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The Alpha-Model of Nuclear Structure, and Nuclear Moments

D. R. INGLIS
Johns Hopkins University, Baltimore, Maryland
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A criticism has been directed against the plausibility of the alpha-model, based on the contention that the first-order interaction between alphas is a repulsion. If this were true, the second-order attraction would have to be more than strong enough to overcome it and would be expected to distort and mix the alphas, making the alpha-model implausible even as a fair approximation. The contention is based on the conventional assumption of nuclear forces with several exchange terms and a single range parameter, and is here controverted by means of a calculation based on a more nearly satisfactory assumption of nuclear forces with fewer terms and two range parameters. This sample interaction has more "tail" than has the conventional interaction (a non-exchange tail), and leads to an adequate first-order attraction between alphas. The alpha-model facilitates a qualitative understanding of several well-known regularities among observed nuclear moments. The low degree of degeneracy of neutron states in an alpha-framework may be associated with the existence of surprisingly many cases in which the addition of two neutrons to a nucleus does not appreciably alter its magnetic moment. This, and the expectation that the lack of complete rigidity of an alpha-framework inhibits its rotation, may bring it about that the orbital moment of the odd-proton nuclei is essentially due to protons, in keeping with the observed trend of the magnetic moments. The occurrence of large positive electric quadrupole moments only in the neighborhood of the rare earths has been related to the shape of an alpha-framework, and to the participation of only one or very few particles in the orbital motion.

OF the various proposed methods of approximating the problem of nuclear structure, the alpha-model is perhaps the most plausible.[1–5] The only serious objections which have appeared against its plausibility have been based on a very specialized assumption about nuclear forces.[6,7] This form of the forces—the familiar "Majorana-Heisenberg-Wigner-Bartlett" scheme with a single range parameter—has enjoyed such a

[1] E. Wigner, (a) Proc. Nat. Acad. Sci. 22, 662 (1936); (b) Phys. Rev. **51**, 106, 947 (1937); (c) cf. also W. H. Barkas, Phys. Rev. **55**, 691 (1939).
[2] (a) G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936); (b) E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937); (c) E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937); (d) M. E. Rose and H. A. Bethe, Phys. Rev. **51**, 205 (1937).
[3] (a) W. Wefelmeier, Zeits. f. Physik **107**, 332 (1937); (b) J. A. Wheeler, Phys. Rev. **52**, 1083 (1937), especially pp. 1086–8 where it is shown that the simple alpha-model is probably a good representation of the more general method of resonating group structure for the low excited states (below about $5mc^2$), *a fortiori* for the ground states in which we are most interested.
[4] (a) L. R. Hafstad and E. Teller, Phys. Rev. **54**, 681 (1938); (b) cf. also H. Brown and D. R. Inglis, Phys. Rev. **55**, 1182 (1939).
[5] (a) N. Bohr, Nature **143**, 330 (1939); Phys. Rev. **55**, 418 (1939); (b) N. Bohr and J. A. Wheeler Phys. Rev. **56**, 426 (1939).
[6] (a) B. O. Grönblum and R. E. Marshak, Phys. Rev. **55**, 229 (1939). Cf. also (b) H. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 168 (1936) for an introductory discussion based partly on the earliest preliminary work in this field by W. Heisenberg, Zeits. f. Physik **96**, 473 (1935).
[7] H. Margenau, Phys. Rev. **59**, 37 (1941).

vogue, because of its alleged simplicity, that it was mentioned only parenthetically in reference 6(a), though essential to the conclusions thereof. This form of the forces has always been subject to serious doubt as more than a very rough approximation, the more so in the light of the possibility of meson forces. The objections to the alpha-model are raised directly against the use of the alpha-model in calculating relative binding energies, and only by implication against the association of the alpha-model with a nuclear structure important to nuclear moments and the like. The implication is, however, rather strong, and its denial is the principal purpose of this paper. The objections are based on the tenet that the first-order interaction of two alphas is repulsive.[6,7] In this case, the binding of alphas into nuclei must arise from second-order forces (of no greater range) strong enough to overcome the repulsion. The direction objection is then that such forces would not exhibit the simple additivity employed in reference 4. The implication arises from the circumstance that second-order interactions are accompanied by a distortion of the unperturbed state of the system. If the second-order forces must be more than strong enough to provide the binding, it seems likely that the distortion would be so great that the alphas in a heavy nucleus would not retain even a nebulous identity. In the next four sections we shall show that a different sample force assumption, which is more closely compatible with proton scattering and with the meson theory, gives an *attraction* between alphas in first order.

I. Heavy-Particle Interactions

The suggestions of the meson and other field theories of nuclear forces still leave us in the midst of an empirical search for a formulation of nuclear forces which is satisfactory for as many purposes as possible. One incentive for the search is the hope that its result may provide either a suggestion of the direction in which field theory should develop, or a criterion to distinguish the correct field theory of the forces from the rest. Short of this aim, it may still be very useful as a guide toward crucial experimentation. A very important step in the empirical selection of the most satisfactory forces is the analysis of proton-proton scattering data. This has been carried out by Breit and his successive co-workers,[8] using the data of Tuve, Heydenberg and Hafstad, and of Herb, Kerst, Parkinson and Plain.[9] Three of the assumed forces on the basis of which they carried out their analysis are

