.77\times 10^7\Theta^2\frac{\chi T^2}{F^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}. \qquad (12)$$

The first term of the series in Eq. (12) has been given by Bethe[12] who points out that this result is in agreement with the experimental data of Millikan and Eyring.[13] The temperature dependence is of the same type as that obtained for photo-emission for the simple reason that the electrons in each of these cases originate in the same region B.

For low temperatures the T^2 term in (10) is unimportant, but it may become appreciable for higher temperatures; in any case, it seems that the inclusion of the temperature term in (10) is inconsistent unless this term is modified to include the contribution of those electrons which, because of the temperature dependence of the Fermi distribution function, have energies greater than W_i. However, field emission for which the temperature term is large (say for $T=800$–$1000°$K, $F=2-4\times 10^7$ volts/cm) is not strictly cold emission, but rather may be classed as a transition case between cold and thermionic emission. The current obtained under these "transition conditions" is discussed in Section (c).

(c) Intermediate Fields and Temperatures

In this case the emission from region C of Fig. 1 must be considered. The current from this region should depend upon both the temperature and the field. An expression valid for all of region C is not easily obtained, but the contributions from the upper and lower parts may be given. A particularly important case involving the current from this region has been mentioned in the previous section. If the temperature and field are such that

$$4x_0^2\chi^{1/2}\Theta kT<1 \text{ (atomic units)},$$

i.e.,

$$8.813\times 10^3\Theta\chi^{1/2}T/F<1 \text{ (ev and volt/cm)} \qquad (13)$$

the current from region C is

$$i=Bk^2T^2\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{3/2}\right]\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{1}{n-4x_0^2\Theta\chi^{1/2}kT}. \qquad (14)$$

By adding this current to that from the region B, the total current for the fields and temperatures subject to the condition (12) is obtained

$$i=B\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{3/2}\right]\left\{\frac{1}{16\Theta^2x_0^4\chi}+k^2T^2\sum\frac{(-1)^{n+1}}{n}\times\left(\frac{1}{n+4x_0^2\Theta\chi^{1/2}kT}+\frac{1}{n-4x_0^2\Theta\chi^{1/2}kT}\right)\right\}, \qquad (15)$$

or, putting χ in ev and F in volt/cm

$$i=\left\{1.55\times 10^{-6}\frac{F^2}{\Theta^2\chi}+120.5T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times\left(\frac{1}{n+8.813\times 10^3\Theta\chi^{1/2}T/F}+\frac{1}{n-8.813\times 10^3\Theta\chi^{1/2}T/F}\right)\right\}\times\exp\left[-6.838\times 10^7\Theta\chi^{3/2}/F\right]\text{ amp/cm}^2. \qquad (16)$$

Experimentally it should be possible to make a thorough study of currents under these "transition conditions." If $T=1000°$K, say, then by choosing $\chi=4.531$ ev (the value for tungsten) (16) is valid if $F\geqslant 1.8\times 10^7$ volts/cm. A plot of

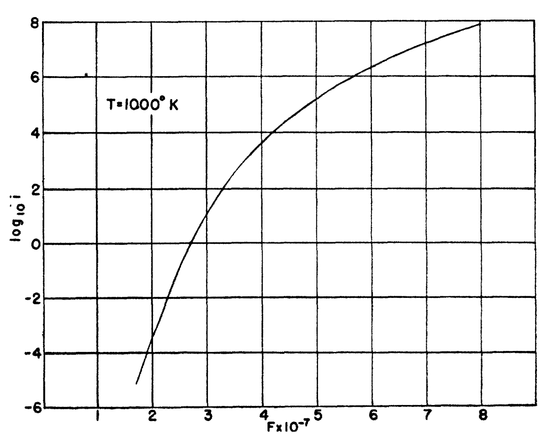

FIG. 3. A plot of the current given by Eq. (16).

[12] A. Sommerfeld and H. Bethe, *Handbuch der Physik* (1933), Vol. 24/2, p. 442.

[13] R. A. Millikan and C. F. Eyring, Phys. Rev. **27**, 51 (1926).

the current obtained from (16) is given in Fig. 3. It may be noted that for fields near the threshold for emission, Eq. (16) gives a current many times that expected from the simpler Nordheim expression given in Eq. (11). In particular, if $T=1000°\mathrm{K}$ and $F=2\times10^7$ volt/cm, (16) gives a current about five times as large as that obtained from (11). If $F=3\times10^7$, (16) yields a value about 1.7 times as large as that given by (11). The larger current is due chiefly to those electrons coming from region C.

