interloper among the $5d^96p$ levels, but is distinguished from them by intensity anomalies as well as by its failure to fit into the theoretical structure of the latter group. This term has therefore been assigned to $5d^86s6p$, as this configuration should be the only other group of odd levels in the neighborhood. None of the other states of $5d^86s6p$ has yet been identified.

In the spectrum of Au II, twelve new terms have been found, and four previously suggested terms have been eliminated (not including the states proposed by Rao). This brings the total number of known terms in Au II to twenty-six even and nineteen odd, and it brings the total number of lines assigned in Au to 203, of which 120 had been previously classified.

Some 1500 of the lines measured in this work remain unclassified. It is hoped that it will be possible in the near future, by use of these lines, to extend further the analysis of Au I and Au II.

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A Statistical Analysis of the Earth's Internal Magnetic Field

WALTER M. ELSASSER
*Blue Hill Meteorological Observatory of Harvard University, Milton, Massachusetts**

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The earth's internal magnetic field consists of the field of a large dipole and an irregular part which is of the order of a few percent of the main dipole field. The field may be represented by a series of spherical harmonics and 42 coefficients are numerically known. The coefficients which characterize the irregular part of the field are here statistically compared with a simple model for this field. It is assumed that an arbitrary number of dipoles are thrown at random inside a spherical shell and mean values are calculated for the probabilities with which the various spherical harmonics appear in the field produced. It is shown that a statistical agreement between the observed and calculated coefficients can be obtained only if the outer radius of the spherical shell is approximately $0.50R$ and is definitely not in excess of the radius of the earth's core, $0.55R$ (R, earth's radius). The result is independent of the number of dipoles producing the field. It seems that if the latter number is not very large, the result does not depend too critically upon the assumption of completely random distribution of the dipoles.

THE gradually increasing evidence[1] for the existence inside the earth, of a liquid core, (radius 3500 km $=0.55R$), presumably made up of molten metals, mainly iron, gives a new stimulus to investigations into the origin and causes of the earth's magnetic field. On the basis of this evidence one may gain a better understanding of the secular variations of the field. These variations, when considered from the point of view of a geological time scale, are exceedingly rapid, so rapid indeed that in the course of a few centuries the non-dipole part of the field will have completely changed its aspect. Changes of this magnitude are hardly conceivable inside a solid body. Now it is found that if the electric conductivity of the core is of an order comparable to that of ordinary metals, the interaction between the field and any fluid masses moving in the core must be intense. Induction currents must be set up and their field will superpose itself upon any field originally present. For various reasons we are inclined to think that the entire field originates inside the core, while the relative importance of the currents set up by the dynamo action of moving masses and of the primary currents caused by electromotive forces of a still unidentified nature is perhaps difficult to estimate.

In the present paper we shall not deal with these dynamical problems of a rather intricate nature, but shall confine ourselves to a question of a much more simple character. It is well known that by a potential analysis of the earth's magnetic field one can divide it into an internal and an external field, the former of which has its

* Part of this investigation was carried out at the California Institute of Technology.

[1] B. Gutenberg, ed. *Internal Constitution of the Earth*, (McGraw-Hill Company, 1939).

source regions entirely inside and the latter entirely outside the earth's boundary. The internal field makes up about 98 percent of the total field. The application of the rigorous machinery of potential theory does not lead to any further information about the sources of the field; it seems interesting, however, to see whether, under reasonably simple assumptions, additional information may be gained regarding the seat of the magnetic source regions. It will be shown here that if very simple assumptions of a statistical character are made, it follows from the observational evidence that the source regions of the field must be located inside the boundary of the core.

We shall make use of the very complete harmonic analysis carried out by Ad. Schmidt[2] in 1895 and referring to the epoch 1885. The coefficients of the development of the internal field in a series of spherical harmonics are given in Table I. The spherical harmonics are normalized, the figures given in Table I are relative values putting the main term (1,0) equal to 10,000. The first line for each m represents the cos terms, the second the sin terms. The signs indicated are the opposite of those given by Schmidt who, in agreement with the ordinary electromagnetic convention, counts the main dipole negative. The coefficients of development as given in Table I were obtained by Schmidt through the use of values of the magnetic field strength read from magnetic charts at 1800 points which are the intersections of a series of meridians with a series of circles of latitudes. Both series progress in steps of 5°, the circles of latitudes going from the equator to lat. $\pm 60°$.