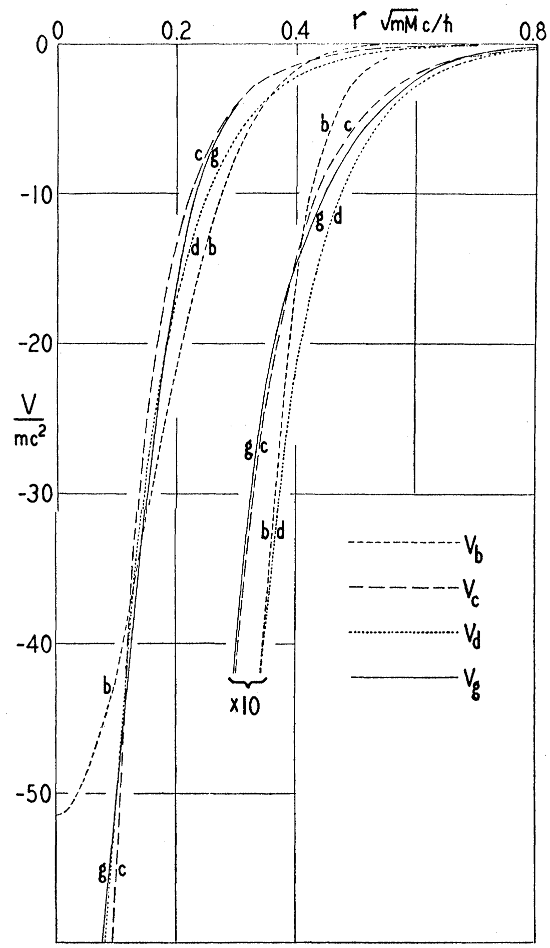

Fig. 1. Possible heavy-particle interactions in the ${}^1S$ state: a meson potential $V_c = -(89.65mc^2)e^{-r/a}a/r$ with $a = 0.42e^2/mc^2 = 0.131\hbar(mMc^2)^{-\frac{1}{2}}$; the exponential potential $V_d = -(137.6mc^2)e^{-2r/b}$ with $b = 0.193\hbar(mMc^2)^{-\frac{1}{2}}$; the Gauss error potential $V_b = -(51.44mc^2)e^{-\alpha r^2}$ with $\alpha = 21.59mMc^2/\hbar^2$; and the double-range potential V_g given by Eq. (1).

[8] (a) G. Breit, E. U. Condon, and R. D. Present, Phys. Rev. **50**, 842 (1936); (b) G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. **55**, 1018 (1939); (c) L. E. Hoisington, S. S. Share, and G. Breit, Phys. Rev. **56**, 884 (1939).

[9] (a) M. A. Tuve, N. P. Heydenberg, and L. R. Hafstad, Phys. Rev. **49**, 402 (1936); **50**, 806 (1936); **51**, 1023 (1937); **53**, 239 (1938); (b) R. G. Herb, D. W. Kerst, D. B. Parkinson, and G. J. Plain, Phys. Rev. **55**, 998 (1939).

shown as V_c, V_d, and V_b of Fig. 1. The deviations of the phase shifts in the case of V_d and V_b are rather definitely outside the range of plausible error, and it may be considered that the analysis, as carried out by Hoisington, Share, and Breit,[8(c)] leads to agreement in the case of V_c only, according to an opinion expressed by Professor Breit.[10] This arises from the fact that the nature of the radial dependence (such features as the ratio of "tail" to "body") determines essentially the curvature of the phase shift as a function of energy, so that a fit cannot be brought about for V_d and V_b by a mere alteration of the parameters.[10]

For the purpose of calculating other nuclear properties, the meson potential V_c is rather awkward, since its use would at best lead to a great deal of numerical integration. Yet we would like, in what follows, to adopt a potential which is in agreement with the scattering data, and which may, in spite of drastic simplifications for the sake of practicability, retain some of the qualitative characteristics of the meson force. In particular we wish to retain the "tail," which seems to be the most important feature in what follows, while attaining saturation entirely by exchange, without tensor forces. We retain the tail, without losing the possibility of analytic integration, by means of the device of approximating the radial dependence V_c which was successful in the scattering problem by a *superposition of two Gauss error curves*, one for the "body" and one for the "tail." This double Gauss error curve, with the parameters chosen to fit V_c reasonably closely through almost all of its range, is

$$V_g = -70mc^2 \exp(-\alpha r^2) - 6mc^2 \exp(-\alpha' r^2) \quad (1)$$

with

$$\alpha = 45mMc^2/\hbar^2 \quad \text{and} \quad \alpha' = 9mMc^2/\hbar^2.$$

It is compared with V_c, as well as with V_b and V_d, in Fig. 1. The part of the potential at very small r (where the function $F^2E^{-\frac{1}{2}}$ plotted in Fig. 4 of reference 8(c) is both small and almost independent of E) is relatively unimportant in the scattering. Furthermore, the meson theory is most ambiguous about the forces in this region of small r—no cut-off is required, and none has been introduced, in the scattering analysis, although a cut-off is required in other problems.[11] It is seen that V_g differs from V_c more than does V_d only in this presumably less important region. Through the main part of the range of r, V_g differs from the meson potential V_c much less than do the others, and it wavers on both sides so that the differences could be expected partially to annul one another. Since the other potentials V_b and V_d can only be excluded after considering the last refinements of accuracy of the scattering data, it seems almost certain, without repetition of the rather lengthy analysis, that the scattering data would not exclude V_g.

V_g in (1) is our assumed ^1S interaction. The more general interaction between two particles, from which (1) follows, which we assume for the purposes of calculation is

$$\begin{aligned} V &= -(A_q P^q + A_\sigma P^\sigma) \exp(-\alpha r^2) - A' \exp(-\alpha' r^2) \\ &= -(105mc^2 P^q + 35mc^2 P^\sigma) \exp(-\alpha r^2) - 6mc^2 \exp(-\alpha' r^2), \quad (2) \end{aligned}$$

with $\alpha = 45mMc^2/\hbar^2$ and $\alpha' = 9mMc^2/\hbar^2$ as before. Here P^q is the space-exchange operator ("Majorana") and P^σ is the spin-exchange operator ("Bartlett"). It will be appreciated that (2), in spite of having two ranges, is somewhat simpler than the four-operator interaction which has often been assumed. As stated above, we introduce the exchange operators principally for the sake of saturation, expecting that this will in some way correspond to the saturation properties of the actual (perhaps partly tensor) interaction. The saturation conditions are derived from consideration of collapsed nuclei, in which the range plays no role, so they apply to (2) as well as to a single-range interaction. The principal saturation conditions[12] are in our case

$$\begin{aligned} A^q &\geqslant 2A^\sigma + 4A', \\ A^q &\geqslant 2A^\sigma + 2A'. \end{aligned} \quad (3)$$

They are satisfied by (2) as inequalities with a

[10] G. Breit, Washington Physics Colloquium, February 26, 1941.

[11] H. A. Bethe, Phys. Rev. **57**, 260, 390 (1940) *et al.*

[12] (a) G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936); (b) N. Kemmer, Nature **140**, 192 (1937); (c) Cf. also G. Breit and E. Wigner, Phys. Rev. **53**, 998 (1938).

