# On Tensor Forces and the Theory of Light Nuclei

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The quadrupole moment of the deuteron indicates the existence of non-central tensor forces in nuclei which destroy the constancy of the total orbital angular momentum. With simple operational representations of the wave functions, the influence of two-body tensor forces on the ground state eigenfunctions of the light nuclei $H^3$ and $He^4$ has been calculated. In $H^3$, the tensor forces directly couple to the fundamental $^2S_{\frac{1}{2}}$ state a $^4D_{\frac{1}{2}}$ state, which in turn interacts with $^2P_{\frac{1}{2}}$ and $^4P_{\frac{1}{2}}$. To the fundamental $^1S_0$ state of $He^4$ is admixed a $^5D_0$ state which is coupled by the tensor forces with $^3P_0$. All states consistent with the total angular momentum and parity conservation rules occur in the ground state eigenfunctions, and these nuclei therefore constitute the simplest examples of the complete breakdown of spin and orbital angular momentum conservation laws. Rarita and Schwinger have satisfactorily accounted for the properties of the deuteron by including the tensor force in a simple interaction represented by a rectangular well potential. With this interaction to describe the forces between all pairs of nuclear particles, the binding energies of $H^3$ and $He^4$ have been estimated by a variation method The trial functions are of the form $^2S_{\frac{1}{2}}+{}^4D_{\frac{1}{2}}$ for $H^3$ and $^1S_0+{}^5D_0$ for $He^4$, with Gaussian radial functions. The calculations yield 32 and 50 percent of the binding energy for $H^3$ and $He^4$, respectively, while a similar test calculation for the deuteron gives 54 percent of the binding energy. The probability that these nuclei are in a $D$ state is found to be 4 percent for all three nuclei, in agreement with the exact deuteron computations. Improvement of the radial dependence of the trial functions increases the estimated binding energy of the deuteron to 76 percent of the known value but does not materially affect either the estimated binding energies of $H^3$ and $He^4$, or the amount of $D$ state admixture of the three nuclei. An analysis of the results shows that the tensor forces, which produce all the binding in the deuteron, are relatively ineffective in binding $H^3$ and $He^4$. This apparently indicates that the assumption of ordinary and tensor forces of the same range is not adequate to represent the properties of $H^3$ and $He^4$.

## I. INTRODUCTION

THE theory of nuclei attempts to interpret nuclear properties in terms of two-body forces. Current nuclear theories further postulate equal interactions between all pairs of nuclear particles, and until very recently it has been customary in the theory of light nuclei to construct the interaction as a linear combination of Majorana, Heisenberg, Wigner, and Bartlett forces operating through similar potentials of the same range. That this is insufficient was, however, demonstrated by Rarita and Present,[1] and their conclusions substantiated by the analysis of proton-proton scattering data.[2] An interaction of the sort described, when fitted to represent the experimental binding energies of $H^2$ and $H^3$, as well as the cross section for slow neutron-proton scattering, will predict a binding energy of $He^4$ which is about 20 percent too large. The existence of the quadrupole moment of the deuteron has established that the neutron-proton interaction must involve tensor spin-orbit coupling terms of the form

$$S_{12}=\frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r}_{12})(\boldsymbol{\sigma}_2\cdot\mathbf{r}_{12})}{r_{12}{}^2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$

and spin dependent forces of this character are predicted by various current mesotron field theories of the neutron-proton interaction. More recently Rarita and Schwinger[3] have examined the possibility of representing the properties of the deuteron by means of the interaction operator

$$V=-\{1-\tfrac{1}{2}g+\tfrac{1}{2}g\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\gamma S_{12}\}J(r_{12}).\quad(1)$$

For the ground state of the deuteron, which is of even parity, the omission of space exchange operators from (1) is of no consequence, and therefore, for this state, $V$ is equivalent to the most general linear combination of Majorana, Heisenberg, Wigner, Bartlett, and tensor operators. Indeed, it has been shown that, with $J(r_{12})$ a square well of range $r_0=2.80\times10^{-13}$ cm and depth $V_0=13.89$ Mev, and $g=0.0715$ and

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[1] William Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937).
[2] G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. **55**, 1018 (1939).
[3] William Rarita and Julian Schwinger, Phys. Rev. **59**, 436 and 556 (1941).

$\gamma=0.775$, (1) will account for all the properties of the neutron-proton system which depend only on the neutron-proton forces.