The major part of this field may be represented by a dipole whose axis is inclined by about $11\frac{1}{2}°$ with respect to the earth's axis. The magnitude and inclination of the dipole is determined by the three first-order harmonics. One can moreover reduce the magnitude of the second-order harmonics by shifting this dipole to a position about 340 km off the center of the earth while leaving its direction unchanged.[3] This operation, introducing 3 additional parameters, does not completely eliminate the 5 quadrupole components and seems to have little physical

TABLE I.

m	$l=1$	2	3	4	5	6
0	10000	128	−193	−145	28	−3
1	742	−690	254	−93	−64	−3
	−1886	175	61	−51	44	−14
2		−165	−295	−94	−47	8
		−366	−5	31	5	−9
3			−82	56	−3	25
			−141	41	−3	10
4				−29	1	3
				13	13	4

meaning, especially since Table I does not show any discontinuity between the quadrupole and the higher order moments. On the other hand, there is a certain discontinuity between the dipole and quadrupole terms, as we shall see later on. In this paper, if we speak of the main dipole, we mean a *centered* dipole, either the (1,0) component alone or the inclined dipole determined by the three first-order harmonics.

Recently McNish[4] has shown by a method of empirical trials that the field which remains after the subtraction of the excentric dipole may be approximated by the combined fields of 14 small dipoles located somewhat arbitrarily at a depth of one-half of the earth's radius. Each of these dipoles has a strength of about 1 percent of the main dipole and radial direction; some of them have positive, some negative signs.

The statistical problem which we propose to treat here is allied to this last method, although more general. Let N dipoles be placed at points which are distributed at random inside a spherical shell; we may then ask for the probability of finding any given value for the (l, m)th harmonic component of the field outside the shell. We shall later make a number of appropriate assumptions regarding the magnitude and direction of these dipoles.

Before proceeding to analyze this problem we shall introduce a few definitions of which we shall have to make frequent use. Let $x_1, x_2 \cdots x_n$

[2] Ad. Schmidt, Munich Akad. Abh. **19**, 1–66 (1895).
[3] H. Bartels, Terr. Mag. **41**, 225 (1936).
[4] A. G. McNish, Trans. Am. Geophys. Union **2**, 287 (1940).

be N quantities; we define the linear, quadratic and generally, the kth order mean by

$$[x_1\cdots x_n]_1=N^{-1}(x_1+x_2+\cdots x_n),$$
$$[x_1\cdots x_n]_2=N^{-\frac{1}{2}}(x_1^2+x_2^2+\cdots x_n^2)^{\frac{1}{2}}, \quad (1)$$
$$[x_1\cdots x_n]_k=N^{-1/k}(x_1^k+x_2^k+\cdots x_n^k)^{1/k}.$$

These formulae are connected with the ordinary definition of the kth moment M_k of a statistical distribution by

$$[x_1\cdots x_n]_k=(M_k)^{1/k}.$$

If x is a continuous rather than a discrete variable and if its density distribution is $\rho(x)$, we can generalize (1) by putting

$$[x]_k=\left\{\int x^k\rho dx\right\}^{1/k}\left\{\int \rho dx\right\}^{-1/k}. \quad (2)$$

One Dipole

Let us now consider one dipole which is thrown at random into a spherical shell concentric with the earth. Let x, y, z be the coordinates of the dipole where the z axis coincides with the earth's axis, and let r, ϑ, φ be the corresponding polar coordinates. Further, let R, θ, ϕ be the polar coordinates of a point at the earth's surface. If, first, a *unit monopole* is located at r, ϑ, φ, we have for the potential at the outside:

$$U_0=\sum_{l=0}^{\infty}(r^l/R^{l+1})P_l(\cos A), \quad (3)$$
$$\cos A=\cos\vartheta\cos\theta+\sin\vartheta\sin\theta\cos(\varphi-\phi).$$