TABLE I. *Deuteron and alpha-energies.*

	DEUTERON	ALPHA
Energy due to V	$-3._2\ mc^2$	$-51\ mc^2$
Discrepancy with observed energy (allowed for admixture)	$-mc^2$	$-\ 3._7\ mc^2$

considerable margin. It has been customary to assume that the saturation conditions should be satisfied as equalities or near-equalities, following a suggestion[13] based on the desirability of obtaining as much Li^6 binding as possible with the usual *single-range* interaction. There is thus no reason why the double-range interaction (2) should be made to satisfy the conditions (3) as near-equalities. The greater freedom of choice of parameters thus afforded seems desirable, and is perhaps an advantage of the form (2) over the usual form of interaction—it seems unlikely that the actual interactions, which we hope to approximate, would happen to correspond to such stringent conditions as (3) as equalities, or even as near-equalities. Something corresponding to this greater freedom of choice is probably also afforded by the tensor nature of the forces, which is expected to depress the Li^6 energy more than the alpha-energy by intermixing states. With most interactions which, like (2), are more than saturated, the alpha-model appears to be almost essential in interpreting the linear trend of nuclear binding energies.

The choice of the parameters in V has been further limited by the requirement that it shall give a satisfactory value of the deuteron binding and of the alpha-binding. Though the interaction which V is intended to approximate is probably tensorial, V can reproduce only its direct contributions to the energy of these simple systems, not the indirect contributions arising from admixture of states of differing L. We therefore make an arbitrary, or very roughly estimated, allowance for the indirect contributions. Since the energies are quite sensitive to the choice of parameters, we need not demand great accuracy. The energies derived from V, and the allowances to be made for admixture of states in order to give the observed binding, are listed in Table I. The allowances for admixture are, in order of magnitude at least, compatible with indications from the deuteron quadrupole moment.[11]

The energy due to V is in each case estimated by means of perturbation theory. In the case of the deuteron, the convergence question is more serious[14] (because of the greater penetration of the barrier), and attainment of a significant result in the double-range calculation depends on comparison between perturbation and exact results in the single-range case. Warren and Margenau[15] have carried out the second-order Schroedinger perturbation theory of the deuteron for a single-range interaction equivalent to (2) with $A'=0$. Using a rather long-range interaction with $\alpha=20mMc^2/\hbar^2$, they found that the difference between $-E$, the binding energy obtained from the assumed interaction by a correct computation, and $-E^{(1+2)}$, the binding energy in second order, is about mc^2. The difference is to be attributed to the higher orders of the perturbation calculation. Its magnitude depends on how poorly the zero-order functions fit the actual functions. The fit is worse when there is much penetration of the barrier, so one may expect the higher-order contribution to be larger for shorter range of interaction and for smaller binding energy $-E$. The extent of these variations of the higher-order contributions may be judged by the three cases[16] listed in Table II. One sees that the dependence on E is less important than the range dependence, and that we should attribute about $2mc^2$ of the deuteron binding to higher orders of the perturbation theory when using the interaction (2), of which the dominant term is quite short-ranged. A second-order perturbation calculation similar to that sketched in the appendix of reference 15, but generalized to accommodate the double-range interaction (2), leads to the energy of

[13] D. R. Inglis, Phys. Rev. **51**, 531 (1937).

[14] For this reason we disagree with the remarks, toward the end of p. 1028 of the otherwise admirable paper quoted below (reference 15), criticizing alpha-results on the basis of deuteron behavior.

[15] D. T. Warren and H. Margenau, Phys. Rev. **52**, 1027 (1937).

[16] The values of E are taken from E. Feenberg and J. K. Knipp, Phys. Rev. **48**, 906 (1935), Table I, and from E. Feenberg and S. S. Share, Phys. Rev. **50**, 253 (1936), Table II.

the deuteron in second order:

$$E^{(1)}+E^{(2)}=(9\hbar^2/8M)\sigma\alpha -A\{1-3/[4(\sigma+1)]\}[\sigma/(\sigma+1)]^{\frac{3}{2}} -A'\{1-3/[4(s+1)]\}[s/(s+1)]^{\frac{3}{2}} -(M/2\hbar^2\alpha\sigma)\{A^2[\sigma/(\sigma+1)]^3 \times[a-\log(1+1/a)+\log 2-1] +2AA'[\sigma/(\sigma+1)]^{\frac{3}{2}}[s/(s+1)]^{\frac{3}{2}} \times[b-\log(1+1/b)+\log 2-1] +A'^2[s/(s+1)]^3[a'-\log(1+1/a') +\log 2-1]\}, \quad (4)$$

where $a=(\sigma+1)/[(\sigma+1)^2-1]^{\frac{1}{2}}$, $a'=(s+1)/[(s+1)^2-1]^{\frac{1}{2}}$, $s=\sigma\alpha/\alpha'$, and $b=\{(\sigma+1)(s+1)/[(\sigma+1)(s+1)-1]\}^{\frac{1}{2}}$. The second-order contributions of the term in A' are here, of course, relatively small, with an interaction such as (2) which has the term in A' considerably weaker and longer-ranged than that in A. Indeed, in the limit of very long range of the term in A'—that is, very large s—the second-order and higher-order contributions of this term vanish, since the addition of a constant potential merely depresses the energy-level scheme by a constant amount (A') without altering the wave functions. Because of the weakness and long range of the term in A' we may expect its effect on the rate of convergence to be slight. Evaluation of (4) with the values of the parameters given in (2) leads to the minimum value $E^{(1+2)}=-1.24mc^2$, the minimizing value of σ being 0.26. This and $-2mc^2$ just attributed to higher orders give the value in Table I.

The alpha is a much more compact nucleus, with comparatively little penetration of the barrier, and for it a perturbation theory[17] based on Hermite functions is very suitable.[13] The second-order contribution is only about $2mc^2$ out of a total $55mc^2$ of alpha-binding.[13]

The first-order energy of the low ^1S state ("ground state") of the alpha with the double-range interaction (5) is

$$E_\alpha=(9\hbar^2/4M)\alpha\sigma-6A_q[\sigma/(\sigma+2)]^{\frac{3}{2}} -6A'[s/(s+2)]^{\frac{3}{2}}+\tfrac{1}{4}(\alpha\sigma\hbar^2/mMc^2)^{\frac{1}{2}}mc^2. \quad (5)$$

With the parameters given in (2), this has a minimum $E_\alpha=-48.3mc^2$ at $\sigma=1.6$. A complete calculation of the alpha-binding with this interaction might then be expected to give about $51mc^3$. The allowance for admixture (Table I) is then relatively small, as seems compatible with the high excitation of other states of the alpha.[18] Even if this allowance might turn out to be considerably too small,[19] the result indicates that the simple interaction (2) is reasonably adequate for our purposes.

[17] W. Heisenberg, Zeits. f. Physik **96**, 473 (1935).