The operator $S_{12}$ couples the spin and orbital motion of the particles, and is not invariant under separate rotation of space or spin coordinates. Its inclusion in the Hamiltonian therefore results in leaving as constants of the motion, available for the description of stationary states, in general only the parity, the total angular momentum $J$, and the total magnetic quantum number $m$. The eigenfunction of the ground state of a many-particle system will consequently contain terms corresponding to higher spins and orbital angular momenta than predicted in the absence of spin-orbit coupling, or, in other words, including $S_{12}$ in the neutron-proton interaction implies a breakdown of the previously assumed nuclear Russell-Saunders coupling. The extent of this breakdown is of immediate interest with reference to the magnetic dipole moments and other multipole moments of nuclei, as well as to the problem of selection rules in nuclear transmutations. In this paper we investigate the influence of interaction terms of the type $S_{12}$ on the eigenfunctions of the ground states of the light nuclei $H^3$ and $He^4$ by first, in Section II below, determining the types of states which may appear in these eigenfunctions, and then, in the last section, using the operator (1) with the constants determined by Rarita and Schwinger to estimate by a variational method the amount of admixture and binding which may be expected. As described in more detail below, our calculations indicate that only the $D$ state has an appreciable probability of being included in the ground states of $H^3$ and $He^4$ along with the $S$ state, and that the probability of finding either of these nuclei in a $D$ state is about four percent. Our variational estimates of the binding energies of $H^3$ and $He^4$ are rather low, and an analysis of our results shows that, with respect to their influence on the binding, the tensor forces are, in $H^3$ and $He^4$, relatively ineffective when compared with the deuteron. In the deuteron the tensor force is so effective that its introduction in sufficient quantity to give the known quadrupole moment of the deuteron reduces the magnitude of the ordinary interaction necessary to fit the binding energy with the range $2.80\times10^{-13}$ cm from 21.22 Mev to 13.89 Mev. This reduction in the magnitude of the ordinary interaction greatly diminishes the amount of $S$ state binding, and since the tensor forces do not seem to compensate for this diminution in $H^3$ and $He^4$, it follows that our low estimates of the binding energies of these nuclei probably indicate real deficiencies. In other words, the simple introduction of the $S_{12}$ coupling in a symmetric Hamiltonian in which all the forces are of equal range is apparently not yet adequate to reconcile the known properties of the two-particle system with the experimental mass defects of $H^3$ and $He^4$. The validity and limitations of these conclusions are discussed more fully in the last section.

## II. GENERAL FORMULAE

With the breakdown of spin and orbital angular momentum conservation, an eigenfunction of definite total angular momentum $J$ is most conveniently regarded as a mixture of many states classified in terms of their total spin and total orbital angular momentum. The classifications possible are just those consistent with the rules for compounding angular momenta, limited however by a definite maximum total spin. Thus in $H^3$ the ground state is $J=\frac{1}{2}$ and the maximum spin $\frac{3}{2}$, and therefore the eigenfunction is a mixture of ${}^2S_{\frac{1}{2}}$, ${}^2P_{\frac{1}{2}}$, ${}^4P_{\frac{1}{2}}$ and ${}^4D_{\frac{1}{2}}$ states. Similarly in $He^4$ the ground state eigenfunction is composed of ${}^1S_0$, ${}^3P_0$ and ${}^5D_0$ states. In a many particle system it is generally possible to form states of either parity and any total orbital angular momentum, so that in $H^3$ and $He^4$ the parity selection rule merely reduces the number of states in the eigenfunction included in any particular classification such as ${}^2P_{\frac{1}{2}}$ or ${}^3P_0$, but does not completely eliminate any classification. These nuclei are consequently more complicated than the simple deuteron system, in which the ground state, as in $H^3$ and $He^4$, is of even parity and the odd ${}^3P_1$ state is therefore absent from the eigenfunction. A further simplification in the deuteron which does not appear in $H^3$ and $He^4$ is the conservation of total spin which, for a two-particle system only, is a consequence of the symmetry of $S_{12}$ in the operators $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$.

The elucidation of these statements and the more detailed consideration of the states which actually appear in the eigenfunctions necessitate actually obtaining the wave functions of the states in question. These can be readily written down in an operational representation such as developed by Rarita and Schwinger[3] in the theory of the deuteron. We first introduce the usual relative coordinate systems for $H^3$ and $He^4$, defined by:

In $H^3$ $\quad \boldsymbol{\varrho}=\mathbf{r}_2-\mathbf{r}_1, \quad \mathbf{r}=\mathbf{r}_3-(\mathbf{r}_1+\mathbf{r}_2)/2.$

In $He^4$ $\quad \boldsymbol{\varrho}_1=\mathbf{r}_2-\mathbf{r}_1, \quad \boldsymbol{\varrho}_2=\mathbf{r}_4-\mathbf{r}_3,$

$$\mathbf{r}=\frac{\mathbf{r}_4+\mathbf{r}_3}{2}-\frac{\mathbf{r}_1+\mathbf{r}_2}{2}.$$

In both nuclei the subscripts 1 and 2 refer to neutrons, and 3 and 4 to protons. With the abbreviation $\boldsymbol{\sigma}_{ij}=(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_j)/2$, the wave functions of the states that can occur in the ground state eigenfunctions of $H^3$ and $He^4$ are, as explained below:

$H^3$:

$$^2S_{\frac{1}{2}}{}^m:\ \frac{1}{4\pi}\frac{1}{\sqrt{2}}(\chi_1{}^+\chi_2{}^--\chi_1{}^-\chi_2{}^+)\chi_3{}^m=\psi, \tag{2}$$

$$^2S_{\frac{1}{2}}{}^m:\ \frac{1}{\sqrt{3}}(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\psi, \tag{3}$$

$$\left.\begin{aligned} &^2P_{\frac{1}{2}}{}^m:\ \left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{\boldsymbol{\sigma}_3\cdot\mathbf{r}\times\boldsymbol{\varrho}}{r\rho}\psi,\\ &^2P_{\frac{1}{2}}{}^m:\ \frac{1}{\sqrt{2}}\frac{1}{r\rho}[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)\cdot\mathbf{r}\times\boldsymbol{\varrho}](\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\psi=\frac{1}{\sqrt{2}}\frac{1}{r\rho}\{\boldsymbol{\sigma}_{12}\cdot\mathbf{r}\times\boldsymbol{\varrho}+i\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_{12}\cdot(\mathbf{r}\times\boldsymbol{\varrho})\}\psi, \end{aligned}\right\} \tag{4}$$