We introduce now the addition theorem of the spherical harmonics. It is convenient, in this connection, to use the normalized spherical harmonics defined by

$$Y_l^0=(4\pi)^{-\frac{1}{2}}(2l+1)^{\frac{1}{2}}P_l(\cos\vartheta),$$
$$\left.\begin{matrix}Y_l^{mc}\\ Y_l^{ms}\end{matrix}\right\}=(2\pi)^{-\frac{1}{2}}(2l+1)^{\frac{1}{2}}\times\left\{\frac{(l-m)!}{(l+m)!}\right\}^{\frac{1}{2}}P_l(\cos\vartheta)\left\{\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}\right..$$

The addition theorem reads then

$$\frac{2l+1}{4\pi}P_l(\cos A)=Y_l^0(\vartheta,\varphi)Y_l^0(\theta,\phi)+\sum_{m,c}^{m\leq l}Y_l^{mc}(\vartheta,\varphi)Y_l^{mc}(\theta,\phi), \quad (4)$$

where the summation index c indicates that the sum is to be extended over both the cos and the sin terms. The potential of a *unit dipole* with its axis parallel to the z axis is now according to (3) and (4),

$$U_z=\sum_{l,m,c}c_l^{mc}Y_l^{mc}(\theta,\phi), \quad (5)$$

where the sum symbols have been abbreviated in an easily understandable way, and where:

$$c_l^{mc}=4\pi(2l+1)^{-1}R^{-l-1}(\partial/\partial z)r^lY_l^{mc}(\vartheta,\varphi). \quad (6)$$

Correspondingly we have for dipoles with axes parallel to the x and y axis

$$U_x=\sum_{l,m,c}a_l^{mc}Y_l^{mc}(\theta,\phi),$$
$$U_y=\sum_{l,m,c}b_l^{mc}Y_l^{mc}(\theta,\phi) \quad (7)$$

with

$$a_l^{mc}=4\pi(2l+1)^{-1}R^{-l-1}(\partial/\partial x)r^lY_l^{mc}(\vartheta,\varphi),$$
$$b_l^{mc}=4\pi(2l+1)^{-1}R^{-l-1}(\partial/\partial y)r^lY_l^{mc}(\vartheta,\varphi). \quad (8)$$

In order to calculate the expressions (6) and (8), it is convenient to introduce complex spherical harmonics which are connected with the real harmonics introduced above by

$$\sqrt{2}\,Y_l^{\pm m}=Y_l^{mc}\pm iY_l^{ms}\quad (m\neq 0), \quad (9)$$

while for $m=0$ they reduce to the Y_l^0 given above. The factor $\sqrt{2}$ is here necessary in order that the Y_l^m may be normalized in the complex, Hermitian sense if the real harmonics are normalized in the ordinary sense. Now we have[5]

$$(\partial/\partial z)r^lY_l^m=f_l^m(2l+1)^{\frac{1}{2}}r^{l-1}Y_{l-1}^m,$$
$$(\partial/\partial x\pm i\partial/\partial y)r^lY_l^m=\mp g_l^{\pm m}(2l+1)^{\frac{1}{2}}r^{l-1}Y_{l-1}^{m\pm 1}, \quad (10)$$

where

$$f_l^m=\{(l-m)(l+m)/(2l+1)^2(2l-1)\}^{\frac{1}{2}},$$
$$g_l^m=\{(l-m)(l-m-1)/(2l+1)^2(2l-1)\}^{\frac{1}{2}}. \quad (11)$$

From these relations the values of the coefficients (6) and (8) may immediately be obtained by passing from the complex to the component real quantities. We need not write down the explicit formulae.

[5] See for instance H. Bethe, *Handbuch der Physik*, Vol. 24, I, p. 558.