The expression (14) for the current from the lower part of C shows, as was expected, a larger temperature dependence and a smaller field dependence than was had for cold emission.

Photoelectric emission could be obtained from the region C by applying fields high enough to make the electrons struck with low frequency photons tunnel the barrier in the region adjacent to C. The photo-current from this region is

$$i=\exp\left[-6.838\times10^7\Theta(\chi-h\nu)^{3/2}/F\right]\left\{1.55\times10^{-6}\frac{F^2}{\Theta^2(\chi-h\nu)}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times\frac{k^2T^2}{n+8.81\times10^3\Theta(\chi-h\nu)^{1/2}T/F}\right\}. \quad (17)$$

This current is very similar to that for cold emission as given by Eq. (9). However, the photons have the effect of reducing the work function by an amount $h\nu$, so that really the electrons from region B are effectively raised to region C and tunnel through the barrier from this region.

III. THE EFFECT OF PATCHES ON THE EMISSION[14]

In all the above work the electrons are assumed to be emitted from a surface with one uniform "effective" work function. If, instead, the current is assumed to come from a number of crystal faces (hereafter called patches) with different work functions, the previous expressions for the current must be modified. If the applied fields used are high ($x_0\ll$patch diameter $\sim100\mathrm{A}$) as is generally true above, each patch may be assumed to emit independently, so that the resulting current is

$$i=\sum f_n i_n, \quad (18)$$

where i_n is the current from the nth crystal face, and f_n is the fractional area of this patch.

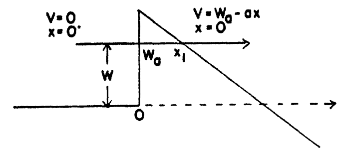

FIG. 4. The potential barrier at the metal surface if the image force is neglected.

(a) Thermionic Emission

If it is assumed that a certain number p of patches have work functions very nearly equal, the work function of the group being called χ, and if q patches have a work function $\chi+\delta$, the current from the first group is

$$i_1=\sum_{n=1}^{p} f_n i_n=\sum_{n=1}^{p} f_n A_0\bar{D}T^2e^{-(\chi-\Delta\chi)/kT} \quad (19)$$

and from the second

$$i_2=\sum_{n=1}^{q} f_n i_n=e^{-\delta/kT}\sum_{n=1}^{q} f_n A_0\bar{D}T^2e^{-(\chi-\Delta\chi)/kT}. \quad (20)$$

If $\delta>kT$, $i_2\ll i_1$ so that the principal contribution to the current comes from the patches of lowest work function. In this case the "ideal surface" emission constant $+A_0$ must be replaced by

$$A_0\sum_{n=1}^{p} f_n$$

which is always less or equal to A_0, the equality holding only if the entire surface has the same work function. For fields high enough that the patches emit independently, the Schottky line is the same as for an ideal surface, i.e.,

$$\ln i(F)-\ln i(0)\sim-\Delta\chi/kT=e^{3/2}F^{1/2}/kT. \quad (21)$$

If the various groups of crystal faces have work functions which differ by an amount $\delta>kT$, the current comes almost exclusively from the facets of lowest χ, and the periodic deviations found in Part II are unaffected. However, if there exists a δ for which the condition $\delta>kT$ is not fulfilled,

[14] For a complete discussion of the theory of patch emission see M. H. Nichols, reference 9; cf. also C. Herring, Phys. Rev. **59**, 889 (1941). We are indebted to Drs. Herring and Nichols for letting us have the manuscripts of their papers before publication.

then a current contribution from the corresponding group of patches must be considered. However, a δ this small would not affect the periodic deviations of Part II (which depend upon W_a), since their dependence upon χ is a rather insensitive one, and the only effect of this patch group is to increase the current. Thus, it is seen that the periodic deviations are never destroyed by "interference" between the emissions of different patches.

The arguments applied above about the effect of patches on thermionic emission may be applied also to cold emission (here there is no periodic term, of course) which, according to Eq. (9) depends exponentially upon $\chi^{\frac{3}{2}}$.