$$^4P_{\frac{1}{2}}{}^m:\ \frac{1}{r\rho}\left\{\boldsymbol{\sigma}_{12}\cdot\mathbf{r}\times\boldsymbol{\varrho}-\frac{i}{2}\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_{12}\cdot(\mathbf{r}\times\boldsymbol{\varrho})\right\}\psi, \tag{5}$$

$$\left.\begin{aligned} &^4D_{\frac{1}{2}}{}^m:\ \left(\frac{3}{2}\right)^{\frac{1}{2}}\left\{\frac{(\boldsymbol{\sigma}_{12}\cdot\mathbf{r})(\boldsymbol{\sigma}_3\cdot\mathbf{r})}{r^2}-\frac{1}{3}(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\right\}\psi,\\ &^4D_{\frac{1}{2}}{}^m:\ \left(\frac{3}{2}\right)^{\frac{1}{2}}\left\{\frac{(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\varrho})(\boldsymbol{\sigma}_3\cdot\boldsymbol{\varrho})}{\rho^2}-\frac{1}{3}(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\right\}\psi,\\ &^4D_{\frac{1}{2}}{}^m:\ \frac{3}{2(5)^{\frac{1}{2}}}\frac{1}{r\rho}\{(\boldsymbol{\sigma}_{12}\cdot\mathbf{r})(\boldsymbol{\sigma}_3\cdot\boldsymbol{\varrho})+(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\varrho})(\boldsymbol{\sigma}_3\cdot\mathbf{r})-\tfrac{2}{3}(\mathbf{r}\cdot\boldsymbol{\varrho})(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\}\psi, \end{aligned}\right\} \tag{6}$$

$$^4D_{\frac{1}{2}}{}^m:\ \frac{15}{4(5)^{\frac{1}{2}}}\frac{1}{r^2\rho^2}\{(\boldsymbol{\sigma}_{12}\cdot\mathbf{r}\times\boldsymbol{\varrho})(\boldsymbol{\sigma}_3\cdot\mathbf{r}\times\boldsymbol{\varrho})-\tfrac{1}{3}(\mathbf{r}\times\boldsymbol{\varrho})^2(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\}\psi. \tag{7}$$

$He^4$:

$$^1S_0:\ \frac{1}{(4\pi)^{\frac{3}{2}}}\tfrac{1}{2}(\chi_1{}^+\chi_2{}^--\chi_1{}^-\chi_2{}^+)(\chi_3{}^+\chi_4{}^--\chi_3{}^-\chi_4{}^+)=\varphi$$

$$^1S_0:\ \frac{1}{\sqrt{3}}(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_{34})\varphi,$$

$$\left.\begin{aligned} &^3P_0:\ \left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{(\boldsymbol{\sigma}_{12}\cdot\mathbf{r}\times\boldsymbol{\varrho}_1)}{r\rho_1}\varphi\\ &^3P_0:\ \left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{(\boldsymbol{\sigma}_{34}\cdot\mathbf{r}\times\boldsymbol{\varrho}_1)}{r\rho_1}\varphi\\ &^3P_0:\ \left(\frac{3}{4}\right)^{\frac{1}{2}}\frac{\boldsymbol{\sigma}_{12}\times\boldsymbol{\sigma}_{34}\cdot(\mathbf{r}\times\boldsymbol{\varrho}_1)}{r\rho_1}\varphi \end{aligned}\right\}\text{etc.}$$

$$\left.\begin{aligned} &{}^5D_0: \left(\frac{3}{2}\right)^{\frac{1}{2}}\left\{\frac{(\boldsymbol{\sigma}_{12}\cdot\mathbf{r})(\boldsymbol{\sigma}_{34}\cdot\mathbf{r})}{r^2}-\tfrac{1}{3}(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_{34})\right\}\varphi,\\ &{}^5D_0: \frac{3}{2(5)^{\frac{1}{2}}}\frac{1}{r\rho_1}\{(\boldsymbol{\sigma}_{12}\cdot\mathbf{r})(\boldsymbol{\sigma}_{34}\cdot\boldsymbol{\varrho}_1)+(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\varrho}_1)(\boldsymbol{\sigma}_{34}\cdot\mathbf{r})-\tfrac{2}{3}(\mathbf{r}\cdot\boldsymbol{\varrho}_1)(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_{34})\}\varphi, \end{aligned}\right\}\text{etc.} \tag{8}$$

$$ {}^3P_0: \frac{3}{\sqrt{2}}\frac{1}{r^2\rho_1\rho_2}(\boldsymbol{\varrho}_1\times\boldsymbol{\varrho}_2\cdot\mathbf{r})(\boldsymbol{\sigma}_{12}\cdot\mathbf{r})\varphi, \text{ etc.} \tag{9}$$