We shall later on be interested in the second-order mean, as defined by (2) of the coefficients (6), taken over the interior of the spherical shell inside of which our dipole is located. Let V designate the volume of the shell and r_0 and r_1 its outer and inner radius; we find from (6) and (10) on carrying out the integration over the shell:

$$[c_l^{mc}]_2=\left\{V^{-1}\int(c_l^{mc})^2dv\right\}^{\frac{1}{2}}$$
$$=4\pi V^{-\frac{1}{2}}f_l^mR^{-l-1}(r_0^{2l+1}-r_1^{2l+1})^{\frac{1}{2}} \quad (12)$$

and similar, somewhat more complex, expressions for the a_l^{mc} and b_l^{mc}.

Now the quantity (12) has a very simple statistical meaning. If a dipole is thrown into the spherical shell and falls upon the point r, ϑ, φ then c_l^{mc} represents the amplitude with which the (l, m)th spherical harmonic appears in the potential produced by the dipole. Hence we may presume that $[c_l^{mc}]_2$ is the quadratic mean, (the square root of the second moment) of the probability distribution produced by throwing the dipole at random into the spherical shell. We shall now justify this presumption.

Suppose we want to determine the probability of finding a coefficient between c and $c+dc$. This probability is equal to $\delta v/V$ where δv is that part of the volume in which the dipole must be located in order to produce a (l, m)th harmonic component of the potential with a value between c and $c+dc$. Hence it may be seen that c is the independent and δv the dependent variable of the problem. To determine the probability distribution means to find δv as function of c. We write δv rather than dv in order to indicate that δv is not necessarily an exact differential. Let us illustrate these statements by an example. If we seek the probability of an almost vanishing amplitude c_l^{mc} of the (l, mc)th harmonic, the corresponding volume will consist of a number of narrow strips which run along the nodal surfaces subtended in the spherical shell by the solutions of the equation $Y_l^{mc}(\vartheta, \varphi)=0$. For any other value of c_l^{mc} the corresponding δv will consist of a number of similar strips. It is difficult to find the analytical form of this probability distribution; however, it is now evident from (2) that

$$[c_l^{mc}]_k=\left\{V^{-1}\int(c_l^{mc})^kdv\right\}^{1/k}$$

represents the kth mean of this distribution. We may write here dv under the integral in place of δv, since the way in which the elements δv are summed up does not influence the result of the integration.

It is readily seen that the first-order moments of all the c, and a, b, vanish, because the integrals over the corresponding spherical harmonics vanish, excepting only c_1^0, and also a_1^{1c} and b_1^{1s}. The quadratic means of the c are given by (12) and similar formulae for the a and b. We shall further use the quadratic means of several, say n, coefficients, that is expressions of the form

$$\left\{n^{-1}V^{-1}\sum^{n}\int c^2dv\right\}^{\frac{1}{2}}.$$

In particular, we shall obtain the means of those $(2m+1)$ coefficients belonging to a definite l for which m does not exceed a given value. Using (12) we find

$$[c_l^0,\ \cdots c_l^{mc},\ c_l^{ms}]_2$$
$$=4\pi V^{-\frac{1}{2}}F_l^mR^{-l-1}(r_0^{2l+1}-r_1^{2l+1})^{\frac{1}{2}}, \quad (13)$$

where by means of the relation

$$\sum_1^m m^2=m(m+1)(2m+1)/6$$

we have from (11)

$$F_l^m=[f_l^0,\ 2f_l^1,\ \cdots 2f_l^m]_2$$
$$=\{l^2-m(m+1)/3\}^{\frac{1}{2}}(2l+1)^{-1}(2l-1)^{-\frac{1}{2}}. \quad (14)$$