II. Interaction of Two Alphas

The analysis, along molecular lines, of the problem of two interacting alphas, each treated by use of oscillator functions as above, has been carried out by Margenau,[7] in part following an earlier treatment of Heisenberg.[17] A single-range interaction was used, of course, and the result obtained was a repulsion at all distances, as has already been mentioned. We shall now generalize the analysis to encompass the double-range interaction (2). The generalization is entirely straightforward. We still have the relation $\bar{V}=12\bar{V}_{34}+16\bar{V}_{48}$, where $\bar{V}$ is the average potential energy of the system. In our case, the typical average like-particle interaction (in first order) is

$$\bar{V}_{34}=(A_q-A_\sigma)\int\psi\exp(-\alpha r_{34}^2)P^\sigma\psi d\tau -A'\int\psi\exp(-\alpha' r_{34}^2)\psi d\tau \quad (6)$$

and an average unlike-particle interaction is

$$\bar{V}_{48}=-A_q\int\psi\exp(-\alpha r_{48}^2)P^q\psi d\tau -\tfrac{1}{2}A\int\psi\exp(-\alpha r_{48}^2)\psi d\tau -A'\int\psi\exp(-\alpha' r_{48}^2)\psi d\tau. \quad (7)$$

The wave function ψ of the system is the same as before, a determinant of single-particle functions $u(r)$ of position relative to one center and $v(r-R)$ of position relative to a second center, with a normalization factor $4!/(1-\delta^2)^2$. The single-particle function $u(r)$ is $(\sigma\alpha/\pi)^{\frac{3}{4}}\exp(-\sigma\alpha r^2/2)$, an s wave function with an oscillator po-

TABLE II. *Rapidity of convergence as a function of range $\alpha^{-\frac{1}{2}}$ and deuteron ^3S energy, E.*

$\alpha\hbar^2/(mMc^2)$	A/mc^2	E/mc^2	$E^{(1+2)}/mc^2$	$(E-E^{(1+2)})/mc^2$
30	117	−4	−2.67	−1.33
30	114	−3.5	−2.10	−1.40
44.44	162.6	−4	−2.00	−2.0

[18] D. R. Inglis, Phys. Rev. **55**, 988 (1939).

[19] E. Gerjuoy and J. Schwinger, Phys. Rev. **60**, 158A, (1941)

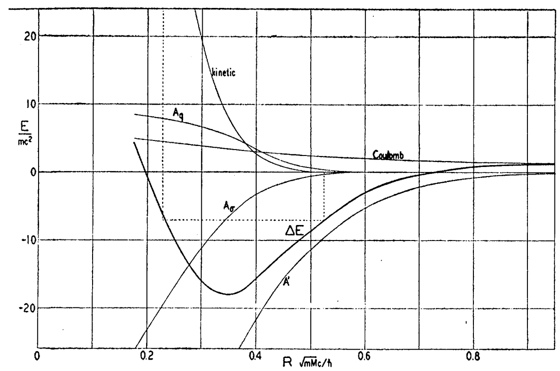

FIG. 2. The interaction of two adjacent alphas, ΔE, and the terms of which it is composed [in Eq. (9)] as functions of the inter-alpha-distance R.

tential, and the "overlapping integral" δ is $\exp(-\sigma\alpha R^2/4)$. The integrations in (6) and (7) which involve α rather than α' are carried out by Margenau in Eqs. (14) to (18) of reference 7, and are stated in terms of the parameters σ and $\rho=\alpha^{\frac{1}{2}}R$. The parameter σ arises in the result as the ratio of $\sigma\alpha$ from the wave function and α from the interaction, so it is replaced by $s=\sigma\alpha/\alpha'$ when one integrates the last terms, involving α', in (6) and (7). Likewise $\rho'=\alpha'^{\frac{1}{2}}R$ replaces ρ in the results of these integrations. From $\bar{V}$, thus calculated, we subtract the first-order average specific nuclear potential of two separated alphas

$$V_0=-12\{A_q\tau^{\frac{3}{2}}+A'\tau'^{\frac{3}{2}}\} \tag{8}$$

with $\tau=\sigma/(\sigma+2)$ and $\tau'=s/(s+2)$, and add the kinetic energy $E_{\mathrm{kin}}=(\sigma\rho\delta\hbar)^2(\alpha/M)/(1-\delta^2)$ and the Coulomb energy $E_C=(4e^2/R)\,\mathrm{erf}[(\sigma\alpha/2)^{\frac{1}{2}}R]$ from reference 7. The result for the first-order energy of interaction of two alphas centered at fixed points R apart, if the particles of which they are composed interact according to (2), is

$$\begin{aligned}E=&(\sigma\rho\delta\hbar)^2(\alpha/M)/(1-\delta^2)\\&-4(1-\delta^2)^{-2}\{A_q[(4\delta^2-1)\lambda_2+(10-\delta^2)\lambda_3\\&-12\delta\lambda_4]+2A_\sigma(1-\delta^2)(\lambda_2-\lambda_3)\\&+A'[(4-\delta^2)\lambda'_2+(5-4\delta^2)\lambda'_3-12\delta\lambda'_4]\}\\&+(4e^2/R)\,\mathrm{erf}[(\sigma\alpha/2)^{\frac{1}{2}}R].\end{aligned} \tag{9}$$

The λ_i are defined in Eq. (15) of reference 7, and the λ'_i are, similarly,

$$\begin{aligned}\lambda'_2&=\tau'^{\frac{3}{2}}\exp(-\tau'\rho'^2),\\\lambda'_3&=\tau'^{\frac{3}{2}}\exp(-\tfrac{1}{2}s'\rho'^2),\\\lambda'_4&=\tau'^{\frac{3}{2}}\exp(-\tfrac{1}{4}(s+3)\tau'\rho'^2).\end{aligned} \tag{10}$$

The term in A' has longer range than the other exponential terms, mainly through the replacement of ρ by ρ', and thus leads to attraction at intermediate distances if it is strong enough to overcome the Coulomb repulsion, as we shall see it is.

The evaluation of (9) with the parameters indicated in (2) is shown, term by term, in Fig. 2. The heavy curve ΔE represents the entire expression (9), the energy of interaction of two alphas at a fixed separation R as a function of R. This has been obtained in a way closely analogous to the Heitler-London treatment of molecular binding. The approximation involving fixed centers is not as good here as in the molecular case, as has been emphasized by Heisenberg,[17] but we may partially take into account the indeterminacy of position of the centers by considering the zero-point vibrations as in molecules. The dotted lines in Fig. 2 represent a typical square-well potential which would just serve to keep two alphas together as a barely stable Be^8 nucleus. The still deeper and broader curve ΔE would thus suffice to make Be^8 quite stable—stable by several mc^2—relative to disintegration into alphas. This means that the attraction between alphas provided by the interaction (2) of the constituent particles is even stronger, already in first order, than required by the energy of Be^8, which is probably unstable by a small fraction of mc^2. Thus we see that a comparatively slight modification of previously-investigated forms of interaction, and one which is also an improvement in some other respects, is sufficient to bring about a strong first-order attraction between alphas. We could take the two ranges more nearly equal than we did explicitly in (2), or introduce some exchange or tensor properties in the long-range term, and still obtain enough alpha-attraction.