The symbolic representations listed above may be derived with the aid of a systematic procedure. Thus, in $H^3$, we wish to describe states of even parity and $J=\frac{1}{2}$. In terms of the spin functions $\chi$ of the particles, a ${}^2S_{\frac{1}{2}}$ wave function of this character can at once be written down, and is just $\psi$ defined by Eq. (2). The magnetic quantum number $m$ of this state is, as indicated, determined by the magnetic quantum number assigned to $\chi_3$. All other wave functions of total angular momentum $J=\frac{1}{2}$ and the same $m$ can now be obtained by operating on $\psi$ by means of scalar functions of the spin vector operators and space vectors of the system, for such operators commute with $\mathbf{J}$. Such rotation invariant functions can only be formed by combining the scalars, vectors, and tensors formed from the spin operators with similar forms built from the position vectors, and because of the commutation properties of the spin operators, the number of independent invariants is actually quite limited. Since the total spin and total orbital angular momentum classification of any state are determined by its rotational properties with respect to spin and space separately, the representation of the wave functions of the even $J=\frac{1}{2}$ states of $H^3$ becomes a straightforward and simple problem. More specifically, as a consequence of the antisymmetry of $\psi$ in the neutron spins, $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\psi=0$, so that, with respect to operations on $\psi$, there are only two independent linear combinations of $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$, which we may choose symmetrically as $\boldsymbol{\sigma}_{12}$ and $\boldsymbol{\sigma}_3$. Therefore, with the exception of the identity operator, the only independent scalar spin operator will be $\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3$, and since this operator is also invariant with respect to space rotations, the wave function (3) will have the same transformation properties as $\psi$, that is, it will represent a ${}^2S_{\frac{1}{2}}$ state. In the same way there are three independent spin vectors, namely $\boldsymbol{\sigma}_{12}$, $\boldsymbol{\sigma}_3$, and $\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_{12}$, which may be combined with the vector $\mathbf{r}\times\boldsymbol{\varrho}$ to form the rotation invariant operators $\boldsymbol{\sigma}_{12}\cdot\mathbf{r}\times\boldsymbol{\varrho}$, $\boldsymbol{\sigma}_3\cdot\mathbf{r}\times\boldsymbol{\varrho}$, and $\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_{12}\cdot(\mathbf{r}\times\boldsymbol{\varrho})$. These operators transform like vectors, in other words, like the spherical harmonics of the first order, under space rotations, and, consequently, when applied to $\psi$, produce $P$ wave functions which, however, in general represent combinations of ${}^2P_{\frac{1}{2}}$ and ${}^4P_{\frac{1}{2}}$ states. To determine then the representations of the pure ${}^2P_{\frac{1}{2}}$ and ${}^4P_{\frac{1}{2}}$ states we employ the device of operating on the two ${}^2S_{\frac{1}{2}}$ wave functions, (2) and (3), with the operator $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)\cdot\mathbf{r}\times\boldsymbol{\varrho}$ which commutes with $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)^2$. We obtain in this way the two wave functions (4) which must have the same total spin as (2) and (3) and can therefore only describe pure ${}^2P_{\frac{1}{2}}$ states. The wave function (5) of the ${}^4P_{\frac{1}{2}}$ state is the linear combination of $\boldsymbol{\sigma}_{12}\cdot\mathbf{r}\times\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_{12}\cdot(\mathbf{r}\times\boldsymbol{\varrho})$ which is orthogonal to (4). That (5) really represents a quartet state may be verified by demonstrating that $\boldsymbol{\sigma}_1\ \boldsymbol{\sigma}_3$ has the eigenvalue 1 when operating on (5). The ${}^4D_{\frac{1}{2}}$ wave functions can be derived by combining the spin tensor $\boldsymbol{\sigma}_{12}\boldsymbol{\sigma}_3$ with the various tensors formed from the space vectors and applying the resultant rotation invariant operators to $\psi$. As before, this will in general furnish a mixture of the $J=\frac{1}{2}$ wave functions of $H^3$, but after the subtraction of the known (2) through (5), the remainder can only represent pure ${}^4D_{\frac{1}{2}}$ states. The necessity for this subtraction may, however, be avoided by making use of the principle that the components of the symmetric traceless tensor $x_iy_j+x_jy_i-\frac{2}{3}\delta_{ij}\sum_k x_ky_k$, $i$, $j$, $k=1$ to 3, transform like the spherical harmonics of the second order, if $\mathbf{x}$ and $\mathbf{y}$ are vectors. Consequently, since (6) and (7) represent $D$ states of $J=\frac{1}{2}$ they are necessarily ${}^4D_{\frac{1}{2}}$, and again it is easy to verify that $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)^2$ has the eigenvalue 15 in the states (6) and (7) and that (6) and (7) are all orthogonal to the wave functions (2) through (5).