In an entirely analogous way we find from (8), (9), (10) and (11) after some calculations that the quadratic mean of the $2(2m+1)$ coefficients a_l^{mc}, b_l^{mc} for which m does not exceed a certain value is given by

$$[a_l^0,\ \cdots a_l^{mc},\ a_l^{ms},\ b_l^0,\ \cdots b_l^{mc},\ b_l^{ms}]_2$$
$$=4\pi V^{-\frac{1}{2}}G_l^mR^{-l-1}(r_0^{2l+1}-r_1^{2l+1})^{\frac{1}{2}} \quad (15)$$

with

$$G_l^m=[G_l^{-m},\ \cdots G_l^0,\ \cdots G_l^m]_2$$
$$=\{l(l-1)+m(m+1)/3\}^{\frac{1}{2}}$$
$$\times(2l+1)^{-1}(2l-1)^{-\frac{1}{2}}. \quad (16)$$

TABLE II.

l	1	2	3	4	5	6
F_l^2	0.192	0.163	0.169	0.157	0.145	0.135
F_l^4	0.192	0.163	0.143	0.128	0.130	0.126
G_l^2	0.272	0.231	0.181	0.157	0.142	0.131
G_l^4	0.272	0.231	0.202	0.181	0.156	0.140
$0.98F_l^2+0.20G_l^2$	0.242	0.206	0.202	0.185	0.170	0.158
$0.98F_l^4+0.20G_l^4$	0.242	0.206	0.180	0.161	0.158	0.151
F_l^{rad}	0.192	0.179	0.162	0.148	0.137	0.128

If our dipoles are directed radially rather than parallel or perpendicular to the z axis we shall get formulae analogous to those given above; we merely have to replace $\partial/\partial z$ in (6) by $\partial/\partial r$. The formation of the quadratic means may be carried out in exactly the same way as before and formula (13) remains valid if now we put in place of F_l^m the quantity

$$F_l^{\text{rad}} = l(2l+1)^{-\frac{1}{2}}, \tag{17}$$

which is independent of m.

Let us finally consider a dipole of arbitrary direction s where

$$\partial/\partial s = \gamma\partial/\partial z + \alpha\partial/\partial x + \beta\partial/\partial y.$$

If we now write for the potential of this dipole

$$U = \sum_{l,m,c} d_l^{mc} Y_l^{mc}(\theta, \phi) \tag{18}$$

we have for the quadratic mean of any one of the coefficients:

$$[d_l^{mc}]_2 = 4\pi(2l+1)^{-1}R^{-l-1}V^{-\frac{1}{2}} \times\left\{\int dv\left[\left(\gamma\frac{\partial}{\partial z}+\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)r^l Y_l^{mc}\right]^2\right\}^{\frac{1}{2}}.$$

If we carry out the square under the integrand we may verify from (9) and (10) that the integrals over the cross-products vanish. Hence we are left with the integral over the sum of three squares. Let us furthermore assume that α and β are themselves statistical variables, that is to say that, while we keep the z component of the dipole constant, we admit all directions of the components perpendicular to the z axis and average over them. Since under these conditions

$$[\alpha]_2 = [\beta]_2 = (1-\gamma^2)^{\frac{1}{2}}$$

we obtain

$$[d_l^{mc}]_2 = \gamma[c_l^{mc}]_2 + (1-\gamma^2)^{\frac{1}{2}}[a_l^{mc}, b_l^{mc}]_2. \tag{19}$$

SEVERAL DIPOLES

Assume now that N equal dipoles are thrown into our shell simultaneously and independently of each other. First, let their axes all be parallel to the z axis. The coefficient c_l^{mc} in the development (5) will then be replaced by

$$c_l^{mc}(1) + c_l^{mc}(2) + \cdots c_l^{mc}(N).$$

If now we form the means of the corresponding probability distribution, it is seen again that all the linear means vanish with the exception of c_1^0 (the means of a_1^{1c} and b_1^{1s} do not vanish, but they are made to vanish by averaging over all directions of the dipoles perpendicular to the z axis). Forming the quadratic means we will have in place of (12)

$$[c_l^{mc}(1) + \cdots + c_l^{mc}(N)]_2 = 4\pi N^{-\frac{1}{2}} V^{-N/2} f_l^m R^{-l-1} \cdot\left\{\int dv_1 \int dv_2 \cdots \int dv_N [r_1^{2l-2} Y_{l-1}^{mc}(\vartheta_1\varphi_1) + \cdots r_N^{2l-2} Y_{l-1}^{mc}(\vartheta_N\varphi_N)]^2\right\}^{\frac{1}{2}}.$$