(b) Photoelectric Emission

In the case of photoelectric emission with rather high fields, neglecting the small periodic term we have

$$i_n \sim Bk^2T^2\left[\frac{\pi^2}{6}+\frac{1}{2}\delta^2-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-n\delta}\right], \quad (22)$$

if

$$\delta=\frac{h\nu-\chi+\Delta\chi}{kT}>0,$$

or

$$i_n \sim Bk^2T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{n\delta} \quad \text{if } \delta<0.$$

It is seen that in this case the current does not fall off exponentially with increasing work function. Again suppose the two groups of patches with work functions χ and $\chi+\delta$ as above. Here it is possible to let δ have an appreciable value and yet to obtain an appreciable current contribution from the patches of work function $\chi+\delta$. Since the phase and amplitude of the periodic deviations found in the photo-current[15] emitted by a metal are dependent somewhat upon the work function of the metal, it is possible that the interference of the several currents from the crystal faces of different work functions might give deviations somewhat different in character from those calculated for an "ideal," single work function emitting surface. However, this interference effect must be small, for in order that two currents from patches with different work functions may have a large interference effect in their periodic terms, the two patches must have a rather large difference in their work functions; but a large difference in work function means that one current is small in comparison with the other, so that the interference effect will be small. This holds in spite of the fact, already pointed out by Nichols, that the Fowler plots are modified by patches. In the case of the Fowler plot a change of the work function by δ is of importance, while the periodic deviations, depending on $W_a(\sim 10$ ev) are rather insensitive to changes in χ of a few percent.

IV. DERIVATION OF THE RESULTS

(a) Low Fields, High Temperatures

In this case Eq. (1) is evaluated, (a) for $W_a'\leq W\leq\infty$, and (b) for $W_a'-\delta\leq W\leq W_a'$; these two ontributions are then added to give the total current from region A.

$$\text{(a)} \quad W_a'\leq W\leq\infty.$$

For this case the transmission coefficient and energy distribution are given by[16]

$$D(F,W)=\frac{1}{1+\exp\left[-2\pi\alpha(W-W_a')\right]}-\frac{W_a^{\frac{1}{2}}}{2}\left(1-\frac{1}{4x_0^2W_a^2}\right)\frac{\exp\left[-\pi\alpha(W-W_a')\right]\cos u_a}{\{1+\exp\left[-2\pi\alpha(W-W_a')\right]\}^{\frac{1}{2}}}, \quad (23)$$

$$u_a=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}-\beta\alpha(W-W_a')$$

$$N(W)\sim BkT\exp\left[-(W-W_i)/kT\right], \quad (24)$$

[15] E. Guth and C. J. Mullin, Phys. Rev. **59**, 867 (1941).

[16] For these results and the calculation of the corresponding current cf. E. Guth and C. J. Mullin, Phys. Rev. **59**, 575 (1941). The transmission coefficient given here is obtained by use of the W.K.B. solution to the wave equation.

$$i=\int_{W_a}^{\infty}D(F,\,W)N(W)dW=\int_0^{\infty}D(F,\,\epsilon)N(\epsilon)d\epsilon,$$

$$\epsilon=W-W'_a=W-W_a+1/x_0,$$

$$i=Bk^2T^2\exp\left(-\chi'/kT\right)\left\{\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n\pi\alpha kT+1}-\frac{W_a^{\frac{1}{4}}\left[1-(1/4x_0^2W_a^2)\right]\cos v_a}{2kT\,\{[\pi\alpha+(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}\right\}, \tag{25}$$

$$v_a=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}-\tan^{-1}\frac{\beta\alpha}{\pi\alpha+(1/kT)}.$$

(b) $W_a'-\delta\leq W\leq W_a'$.

The fields in this region are presumed to be such that the electrons penetrate only the topmost part of the barrier of Fig. 1. Then proceeding as in (a)

$$D(F,\,W)=\frac{\exp\left[-2\pi\alpha(W_a'-W)\right]}{1+\exp\left[-2\pi\alpha(W_a'-W)\right]}-\frac{W_a^{\frac{1}{4}}}{2}\frac{\left[1-(1/4x_0^2W_a^2)\right]\exp\left[2\pi\alpha(W_a'-W)\right]}{\{1+\exp\left[-2\pi\alpha(W_a'-W)\}^{\frac{1}{2}}}\cos u_b, \tag{26}$$

$$u_b=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}+\beta\alpha(W_a'-W),$$

$$N(W)\sim BkTe^{-(W-W_i)/kT}, \tag{27}$$

$$i=\int_{W_a'-\delta}^{W_a'}N(W)D(W,\,F)dW=\int_0^{\infty}N(y)D(y,\,F)dy,$$

$$y=W_a'-W=W_a-W-1/x_0,$$

$$i=Bk^2T^2e^{-\chi'/kT}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{1}{2(n+1)\pi\alpha kT-1}-\frac{W_a^{\frac{1}{4}}}{2kT}\frac{(1-1/4x_0^2W_a^2)}{\{[2\pi\alpha-(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}\cos v_b, \tag{28}$$

$$v_b=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}+\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}+\tan^{-1}\frac{\beta\alpha}{2\pi\alpha-(1/kT)}.$$