Any of the wave functions (2) through (7) may be multiplied by a scalar function of $\mathbf{r}$ and $\boldsymbol{\rho}$ to give a new wave function of the same spectral classification, and it is possible to obtain additional orthogonal wave functions in this fashion. This possibility corresponds to the fact that in $H^3$, as in all many particle systems, there are many different ways of compounding any given value of the total orbital angular momentum out of the orbital angular momenta of the constituent particles. For example, $\psi$ represents a state in which the neutron system and the proton are both in $s$ states. $(\mathbf{r}\cdot\boldsymbol{\rho})\psi$ is a $^2S_{\frac{1}{2}}$ wave function in which the neutrons and the proton are both in $p$ states. In the eigenfunction $\psi$ will of course be multiplied by a scalar function whose angular dependence will determine the relative contribution of each different possible combination of neutron and proton motions to the singlet (in the neutron spins) $^2S_{\frac{1}{2}}$ state of $H^3$. Since in the $H^3$ relative coordinate system there are only three independent vectors, $\mathbf{r}$, $\boldsymbol{\rho}$, and $\mathbf{r}\times\boldsymbol{\rho}$, of which only $\mathbf{r}\times\boldsymbol{\rho}$ is invariant under spatial mirroring, it is quite evident from the foregoing discussion that, except for a multiplicative scalar factor, we have listed all the wave functions of $H^3$ of $J=\frac{1}{2}$ and even parity. As written, the wave functions (2) through (7) are normalized to unity with respect to the summation over all spin coordinates and integration over all angles in the relative coordinate system. Considerations very similar to the above pertain to the derivation of the wave functions of $He^4$ of $J=0$ and even parity. There is only the complication that several independent vectors of even parity are now available, thereby greatly increasing the number of different types of states. We have therefore only listed examples of the kinds of spin dependence possible, indicating by "etc." that additional wave functions may be obtained merely by substituting new vectors for those in the functions listed (i.e., by replacing $\boldsymbol{\rho}_1$ by $\boldsymbol{\rho}_2$, etc.). A further consequence of the increased number of space vectors is the existence of a pseudoscalar $\mathbf{r}\cdot\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2$ which converts wave functions of odd parity into even and vice versa. Thus the wave function (9) is a perfectly acceptable even $^3P_0$ state formed from the odd $(\boldsymbol{\sigma}_{12}\cdot\mathbf{r})\varphi$, and we could of course include in our list one such wave function corresponding to every odd state of $J=0$ of $He^4$. However, states of this character, involving the pseudoscalar $\mathbf{r}\cdot\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2$, in no way enter into the calculations of the following section, so that we have deemed it sufficient to write down the single example (9) and again indicate by "etc." that the list, for $He^4$, is not exhaustive. As in $H^3$ the wave functions are normalized to unity with respect to spin summation and angular integration.

In the absence of spin-orbit coupling the ground state of $H^3$ is $^2S_{\frac{1}{2}}$. The new term in the Hamiltonian is $\sum_{i>j}J(r_{ij})S_{12}(\mathbf{r}_{ij})$ summed over all pairs of particles. Applying this operator to a $^2S_{\frac{1}{2}}$ wave function will obviously produce only $^4D_{\frac{1}{2}}$ wave functions, and it is readily calculated that in $H^3$ $\sum_{i>j}J(r_{ij})S_{12}(\mathbf{r}_{ij})\psi$ or $\sum_{i>j}J(r_{ij})\times S_{12}(\mathbf{r}_{ij})(\boldsymbol{\sigma}_{12}\cdot\boldsymbol{\sigma}_3)\psi$ is a linear combination of the wave functions (6). The further application of $\sum_{i>j}J(r_{ij})S_{12}(\mathbf{r}_{ij})$ to the wave functions (6) produces a complicated mixture of terms among which are included the $P$ states (4) and (5), as well as the remaining $D$ state (7). In other words, in the ground state of $H^3$, to the first approximation of the perturbation theory, the effect of introducing $S_{12}$ coupling in the neutron-proton interaction is to admix the simpler $^4D_{\frac{1}{2}}$ states defined by (6) with the fundamental $^2S_{\frac{1}{2}}$ states. In second approximation all possible types of states appear. Once more very similar statements may be made concerning the ground state of $He^4$. The first effect of the $S_{12}$ coupling is to combine $^5D_0$ wave functions of the type (8), bilinear in the vectors $\mathbf{r}$, $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ with the fundamental $^1S_0$ wave functions. All other states with the exception of those which, like (9), involve the pseudoscalar $\mathbf{r}\cdot\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2$, are included in the eigenfunction through their interaction with the $^5D_0$ states. The states involving the pseudoscalar only appear in still higher approximation. Thus (9) is produced by the application of the Hamiltonian to the $^3P_0$ wave functions of the usual type, such as $(\boldsymbol{\sigma}_{12}\cdot\mathbf{r}\times\boldsymbol{\rho}_1)\varphi$. It is in performing the calculations implied by the statements of this paragraph and Section III below that the advantages of the operational representations we have given for the wave functions become apparent. The evaluation of any matrix element of the interaction becomes merely an exercise in the combination of spin operators, and it is never necessary to make use of the complicated reduction formulas of the rotation group.

### III. VARIATION CALCULATION

As stated in the introductory section, we have employed the interaction (1) with the constants determined by Rarita and Schwinger to estimate by a variational method the binding energies and the amount of admixture of the ground states of $H^3$ and $He^4$. The smallness of $g$ in (1) and the fact that only the states (6) are directly coupled to $\psi$ by $S_{12}$ then suggest that only the states (2) and (6) have an appreciable probability of being represented in the ground state of $H^3$. A suitable normalized, antisymmetric in the neutron coordinates, trial function for the variation calculation in $H^3$ is consequently

$$\Phi_1=\frac{N_S}{(1+C^2)^{\frac{1}{2}}}\exp[-\tfrac{1}{2}\mu(r_{12}^2+r_{13}^2+r_{23}^2)]\psi+\frac{CN_D}{(1+C^2)^{\frac{1}{2}}}\exp[-\tfrac{1}{2}\nu(r_{12}^2+r_{13}^2+r_{23}^2)]$$