The curves in Fig. 2 have not been drawn for very small values of R, as the first-order result of perturbation theory there has very little meaning. The trustworthiness of the result is indicated in part by the smallness of the overlapping, measured by δ^2 relative to unity. This quantity is 0.04 at $R(mM)^{\frac{1}{2}}c/\hbar=0.3$, but is as large as 0.24 at 0.2, where the first-order result is a strong repulsion. In spite of the inapplicability of the first-order result at smaller separations, we may be sure that the repulsion continues to very small values of R, because, as already mentioned in Section I, our interaction (2) is more than saturated. The fact that the repulsion sets in at distances where the overlapping is comparatively small helps to make the alpha-model very plausible.

We have demonstrated the possibility of selecting a simple interaction between heavy particles which satisfies all the usual demands (insofar as this can be done without tensor forces) and at the same time leads to a satisfactory first-order attraction between alphas. This liberates from serious criticism a valuable tool with which we may attempt to correlate and understand nuclear properties, the alpha-model.

III. Empirical Demand for the Alpha-Model. Nuclear Moments

In this section we shall emphasize three striking facts that have previously been pointed out concerning correlations among the moments of odd nuclei, facts which suggest, in a way much too definite to be ignored, that nuclei must have a detailed structure much more simple than one would *a priori* expect in mechanical systems so complex. These facts are, first that there are several cases in which the addition of two neutrons to a nucleus makes practically no difference to the magnetic moment,[20] second, that the magnetic moments seem to be due mostly to a single particle;[21] and, third, that the large electric quadrupole moments are all positive.[22] We shall also discuss other facts related to these but concerning individual nuclei.

Of the twelve odd isotopic pairs with the same nuclear spin of which the magnetic moments are known, seven or eight have the magnetic moments nearly equal in the two isotopes. All of the known magnetic moments of isotopic pairs, including those with different nuclear spins, are listed in Table III. The ratios μ_{A+2}/μ_A are listed for those elements of which the two isotopes have the same nuclear spin, the ones in which we are interested at present. In the case of Ag, and one might also say in K, the magnetic moments are both very small, due presumably to an almost complete cancellation of spin and orbital contributions. In each of these cases we may consider the magnetic

TABLE III. *Magnetic moments of isotopic pairs.**

Odd Proton	I	μ/μ_N	(μ_{A+2}/μ_A)	Odd Neutron	I	μ/μ_N	(μ_{A+2}/μ_A)
$_{17}Cl^{35}$	5/2	1.37	0.83	$_{48}Cd^{111}$	1/2	−0.65	1.0
$_{17}Cl^{37}$	5/2	1.14		$_{48}Cd^{113}$	1/2	−0.65	
$_{19}K^{39}$	3/2	0.39	0.56	$_{50}Sn^{117}$	1/2	−0.89	1.0
$_{19}K^{41}$	3/2	0.22		$_{50}Sn^{119}$	1/2	−0.89	
$_{29}Cu^{63}$	3/2	2.43	1.04	$_{54}Xe^{129}$	1/2	−0.9	
$_{29}Cu^{65}$	3/2	2.54		$_{54}Xe^{131}$	3/2	0.7	
$_{31}Ga^{65}$	3/2	2.11	1.27_0	$_{56}Ba^{135}$	3/2	0.837	1.117
$_{31}Ga^{67}$	3/2	2.69		$_{56}Ba^{137}$	3/2	0.936	
$_{35}Br^{79}$	3/2	2.61	1.0	$_{70}Yb^{171}$	1/2	0.45	
$_{35}Br^{81}$	3/2	2.61		$_{70}Yb^{173}$	5/2	−0.65	
$_{37}Rb^{85}$	5/2	1.34		$_{80}Hg^{199}$	1/2	0.5	
$_{37}Rb^{87}$	3/2	2.74		$_{80}Hg^{201}$	3/2	−0.6	
$_{47}Ag^{107}$	1/2	−0.10	1.9_3				
$_{47}Ag^{109}$	1/2	−0.19					
$_{49}In^{113}$	9/2	5.43	1.002				
$_{49}In^{115}$	9/2	5.43					
$_{51}Sb^{121}$	5/2	3.7					
$_{51}Sb^{123}$	7/2	2.8					
$_{63}Eu^{151}$	5/2	3.4					
$_{63}Eu^{153}$	5/2	1.5					
$_{75}Re^{185}$	5/2	3.3	1.0108				
$_{75}Re^{187}$	5/2	3.3					
$_{81}Tl^{203}$	1/2	1.44	1.0097				
$_{81}Tl^{205}$	1/2	1.45					