If the square under the integral is carried out, the integrals over all the cross-products will vanish; the squares, of which there are N, are equal to each other. Hence it follows that

$$[c_l^{mc}(1) + \cdots + c_l^{mc}(N)]_2 = [c_l^{mc}]_2, \tag{20}$$

where the right-hand side refers to the case of a single dipole. In exactly the same way one can generalize (13) and (15) and one finds that the equality (20) holds not only for a single coefficient like c_l^{mc} but for the quadratic mean of any number of them. Hence, in all cases, the quadratic means of the probability distribution for N dipoles are identical with the corresponding means for one dipole. The same will be true for the more general coefficients d_l^{mc} of (19).

Let us next assume that the N dipoles are not all alike in magnitude, but have dipole moments $\mu_1, \mu_2, \cdots \mu_N$ which we assume as statistically

TABLE III.

l	1		2		3		4		5		6
obs. Q_l^2	1433		370		196		92		43		8.5
Ratio		0.26		0.53		0.47		0.47		0.20	
obs. Q_l^4	1433		370		177		73		32		11
Ratio		0.26		0.48		0.41		0.44		0.34	

independent of the locations of these dipoles. As, in virtue of the last assumption, the probability distribution is a product of the probability for the d_l^{mc} and of the probability for the μ, we have for the quadratic mean:

$$[d_l^{mc}(1), \cdots d_l^{mc}(N)]_2 = [\mu_1, \cdots \mu_N]_2[d_l^{mc}]_2. \quad (21)$$

Comparison with Observation

We are now ready to compare our formulae with the results of the harmonic analysis of the actual field as given in Table I. As the figures of this table are ratios of the coefficients to the main dipole, we shall divide our calculated quadratic means by the coefficient of this dipole. If the latter is located at the earth's center and has the strength M, its potential is

$$MR^{-2}P_1(\cos\theta) = (4\pi/3)^{\frac{1}{2}}MR^{-2}Y_1^0.$$

Now divide (21) by the coefficient of the main dipole and write

$$Q_l^m = (4\pi/3)^{-\frac{1}{2}}M^{-1}[\mu_1 \cdots \mu_N]_2[d_l^{mc}]_2. \quad (22)$$

Assume first that the spherical shell degenerates into a complete sphere. We have then from (13), (15), (19), (20), (21), and (22):

$$Q_l^m = 3M^{-1}[\mu]_2(r/R)^{l-1} \times(\gamma F_l^m + (1-\gamma^2)^{\frac{1}{2}}G_l^m) \quad (23)$$

and if the spherical shell degenerates into a thin shell of thickness Δr

$$Q_l^m = 3M^{-1}[\mu]_2(r/R)^{l-1}\{(2l+1)\Delta r/r\}^{\frac{1}{2}} \times(\gamma F_l^m + (1-\gamma^2)^{\frac{1}{2}}G_l^m). \quad (24)$$

It is assumed in these formulae (and we shall justify the assumption presently) that the dipoles are directed preferentially along the z axis, γ being the mean directional cosine with respect to the z axis. The components perpendicular to the z axis are assumed to have random directions.

The factors F_l^m and G_l^m, defined by (14) and (16), and a linear combination $0.98F_l^m + 0.20G_l^m$ are given in Table II for $m=2$ and $m=4$. Whenever l is smaller than the chosen value of m, we put $m=l$ in (14) or (16). The factor (17) for radial dipoles is also given.

In order to compare our formulae with the results of the harmonic analysis of the actual field as laid down in Table I, we give, in Table III, the quadratic means of a number of observed coefficients with the same l. The first line gives the mean of the coefficients up to and including $m=2$, while the third line gives the quadratic mean of all the coefficients in one column of Table I. The second and fourth lines give the ratios of successive coefficients in the preceding lines.