$$\times(3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_3\cdot\boldsymbol{\rho})+3(\boldsymbol{\sigma}_1\cdot\boldsymbol{\rho})(\boldsymbol{\sigma}_3\cdot\mathbf{r})-2(\mathbf{r}\cdot\boldsymbol{\rho})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3))\psi$$

with

$$N_S=\frac{2(2\mu)^{3/4}(3/2\mu)^{3/4}}{\Gamma(3/2)},\quad N_D=\frac{2(2\nu)^{5/4}(3/2\nu)^{5/4}}{(20)^{\frac{1}{2}}\Gamma(5/2)},$$

$$r_{12}^2+r_{13}^2+r_{23}^2=2r^2+\tfrac{3}{2}\rho^2.$$

The energy is minimized, using (1), with respect to the parameters $\mu$, $\nu$ and $C$. This choice of trial function limits the orbital angular momenta of the constituent neutron and proton motions in the $S$ and $D$ waves to as small values as possible. That is, the $S$ wave, as stated in the previous section, represents a state in which the neutron system and the proton are both in $s$ states, in the $D$ wave the neutrons and the proton are in $p$ states. These are the states whose energies would be expected to lie lowest in a Hartree approximation. Similarly, in $He^4$ we choose the trial function

$$\Phi_2=\frac{N_S}{(1+C^2)^{\frac{1}{2}}}\exp[-\tfrac{1}{2}\mu(2\rho_1^2+2\rho_2^2+4r^2)]\varphi-\frac{CN_D}{(1+C^2)^{\frac{1}{2}}}\exp[-\tfrac{1}{2}\nu(2\rho_1^2+2\rho_2^2+4r^2)]$$

$$\times(3(\boldsymbol{\sigma}_1\cdot\boldsymbol{\rho}_1)(\boldsymbol{\sigma}_3\cdot\boldsymbol{\rho}_2)+3(\boldsymbol{\sigma}_1\cdot\boldsymbol{\rho}_2)(\boldsymbol{\sigma}_3\cdot\boldsymbol{\rho}_1)-2(\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3))\varphi,$$

$$N_S=\frac{2^{3/2}(2\mu)^{3/4}(2\mu)^{3/4}(4\mu)^{3/4}}{[\Gamma(3/2)]^{\frac{3}{2}}},\quad N_D=\frac{2^{3/2}(2\nu)^{5/4}(2\nu)^{5/4}(4\nu)^{3/4}}{(20)^{\frac{1}{2}}\Gamma(5/2)[\Gamma(3/2)]^{\frac{1}{2}}},$$

$$2\rho_1^2+2\rho_2^2+4r^2=r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2.$$

As demanded by the exclusion principle, $\Phi_2$ is antisymmetric with respect to the interchange of neutron or proton coordinates. It is readily verified that, in both $\Phi_1$ and $\Phi_2$, with our symmetric choice of Gaussian radial functions, the $D$ wave function must be restricted to the particular form we assumed in order that the exclusion principle be satisfied.

In order that we may have some estimate of the validity of our results, we have also performed a similar variation calculation for the deuteron, using as a trial wave function for the ground state

$$\Phi_3=\frac{N_S}{(1+C^2)^{\frac{1}{2}}}\exp(-\tfrac{1}{2}\mu\rho^2)\chi+\frac{CN_D}{(1+C^2)^{\frac{1}{2}}}\exp(-\tfrac{1}{2}\nu\rho^2)(3(\boldsymbol{\sigma}_N\cdot\boldsymbol{\rho})(\boldsymbol{\sigma}_P\cdot\boldsymbol{\rho})-\rho^2(\boldsymbol{\sigma}_N\cdot\boldsymbol{\sigma}_P))\chi,$$

$$N_S=\frac{\sqrt{2}(\mu)^{3/4}}{[\Gamma(3/2)]^{\frac{1}{2}}},\quad N_D=\frac{\nu^{7/4}}{2[\Gamma(7/2)]^{\frac{1}{2}}},\quad \boldsymbol{\rho}=\mathbf{r}_P-\mathbf{r}_N.$$

$\chi$ is the fundamental normalized $^3S_1$ state of the deuteron, defined analogously to $\psi$ and $\varphi$. The details of the calculation of the binding energies of these wave functions are quite standard and it is therefore sufficient to write down just the expressions for the binding energies in terms of the variational parameters:

$$\begin{aligned}
\mathrm{H}^3:\quad (1+C^2)\frac{E}{E_0} &= -3.662x^2+35.654m_2(x)-6.1033C^2y^2+9.60C^2m_2(y)-1.632C^2m_4(y) \\
&\quad -5.2800C^2m_6(y)+737.66C\frac{x^3y^5}{(x^2+y^2)^4}m_4\left[\left(\frac{x^2+y^2}{2}\right)^{\frac{1}{2}}\right], \\
\mathrm{He}^4:\quad (1+C^2)\frac{E}{E_0} &= -5.4932x^2+71.3088m_2(x)-7.9346C^2y^2+32C^2m_2(y)-3.264C^2m_4(y) \\
&\quad -3.52C^2m_6(y)+2409.16C\frac{x^{9/2}y^{13/2}}{(x^2+y^2)^{11/2}}m_4\left[\left(\frac{x^2+y^2}{2}\right)^{\frac{1}{2}}\right], \qquad (10) \\
\mathrm{H}^2:\quad (1+C^2)\frac{E}{E_0} &= -1.8310x^2+12.80m_2(x)-4.2725C^2y^2-7.0400C^2m_6(y) \\
&\quad +245.89C\frac{x^{3/2}y^{7/2}}{(x^2+y^2)^{5/2}}m_4\left[\left(\frac{x^2+y^2}{2}\right)^{\frac{1}{2}}\right].
\end{aligned}$$