* The sources of the data in Table III and in Figs. 3 and 4 are as follows: H^1, Li^7, F^{19}, Na^{23}: S. Millman and P. Kusch, Phys. Rev. **60**, 91 (1941). Be^9: P. Kusch, S. Millman, and I. I. Rabi, Phys. Rev. **55**, 666 (1939). B^{11}: S. Millman, P. Kusch, and I. I. Rabi, Phys. Rev. **56**, 1176 (1939). Cf., however, Douglas and Herzberg, Can. J. Res. **18A**, 165 (1940). Their measured intensity ratio 1.42±10% does not include the ratio 1.67 expected from $I=3/2$, which is indicated as questionable in Fig. 3. C^{13}: R. H. Hay, Phys. Rev. **60**, 75 (1941). N^{15}: G. H. Dieke and R. W. Wood, J. Chem. Phys. **6**, 908 (1938); J. R. Zacharias and J. M. B. Kellogg, Phys. Rev. **57**, 570(A) (1940). Al^{27}: S. Millman and P. Kusch, Phys. Rev. **56**, 303 (1939). $Cl^{35,37}$: Shrader, Millman and Kusch, Phys. Rev. **58**, 925 (1940). K^{39}, Cs^{133}: Kusch, Millman and Rabi, Phys. Rev. **55**, 1176 (1939). K^{41}: J. H. Manley, Phys. Rev. **49**, 921 (1936). Sc^{45}: Kopferman and Wittke, Zeits. f. Physik **105**, 16 (1937). Mn^{55}: R. A. Fisher and E. R. Peck, Phys. Rev. **55**, 270 (1939). Co^{59}: K. R. More, Phys. Rev. **46**, 470 (1934). $Cu^{63,65}$: Tolanski and Forester, Proc. Phys. Soc. **50**, 826 (1938). Zn^{67}: Lyshede and Rasmussen, Zeits. f. Physik **104**, 434 (1937). $Ga^{69,71}$: N. A. Renzetti, Phys. Rev. **57**, 753 (1940). As^{75}: (a) Crawford and Bateman, Can. J. Res. **10**, 701 (1934); (b) Schüler and Marketu, Zeits. f. Physik **102**, 703 (1936). The value 1.72 found in (b) disagrees so badly with 1.1 found in (a) and with 0.78 discarded in (b) that this perhaps discordant datum might legitimately have been omitted from Fig. 3. Kr^{83}: Kopfermann and Wieth-Knudsen, Zeits. f. Physik **85**, 353 (1933); H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936). $Rb^{85,87}$: P. Kusch and S. Millman, Phys. Rev. **56**, 527 (1939). Sr^{87}: Heyden and Kopfermann, Zeits. f. Physik **108**, 232 (1938). Cb^{93}: W. W. Meeks and R. A. Fisher, appearing soon in *The Physical Review* (here by their kind permission). $Ag^{107,109}$: Jackson and Kuhn, Proc. Roy. Soc. **158**, 372 (1937). $Cd^{111,113}$: Schüller and Keyston, Zeits. f. Physik **71**, 413 (1931). In^{113}: T. C. Hardy, Phys. Rev. **60**, 167A (1941). In^{115}: Millman, Rabi and Zacharias, Phys. Rev. **53**, 384 (1938); D. R. Hamilton, Phys. Rev. **56**, 30 (1939); T. C. Hardy, Phys. Rev. **59**, 686A (1941). $Sn^{117,119}$: Schüler and Westmeyer, Naturwiss. **21**, **660** (1933). $Sb^{121,123}$; La^{139}; $Eu^{151,153}$; Au^{197}: Th. Schmidt, Zeits. f. Physik **108**, 408 (1938). I^{127}: Th. Schmidt, Zeits. f. Physik **112**, 199 (1939). $Xe^{129,131}$: Kopfermann and Rindal, Zeits. f. Physik **87**, 460 (1934); E. G. Jones, Proc. Roy. Soc. **144**, 587 (1934); Bethe and Bacher, Rev. Mod. Phys. **8**, 82 (1936). $Ba^{135,137}$: A. N. Benson and R. A. Sawyer, Phys. Rev. **49**, 867 (1936); C^{13} reference. $Yb^{171,173}$: Schüler and Korsching, Zeits. f. Physik **111**, 386 (1939). Lu^{175}: H. Gollnow, Zeits. f. Physik **103**, 443 (1936). Pt^{195}: Th. Schmidt, Zeits. f. Physik **101**, 486 (1936). $Hg^{199,201}$: Schüler and Jones, Zeits. f. Physik **74**, 631 (1932). $Tl^{203,205}$: Schüler and Schmidt, Zeits. f. Physik **104**, 468 (1937). Pb^{207}: G. Breit and L. A. Wills, Phys. Rev. **44**, 470 (1933). Bi^{209}: A. B. McLay and M. F. Crawford, Phys. Rev. **44**, 986 (1933).

[20] H. Schuler and H. Korsching, Zeits. f. Physik **105**, 485 (1937).

[21] (a) A. Landé, Phys. Rev. **46**, 477 (1934); (b) Th. Schmidt, Zeits. f. Physik **106**, 358 (1937).

[22] H. Schuler, J. Roig, and H. Korsching, Zeits. f. Physik **111**, 173 (1938).

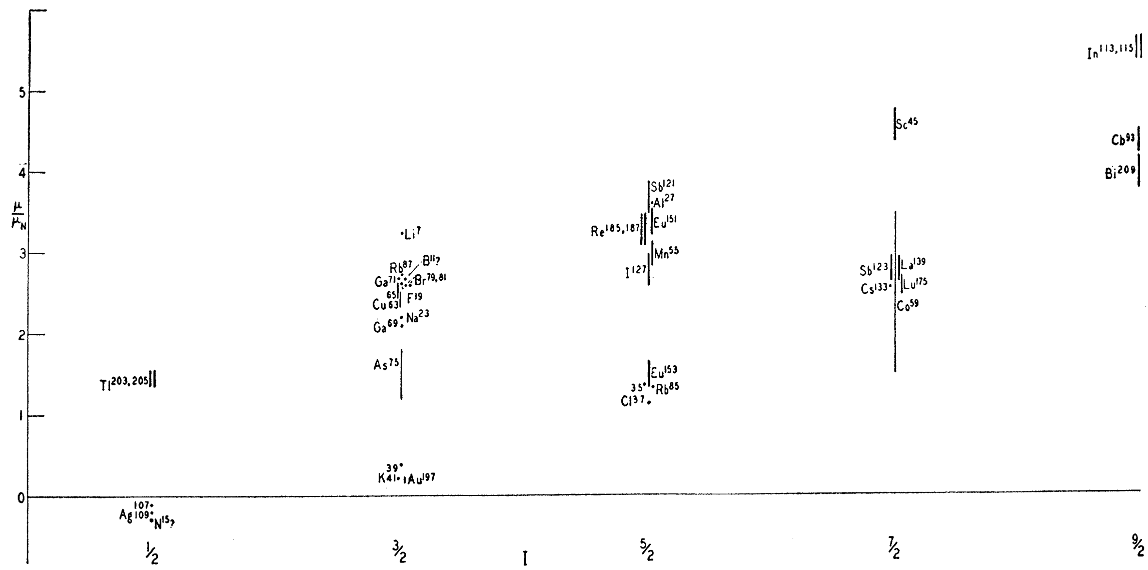

FIG. 3. Observed magnetic moments of the odd-proton nuclei. For sources of data, see footnote to Table III.

moments almost equal, although their ratio is far from unity. In three cases, the ratios are conservatively given as 1.0, although they are very probably more accurately equal to unity than so few digits imply. They are cases in which the hyperfine splittings due to the two isotopes have not been separately resolved, but are almost surely superposed.