Table IV.

$(l+1)/l$	2/1	3/2	4/3	5/4	6/5
$m \leqq 2$	0.30	0.54	0.51	0.51	0.22
$m \leqq 4$	0.30	0.55	0.46	0.45	0.36
radial	0.28	0.53	0.45	0.48	0.36

There is also visible, in Table I, a decrease of the coefficients downwards, i.e., with increasing m for constant l, although this feature is not very marked. Formula (11) shows that for dipoles parallel to the c axis the coefficients, for fixed l, decrease with increasing m, while for dipoles perpendicular to the z axis they increase with increasing m. If we form a linear combination of both we can only permit a rather small admixture of perpendicular components if we want to retain a decrease of the coefficients with increasing m. Although the statistical evidence of Table I is hardly in itself adequate to justify any very definite conclusion, the result seems plausible from the physical viewpoint. The linear combination which has been used in Table II involves a mean directional cosine corresponding to a deviation of $11\frac{1}{2}°$ from the earth's axis.

Going back now to our formulae (23) and (24) we have

$$Q_{l+1}^m / Q_l^m = C \cdot r/R \quad (25)$$

where C is a numerical factor. In the case given by (23), (full sphere) C is equal to the ratio of two successive figures in one line of Table II. We may effect a comparison of (25) with observation by substituting, on the left-hand side of (25), the ratios of the observed quadratic means as given in the second and forth line of Table III. If we use the numerical factors resulting from the last three lines of Table II we obtain the values for r/R which are contained in Table IV. They are rather uniform and although there is perhaps not much sense in taking a mean of them, it appears that the true value of r/R is near 0.5 and is definitely smaller than $r/R=0.55$ which is the boundary of the core. In the case of a thin spherical shell given by (24) the resulting values of r/R are smaller by about 10 percent.

The figures in the first column of Table IV are too small in comparison to the other figures. They have of course a physical sense only if it is assumed that the dipole components perpendicular to the z axis are due to the superposition of the auxiliary dipoles rather than to an inclination of the main dipole. One cannot, however, judge these figures without a knowledge of the absolute number of dipoles which cooperate in creating the irregular field and this number does not appear in Eq. (25). Later on we shall try to determine the number of dipoles.

We might now ask whether our formulae would be essentially modified if the dipoles were not distributed at random, but with a certain regularity, approximating perhaps the structure of a lattice. Qualitative arguments might be advanced which indicate that there is not much difference between the two cases, that a formula similar to (25) with a C near unity should also hold if a certain degree of regularity prevails. Indeed, each individual dipole produces coefficients, c_l^{mc}, etc. and their magnitude is limited to a definite interval of values. If several dipoles are involved, a single coefficient can become large only if the dipoles are distributed inside the shell according to a pattern which has the same type of symmetry as the spherical harmonic in question. We can hardly expect this singular condition to be fulfilled for several harmonics simultaneously, except perhaps for a set of them having for instance the same m and different values of l. There is, however, no sign, in Table I, of a preponderance of any one coefficient or of any limited set of coefficients. We might infer that a pronounced "resonance" of the actual distribution with any particular spherical harmonic of second or higher degree, improbably already *a priori*, is not indicated by the empirical findings. Excluding the case of "resonance," we might expect that the quadratic means produced by a distribution of dipoles endowed with some degree of regularity do not differ widely from the means produced by a random distribution. This argument will apply when the number N of dipoles involved is moderate; its application becomes doubtful when the number of dipoles is very large. It is indeed clear from a purely mathematical viewpoint that one can always represent the observed field as produced by a magnetically polarized spherical shell, and when the polarization as function of ϑ, φ is properly chosen, the shell may be located at any depth, outside or inside the core. The distribution of polarization in the shell, however, will then have very little resemblance with a random distribution of dipoles.

If the number of dipoles is large, a model involving extensive areas of magnetization seems appropriate. It may be shown that in this case, again, the magnetized regions are, if not below, at least not much above the boundary of the core.