In $\mathrm{H}^3$: $x=r_0(3\mu)^{\frac{1}{2}}$, $y=r_0(3\nu)^{\frac{1}{2}}$. In $\mathrm{He}^4$: $x=r_0(4\mu)^{\frac{1}{2}}$, $y=r_0(4\nu)^{\frac{1}{2}}$. In $\mathrm{H}^2$: $x=r_0(2\mu)^{\frac{1}{2}}$, $y=r_0(2\nu)^{\frac{1}{2}}$.

Here $E_0$ is the binding energy of the deuteron, 2.17 Mev, and $E$ the binding energy of the system associated with the wave function under consideration.[4] The expression on the right side of the equation, consequently, is positive when there is binding, and must be maximalized with respect to $C$, $x$, and $y$. This part of the computations must of course be performed numerically, and is greatly facilitated by the introduction of the functions $m_n(u)$, the incomplete normal moment functions, defined, for $n$ even, as[5]

$$m_n(u)=\frac{1}{2}\frac{\int_0^u dv v^n \exp\left(-\frac{1}{2}v^2\right)}{\int_0^\infty dv v^n \exp\left(-\frac{1}{2}v^2\right)}.$$

From (10), the best values of the binding energy, expressed for each of the nuclei in units of the binding energy of the deuteron, and the corresponding values of the parameters, are given in Table I. We have included in Table I the percent of actual binding to which our extremal energies correspond, assuming the binding energies of $\mathrm{H}^3$ and $\mathrm{He}^4$ to be 8.3 Mev and 28 Mev, respectively. The correct value of $C$ in the ground state of the deuteron is, according to the exact computations of Rarita and Schwinger, $C=0.197$. The calculations were performed by finding the best value of $C$ for any pair of values of $x$ and $y$, which measure the effective ranges of the $S$ and $D$ radial functions, and it is noteworthy that in all three nuclei

TABLE I.

| | $E$ | $x$ | $y$ | $C$ | Percent binding |
|---|---|---|---|---|---|
| $\mathrm{H}^3$ | 1.216 | 1.6 | 2.5 | 0.168 | 32 |
| $\mathrm{He}^4$ | 6.518 | 1.8 | 2.3 | 0.193 | 50 |
| $\mathrm{H}^2$ | 0.536 | 1.5 | 3.0 | 0.195 | 54 |

[4] We may remark that the forms of the expressions (10) show that with the introduction of tensor forces it is no longer possible to use the "equivalent" two-body method to estimate the binding energy.

[5] These functions are tabulated in the *Tables for Statisticians and Biometricians*, edited by Karl Pearson, Cambridge University Press (1914). With the aid of these tables it is possible to combine, in calculations of the sort described here, the advantages of Gaussian radial functions and square well potentials.

this best value of $C$ was almost independent of the choice of $x$ and $y$ in the neighborhood of the extremum, and differed surprisingly little from the extremal value for even quite bad choices of $x$ and $y$. Thus in $H^3$ for $x=2.2$ and $y=2.7$ the best value of $C$ is 0.222, although this set of values fails to give any binding by 1.156 deuteron energies. By actually evaluating the analytical expressions for $\partial E/\partial x$, $\partial E/\partial y$ and $\partial E/\partial C$ from (10) and showing that these derivatives passed through zero in the immediate neighborhood of the extremal points given in Table I, it was possible to establish definitely that we had found the extremum. The variation of $E$ with $x$ or $y$ is, as might be expected, a good deal more rapid than its variation with respect to $C$.

We have also attempted to estimate the effect on the extremal value of $C$ of improving the radial dependence of the trial functions. That is, in the trial functions $\Phi_1$, $\Phi_2$, and $\Phi_3$, we replace the single Gaussian radial function of the $S$ terms by a sum of two Gauss functions of the form $\exp(-\frac{1}{2}\mu(\sum r_{ij}^2))+A\exp(-\frac{1}{2}\tau(\sum r_{ij}^2))$, suitably normalized, and then, retaining the previous values of $x$ and $y$ from Table I in the trial function, varying with respect to $A$, $\tau$, and $C$. The new positions of the extremum, with the values of $x$ and $y$ fixed as in Table I, are listed in Table II. $z$ is defined like $x$ and $y$; $z=r_0(3\tau)^{\frac{1}{2}}$ in $H^3$, $z=r_0(4\tau)^{\frac{1}{2}}$ in $He^4$, $z=r_0(2\tau)^{\frac{1}{2}}$ in $H^2$. Finally, in the deuteron, we replaced the $D$ radial function by the normalized form $\exp(-\frac{1}{2}\nu(\sum r_{ij}^2))+B\exp(-\frac{1}{2}\omega(\sum r_{ij}^2))$ and, with $x$, $y$, $z$, and $A$ as in Tables I and II, varied with respect to $B$, $C$, and $\omega$. The extremal values were $E=0.759$, $C=0.19$, $B=0.01$.