The region of integration is taken as $0\leq y\leq\infty$ since it is assumed that the integral vanished at the upper limit. Actually, the integration range should be $0\leq y\leq\delta$ corresponding to $W_a'-\delta\leq W\leq W_a'$.

Adding the results of (a) and (b) gives for the current from the region A

$$i=Bk^2T^2e^{-\chi'/kT}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)\mu}{n(n+1)\mu^2+\mu-1}\right.$$
$$\left.-\frac{W_a^{\frac{1}{4}}}{2kT}\left(1-\frac{1}{4x_0^2}W_a^2\right)\left[\frac{\cos v_a}{\{[\pi\alpha+(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}+\frac{\cos v_b}{\{[2\pi\alpha-(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}\right]\right\}. \tag{29}$$

If the transmission coefficient given by Eq. (7) is used and only electrons for which $W\geqslant W_a'$ are counted

$$i=BkTe^{-\chi'/kT}\int_0^{\infty}D_c(1-e^{-\epsilon/\omega})e^{-\epsilon/kT}d\epsilon$$

$$=BkTe^{-\chi'/kT}\int_0^{\infty}D_ce^{-\epsilon/kT}d\epsilon-\int_0^{\infty}D_ce^{-[(1/kT)+(1/\omega)]\epsilon}d\epsilon.$$

The first integral has been evaluated above, and the second integral may be obtained from the first by replacing $1/kT$ with $1/kT+1/\omega$. Hence

$$i=Bk^2T^2e^{-\chi'/kT}\left\{\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n\mu+1}-\frac{1}{n\mu+1+(kT/\omega)}\right)\right.$$
$$\left.-\frac{W_a^{\frac{1}{2}}}{2kT}\left(1-\frac{1}{4x_0^2}W_a^2\right)\left[\frac{\cos u_1}{\{[\pi\alpha+(1/kT)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}-\frac{\cos u_2}{\{[\pi\alpha+(1/kT)+(1/\omega)]^2+\beta^2\alpha^2\}^{\frac{1}{2}}}\right]\right\}, \tag{30}$$

$$u_1=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}-\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}-\tan^{-1}\frac{\beta\alpha}{\pi\alpha+(1/kT)},$$

$$u_2=\frac{4\sqrt{2}}{3}x_0^{\frac{1}{2}}-\frac{2}{W_a^{\frac{1}{2}}}-\tan^{-1}\frac{W_a^{\frac{1}{2}}}{4}-\tan^{-1}\frac{\beta\alpha}{\pi\alpha+(1/kT)+(1/\omega)}.$$

(b) High Fields, Low Temperatures

For this case only electrons from region B need be considered. Here

$$D\sim\exp(-2Q);\quad Q=\int_{x_2}^{x_3}\left(W_a-W-\frac{1}{2x}-\frac{x}{2x_0^2}\right)^{\frac{1}{2}}dx. \tag{31}$$

The effect of the image force on electrons from region B is small. Hence, to evaluate Q it is legitimate to set $2Q=2\Theta Q_0$, where

$$2Q_0=2\int_{x_2}^{x_3}\left(W_a-W-\frac{x}{2x_0^2}\right)^{\frac{1}{2}}dx=\frac{8x_0^2}{3}(W_a-W)^{\frac{3}{2}}$$

is the exponent which is obtained in the transmission coefficient if the effect of the image force is neglected, and Θ is the correction function which has been used previously. The potential barrier used in the determination of Q_0 is shown in Fig. 4. Since the transmission coefficient decreases exponentially, the current vanishes when $\epsilon=W_i-W$ is large. Hence, Q_0 may be expanded in powers of ϵ:

$$Q_0\sim(4x_0^2/3)\chi^{\frac{3}{2}}+2x_0^2\chi^{\frac{1}{2}}\epsilon$$

and since

$$N(\epsilon)=B\left[\epsilon+kT\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n\epsilon/kT}\right],$$