There are almost as many cases in which the two isotopic nuclear spins are different as the same. This does not seem at all surprising: The addition of two neutrons may change the nuclear spin either by changing the "orbital" angular momentum $L\hbar$ (of the (LS) coupling state which is predominant in the ground state) or the sign of the spin-orbit coupling, or both. (Change of the spin angular momentum $S\hbar$ seems not to occur, S remaining $\frac{1}{2}$.) But in the cases in which the addition of two neutrons does not change the nuclear spin, the two neutrons make practically no difference in the magnetic moment either, in about seven out of twelve cases, and this is quite astonishing.

In attempting to interpret this situation, one might first consider that the composition of the "orbital" angular momentum $L\hbar$ is so complex that all of the particles contribute about equally to it on the average.[23,24] In the extreme case in which this smoothing out is uniform for all nuclei, one would in effect have a "droplet model" of the "orbital" motion, and the "orbital" gyromagnetic ratio would be

$$g_L = N_\pi/(N_\pi + N_\nu) \tag{11}$$

in a nucleus with N_π protons and N_ν neutrons. The addition of two neutrons, if they should have no effect on the spin gyromagnetic ratio, would then merely cause a slight algebraic decrease of the nuclear magnetic moment, such as is observed in the cases Ag and K. In the cases Cu, Re, and Tl, the change is of the wrong sign, but here it is still of the order of magnitude given by (11), and one might consider that the discrepancy in sign is not too serious if he had other reason to consider the "droplet model" a fair approximation. But present ideas of nuclear composition do not justify the opinion that nuclear matter should behave as a vibrating liquid or solid even in the question of magnetic moments, useful though this concept may be in the problem of the stability of excited states.[5] At least some of the particles are considered to move past one another relatively much more rapidly than in liquids. In one elegant treatment of nuclear structure, which was developed by Wigner[1(b)] and applied to the problem of nuclear magnetic moments by Margenau and Wigner,[24] the saturation properties of the nuclear forces and

[23] (a) D. R. Inglis, Phys. Rev. **53**, 470 (1938); (b) K. Way, Phys. Rev. **55**, 963 (1939).

[24] H. Margenau and E. Wigner, Phys. Rev. **58**, 103 (1940).

the approximate conservation of isotopic spin (which would be exact without strong tensor or Coulomb forces), are taken into account in a rather general way which, however, does not envisage the possibility of a geometrical clustering into alphas. Their treatment would lead one to expect an average behavior given by (11), when the average is taken over several nuclei, but that the individual values would deviate rather widely from this average in an apparently random fashion. This is incompatible with the practically equal magnetic moments of several isotopic pairs,[24] as listed in Table II.

The equality of the pairs of magnetic moments may then be taken as an indication of the existence of more detailed structure than was introduced by Margenau and Wigner, a structure of a sort to make it likely that two neutrons may be added to a nucleus without changing the motions of the other particles very much. Both the "Hartree" model and the alpha-model contain such a detailed structure—the former in momentum space and the latter in coordinate space. In the Hartree model one would expect to find a few cases, such as the case of K which has already been discussed elsewhere,[23] in which the two neutrons would be added to a closed shell and have only a rather small effect (in second order) on the magnetic moment. But in many more cases one would expect the two neutrons to be added to an unfilled shell and to change g_L considerably in first order, as in the comparatively simple cases[2(d)] $Be^7 \rightarrow Be^9$ and $N^{13} \rightarrow N^{15}$. This is expected because of the high degree of degeneracy of those individual-particle levels which possess the larger angular momenta $l\hbar$, in the spherically symmetrical case—there are several possible orientations of **l**, all with the same energy. In the alpha-model of most nuclei, the degeneracy of the individual extra-neutron states would not be so great—different orientations of **l** relative to the body axes would involve different energies because of the lack of spherical symmetry (as will be discussed further below). For this reason, the two neutrons would, in the alpha-model, very often be added to an individual-particle state which was previously empty (in first order), and would affect g_L only in higher order. Thus the alpha-model does seem to hold considerable promise of accounting for the observation that the addition of two neutrons makes very little difference in the magnetic moment in a large fraction of the cases in which it does not alter the nuclear spin.

The second striking fact, which is closely related to the first, may be summarized by saying that magnetic moments of odd nuclei seem to be largely due each to the single odd particle.[21] In Fig. 3 are shown the observed magnetic moments μ of the odd-proton nuclei for various values of the "nuclear spin," or total angular momentum, $I\hbar$. In Fig. 4, similarly, are those for the odd-neutron nuclei. The limits of error indicated by the vertical extent of the lines are estimated very roughly, largely on the basis of the degree of reliability which similar measurements on other nuclei have proved to have had, and may in many cases be too broad. It is striking, both that the total range of the moments is not larger than it is (that large negative values do not occur in the odd-proton case, for example), and that the values of the magnetic moments in each case seem to *fall into two groups*. The apparent division into two groups is sufficiently striking that it seems very much more likely to be significant than to be the result of chance. It must, of course, still be regarded as subject to cancelation by extension and refinement of the observations, but it was noted and discussed when the data were much more sparse and uncertain than now, and it has so far stood "the test of time." The further testing of this division into two groups is one of the most interesting tasks for future investigation of the magnetic moments of intermediate and heavy nuclei.

Of the magnetic moments which have already been measured, that of Co^{59} is the most in need of further inves-

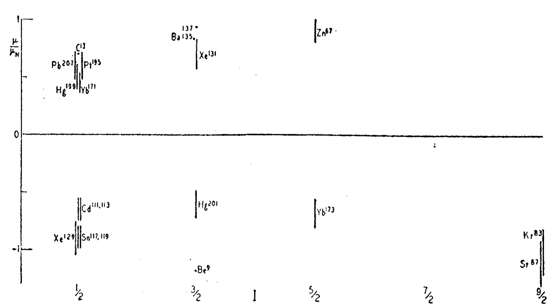

FIG. 4. Observed magnetic moments of the odd-neutron nuclei.

tigation. The electron configurations of the atom are very complex. The present measurement is based on the h.f.s. of one level, in which a rather indefinite allowance was made for coupling of the nucleus to other-than-s electrons by More (see footnote to Table III) who concludes that the value of μ "probably lies somewhere between 2 and 3 nuclear magnetons," not at 3.5, the value to be had by considering the s electron only. The latter value is plotted in Fig. 1 of Margenau and Wigner's paper,[24] and slightly obscures the appearance of a division into two groups in that figure, although one or two exceptions would still leave a rather surprising degree of division.