The results of this section seem to indicate that, as a result of the tensor spin-orbit forces $S_{12}$, the probability of finding either of the nuclei $H^3$ or $He^4$ in a $D$ state is about four percent. This conclusion is supported by the very good agreement between the calculated and exact value of $C$ in the deuteron, and by the small variation of the extremal value of $C$ either with improvement of the trial functions or changes in $x$ and $y$. We may note, however, that our attempts to improve the radial functions were much more effective in the deuteron than in either of the other two nuclei. These "radial" functions are, as we have pointed out,

TABLE II.

| | $E$ | $C$ | $A$ | $z$ | Percent binding |
|---|---|---|---|---|---|
| $H^3$ | 1.477 | 0.15 | 0.05 | 0.8 | 39 |
| $He^4$ | 6.653 | 0.18 | 0.01 | 1.0 | 52 |
| $H^2$ | 0.742 | 0.17 | 0.10 | 0.6 | 74 |

angular dependent in $H^3$ and $He^4$, and the lack of comparable (to the deuteron) improvement in these nuclei is apparently due to our not having taken this angular dependence into account. As a matter of fact, it is actually true that in the deuteron a sum of two Gauss functions is a very good approximation to the $S$ radial function, whereas in view of its omission of angular terms, such a form must be inadequate in $H^3$ and $He^4$. This inadequacy is probably more important, in explaining the relatively low binding obtained for these nuclei, than our failure to include in the trial functions those states interacting only with the $D$ waves, and, in $He^4$, the states like (9) involving the pseudoscalar. Because of the smallness of the $D$ state probability, these other states will appear with still smaller probabilities, and their neglect will not be a bad approximation. We have nevertheless quite a lot of binding to account for, in $H^3$ and $He^4$, and it is questionable whether improvement in the trial functions is all that is necessary. In this connection we may list the relative amounts of binding produced by the various terms in $C$ in (10). That is, we may rewrite (10) as

$$\frac{E}{E_0}=\frac{S}{1+C^2}+\frac{DC^2}{1+C^2}+\frac{IC}{1+C^2},$$

where, for each nucleus, $S/(1+C^2)=S'$ is the binding of the $S$ wave, $DC^2/(1+C^2)=D'$ the binding of the $D$ wave, and $IC/(1+C^2)=I'$ the interaction energy of the two waves. Then, for the parameters given in Table I, with $E$ as usual in units of $E_0$ we have the values given in Table III.

Table III clearly shows that the $D$ wave, which by its interaction with the $S$ wave, produces all the binding in the deuteron, is much less effective in binding the heavier $H^3$ and $He^4$. The 35 percent reduction in the depth of the ordinary potential well which the presence of the tensor force requires in the deuteron, reduces

Table III.

| | $S'$ | $D'$ | $I'$ | $E$ |
|---|---|---|---|---|
| $H^3$ | 0.166 | −0.983 | 2.033 | 1.216 |
| $He^4$ | 4.976 | −1.076 | 2.618 | 6.518 |
| $H^2$ | −1.022 | −1.508 | 3.066 | 0.536 |

by an even larger percentage the $S$ wave binding in $H^3$ and $He^4$, because this binding is the small difference between the kinetic and potential energies of the $S$ state. Consequently, unless the variation approximation is, in $H^3$ and $He^4$, very much worse for the $D$ than for the $S$ wave, the binding energies of these nuclei predicted by (1) must be less than the known experimental values. All these results have of course been obtained with the interaction in the form (1); the justifications for its use are its success in the deuteron system and its simplicity, which reduces the amount of calculation required. This simplicity consists largely in the omission of space exchange operators, a matter which is of no consequence in the deuteron, but can influence the energies of the states in $H^3$ and $He^4$. However, the substitution of exchange forces for the ordinary forces in (1) can only reduce the binding. Furthermore, the neglected forces are known to be principally of the Majorana type, and because of the symmetry of the $S$ waves in our trial functions, the substitution of Majorana forces for the ordinary forces in (1) can only cause a reduction in the $D$ wave binding of our trial function, i.e., a decrease in $D'$ in Table III. These considerations just correspond to the well-known fact that in the calculation of the binding energies of $H^3$ and $He^4$ in the absence of spin-orbit coupling forces, the substitution of Majorana for ordinary forces is a good approximation. In other words, while the omission of space exchange forces in (1) may affect our numerical value of the amount of $D$ state admixture in $H^3$ and $He^4$, it seems to be a justifiable conclusion from our calculations that a simple linear combination of the $S_{12}$, Majorana, Heisenberg, Wigner, and Bartlett forces operating through similar potentials of the same range in a symmetric Hamiltonian probably cannot account for slow neutron-proton scattering, the properties of the deuteron, and the binding energies of $H^3$ and $He^4$.

We wish to stress that, because of the incomplete character of the variation calculation, the conclusions of this section must be considered somewhat speculative. These conclusions could have been made more precise by carrying the variation to a definite completion, but in view of the fundamental theoretical uncertainty in the very form of the neutron-proton interaction, the amount of computational labor which such a calculation would require would scarcely be justified. Our results support the view that the successful introduction of the tensor forces into the two-body interaction will require the assumption of different ranges for the spin-orbit coupling and conservative forces. There exist, however, limitations on the range of the tensor force,[6] and the possibility that many-body forces may be necessary cannot be overlooked. We should like finally to express our thanks to Professor J. R. Oppenheimer for his interest in this work.

[6] J. Schwinger, Phys. Rev. **60**, 164A (1941).