$$i=B\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{\frac{3}{2}}\right]\left\{\int_0^{\infty}\epsilon\exp[-4\Theta x_0^2\chi^{\frac{1}{2}}\epsilon]d\epsilon+kT\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^{\infty}\exp\left[-\left(4\Theta x_0^2\chi^{\frac{1}{2}}+\frac{n}{kT}\right)\epsilon\right]d\epsilon\right\}$$

$$=B\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{\frac{3}{2}}\right]\left\{\frac{1}{16\Theta^2x_0^{\frac{3}{2}}\chi}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{k^2T^2}{n+4x_0^2\Theta\chi^{\frac{1}{2}}kT}\right\}, \tag{32}$$

or if χ is put in ev and F in volt/cm

$$i=\left\{1.55\times10^{-6}\frac{F^2}{\Theta^2\chi}+120.5T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{1}{n+8.81\times10^3(\Theta\chi^{\frac{1}{2}}T/F)}\right\}$$
$$\times\exp[-6.838\times10^7\Theta\chi^{\frac{3}{2}}/F]\ \text{amp/cm}^2. \tag{33}$$

(c) Intermediate Fields and Temperatures

If the fields and temperatures are such that $4x_0^2\Theta\chi^{1/2}kT<1$, the current comes from the lower part of C. Again if we take

$$D\sim\exp\left[-(8x_0^2/3)\Theta\chi^{3/2}+4Vx_0^2\chi^{1/2}\beta\right],\quad \beta=W-W_i$$

as above, and use

$$N(\beta)=BkT\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n\beta/kT}$$

there results

$$i=Bk^2T^2\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{3/2}\right]\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\int_0^{\infty}\exp-\left[4x_0^2\Theta\chi^{1/2}-(n/kT)\right]\beta d\beta$$

$$=Bk^2T^2\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{3/2}\right]\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{1}{n-4\Theta x_0^2\chi^{1/2}kT}. \tag{34}$$

By adding this current to that from region B, the total current for the condition $4x_0^2\Theta\chi^{1/2}kT<1$ is obtained

$$i=B\exp\left[-\frac{8x_0^2}{3}\Theta\chi^{3/2}\right]\left\{\frac{1}{16\Theta^2x_0^4\chi}+k^2T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{1}{n+4\Theta x_0^2\chi^{1/2}kT}-\frac{1}{n-4\Theta x_0^2\chi^{1/2}kT}\right)\right\} \tag{35}$$

or putting χ in ev and F in volt/cm

$$i=\left\{1.55\times10^{-6}\frac{F^2}{\Theta^2\chi}+120.54T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{1}{n+8.81\times10^3(\Theta\chi^{1/2}T/F)}+\frac{1}{n-8.81\times10^3(\Theta\chi^{1/2}T/F)}\right)\right\}$$
$$\times\exp\left[-6.838\times10^7(\Theta\chi^{3/2}/F)\right]\ \text{amp/cm}^2. \tag{36}$$

The current penetrating the very top of the barrier has been given in Part IV (28).

The photoelectric emission from C consists of electrons for which

$$D(W+h\nu,\ F)\sim\exp\left[-\frac{8\Theta x_0^2}{3}(\chi-h\nu)^{3/2}\right]-4\Theta x_0^2(\chi-h\nu)^{1/2}/\epsilon,$$

$$N(W)=B\left[\epsilon+kT\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n\epsilon/kT}\right].$$

Because of the large exponential in the transmission coefficient, the majority of the current comes from electrons for which $y=W_i-W$ is small; hence

$$i=B\exp\left[-\frac{8x_0^2\Theta}{3}(\chi-h\nu)^{3/2}\right]\left\{\frac{1}{16\Theta^2x_0^4\chi}+k^2T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{1}{n+4\Theta x_0^2(\chi-h\nu)^{1/2}kT}\right\} \tag{37}$$

or, using ev and volts per cm,

$$i=\left\{1.55\times10^{-6}\frac{F^2}{\Theta^2(\chi-h\nu)}+120.5T^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{1}{n+8.81\times10^3[\Theta(\chi-h\nu)^{1/2}T/F]}\right\}$$
$$\cdot\exp\left[-6.838\times10^7\frac{\Theta(\chi-h\nu)^{3/2}}{F}\right]\ \text{amp/cm}^2. \tag{38}$$