Assume for a moment that a number of elementary dipoles are located at the surface of the earth. All the mean coefficients Q_l will then be of the same magnitude and the harmonic series diverges. If there is, instead, a number of magnetized areas at the surface, the series will converge slowly, the ratio of means of the successive coefficients being of the order $Q_{l+1}/Q_l=l/(l+1)$. Similarly, if the magnetized areas are located at the surface of a sphere of radius r, this ratio might be presumed to be of the order $Q_{l+1}/Q_l=(r/R)\cdot l/(l+1)$. If on the left-hand side of this relation the figures of Table III are inserted, we obtain larger values of r/R than before, the average r/R as computed from the ratios 3/2, 4/3, and 5/4 being about 0.65. The values of r/R obtained are however not nearly as consistent as those in Table IV. The true model of the sources of the irregular field is probably intermediate between the two models considered, the purely statistical model on the one side and a model with relatively large magnetized areas on the other. The evidence of Table IV points towards the correctness, or near correctness, of the statistical model treated in this paper. Speaking generally we might conclude that, except for highly artificial models, the sources of the irregular part of the internal field must be either very near the boundary of the core or below it.

TABLE V.

l	1	2	3	4	5	6
parallel z	4.5	8.4	7.6	8.3	9.3	12.9
radial	4.0	7.3	6.9	7.6	8.0	10.9

APPENDIX

One might consider the earth's main dipole from two different viewpoints. One can assume that it is physically a single dipole and then treat the remaining irregular field as made up of a number of small additional dipoles. One can also suppose that the main dipole itself results from the superposition of a number of dipoles aligned mainly parallel to the earth's axis. This hypothesis would seem acceptable if one could assume that the irregular part of the field is produced by the same dipoles. We may show that such a picture is at least not inconsistent with our statistical results. Let all dipoles be inside a sphere of radius r. On the assumption that the perpendicular components are small we have for the resultant magnetic moment

$$M=\mu_1+\cdots\mu_N=N[\mu]_1. \tag{26}$$

The dipole part of the field is the same as if all the dipoles were concentrated in the center. Indeed, from (6) and (10) we have on forming a linear mean

$$N[c_1{}^0]_1=(4\pi/3)^{\frac{1}{2}}MR^{-2}$$

and this is identical with the expression for the field of the main dipole used before. Introducing (26) into (23) we obtain

$$N[\mu]_1/[\mu]_2=(r/R)^{l-1}(\gamma F_l{}^m+(1-\gamma^2)^{\frac{1}{2}}G_l{}^m)/Q_l{}^m. \tag{27}$$

On the right-hand side we may substitute the numerical values obtained above. Put $r/R=0.50$. For the other factor in the numerator we may use the values contained in the last two lines of Table II, and for the denominator we might use the figures in the third line of Table III. We obtain then the values for the quantity (27) given in Table V. The ratio of the linear to the quadratic means depends upon the distribution in magnitude of the dipole moments. If the moments show very little scatter in magnitude around a mean, we have nearly $[\mu]_1=[\mu]_2$. As another extreme we consider a Poisson probability distribution

$$p(\mu)=e^{-a\mu}/a.$$

We find in this case

$$[\mu]_1/[\mu]_2=2^{-\frac{1}{2}}=0.71.$$

The true ratio will almost certainly lie between these limits and, hence, N will be somewhat larger than the values of Table V, in the neighborhood of ten, say. Again, the numbers for $l=1$ are smaller than the other figures, but since they are based on two observed components only they can hardly be given a statistical significance and are carried here merely for the sake of completeness.

The number of ten dipoles obtained in this way is smaller than the number of fourteen dipoles used by McNish[4] to represent the irregular field. It is noteworthy that the latter figure has been obtained from the irregular field alone while the values in Table V result from a comparison of the magnitude of the higher order harmonics with the main dipole. Whether the discrepancy is due to the crudeness of the assumptions involved in all of these models or whether it represents an argument against a representation of the main dipole field by a number of smaller components is difficult to decide in view of the very limited size of the statistical data.