The mere division into two groups would be expected from such a "structureless" model as the "droplet model." The predominant occurrence of the spin (that is, spin part of the angular momentum, not "nuclear spin") quantum number $S=\frac{1}{2}$ is almost inevitable for any forces leading to saturation and to vanishing nuclear spins $I\hbar$ of even-even nuclei. If the spin gyromagnetic ratio g_S is never very small, and if g_L does not vary very much from one odd-proton nucleus to another, one would expect a division into two groups, one group corresponding to the occurrence of the nuclear spin quantum number $I=L+\frac{1}{2}$ and the other to $I=L-\frac{1}{2}$, that is, to the two possible orientations of the spin $\frac{1}{2}\hbar$ relative to the orbital angular momentum. The division into two groups is also compatible with the "Hartree" and alpha-models, and with the treatment of Margenau and Wigner[24] so long as one sticks to supermultiplets with $S=\frac{1}{2}$ and to some semblance of (LS) coupling—the deviations from the value of g_L given by (10) would not necessarily be large enough to make the two groups merge. The division into two groups is thus not alone a reason for preference of one model over another, but is an indication of a rather close approximation to (LS) coupling in nuclei. This in itself is slightly surprising in the light of other indications of tensor interactions, which, if strong, would tend to destroy this coupling scheme by mixing states of various values of L and S. It is because of the division into two groups that we are justified in attributing the nuclear magnetic moment to an "orbital" part $g_L\mathbf{L}\mu_N$ and to a spin part $g_S\mathbf{S}\mu_N$, compounded as vectors by use of the familiar "quantum-mechanical cosines" (Landé formula).

The most astonishing aspect of the division into two groups is that the two groups are so situated as to indicate an average value of g_L as high as about $\frac{7}{8}$ in the odd-proton case and as low as $\frac{1}{8}$ in the odd-neutron case,[23(a)] very near the values 1 and 0 which would be caused by a single proton and by a single neutron, respectively. That is, the magnetic moments of the odd-proton nuclei are in general considerably higher, and those of the odd-neutron nuclei are considerably lower, than they would be if the average value of g_L were given by (10), and this can be accounted for by attributing the orbital angular momentum almost entirely to one or several protons in the odd-proton case and to one or several neutrons in the odd-neutron case.

This apparent participation of predominantly one kind of particle in the "orbital" motion of a nucleus is another strong indication of the presence of some detailed structure in nuclei. Both the alpha-model and the "Hartree" model contain some structure of the sort required to keep the orbital moments of the protons and neutrons separated,[23(a)] but of the two the alpha-model seems to be much the better suited to account for the situation.

In the central-field or "Hartree" model, one has some of the required structure only in nuclei with a sufficiently large neutron excess that the highest-energy neutrons are in a different shell from the highest-energy protons. In these cases one might, as a rough approximation, consider the like-particle orbits in unfilled shells to be coupled together somewhat more strongly than the unlike-particle orbits, because of the effect of orthogonality of single-particle wave functions in reducing the value of matrix elements containing them (an effect which is extreme in the long-range and short-range limits, but might almost disappear in between these limits).[23] At least with some types of attractive interaction, there is a tendency of an even number of like particles to form an S state (with their total L equal to zero).[2(b, c)] In this rough approximation the orbital moment of only the odd particle would remain. The roughness of the approximation used, the expectation that second-order modifications would be important in heavy nuclei, and the limited applicability cast grave doubt on the plausibility of this explanation in terms of the "Hartree" model.

In the application of the alpha-model to this problem, the considerations of Teller and Wheeler,[25] concerning the related problem of the rotation of nuclei made up solely of alphas, are of decisive importance. They have shown that a really rigid alpha-framework is not compatible with the apparent lack of rotational states of very low excitation energy in intermediate and heavy nuclei, but that a non-rigid assembly of alphas would have no rotational states lying between the zero state and about the eighth (for heavy nuclei). In this non-rigid assembly, the binding between alphas must be sufficiently elastic that the interchange of alphas, from one position to an equivalent position, by means of a combined vibrational and small rotational motion does not involve the penetration of a high potential barrier. A potential trough so broad that its breadth is comparable to its range, such as ΔE in Fig. 2, acting between the pairs of alphas, would be expected to yield a sufficiently non-rigid framework.

The problem of the disposal of excess neutrons in such a non-rigid framework is a complex one which has not been adequately solved. One may expect them to act to some extent like conduction electrons in a metal—perhaps a molten metal—providing an additional binding between the alphas (as is required to oppose the Coulomb repulsion in heavy nuclei). Any analogy of alphas to bricks and neutrons to mortar is, of course, meant in a non-rigid manner—the additional binding would be expected to be even less rigid than that provided by ΔE of Fig. 2, for example. Nor is it implied that the neutrons move only through and between the alphas—it is quite likely that most of each single α-particle wave function may correspond to motion around the outside of the framework of alphas. Judging by the binding energies and the opinion that the inter-particle interaction is deeper than the inter-alpha-interaction, one expects that an excess neutron has much more kinetic energy than is associated with the center-of-mass motion of one alpha. One may thus consider the approximation in which the period of the "orbital" motion of a single neutron is short compared with the period of vibration of the alpha-framework from one configuration of minimum energy to another. The single-particle wave functions would then be determined by a field varying "slowly" between two extreme positions of the same symmetry but differing orientations. The system would remain in the extreme positions most of the time, and the disposal of the excess particles would be similar to that in a rigid framework.

[25] E. Teller and J. A. Wheeler, Phys. Rev. **53**, 778 (1938).

The problem of the interacting neutrons contains some features of the problem of the existence of ferromagnetism in metals (competition between the effects of the Pauli antisymmetry on the interaction of the particles with each other and on the interaction of the particles with the field), and some features of the problem of the order of S, P, D, $\cdots$, states in atoms (Hund's rule) or of Σ, Π, Δ, $\cdots$, states in molecules. The non-ferromagnetic state of lowest multiplicity has lowest energy in most metals because the tendency toward ferromagnetism due to the *repulsive* interaction of the electrons is weaker than the effect of the Pauli antisymmetry on the interaction of the electrons with the atoms. In nuclei the same result may be expected for quite a different reason: the states with $S_\nu = 0$ (if the number of neutrons is even, or $\frac{1}{2}$ if it is odd) lie lowest because the *attractive* interaction of the neutrons makes a tendency away from ferromagnetism, and this tendency is expected to be stronger than any effect of their interactions with the field of the alphas, because the saturation of the forces makes the neutron-alpha-interactions weaker than those between neutrons. This reason is not quite adequate to deal with all types of interactions—it is rather general but may be modified by exchange operators. The coupling anticipated between an odd-proton spin and the neutron spins is weaker because it is not governed by the Pauli antisymmetry. Thus the spin part Sh of the angular momentum of an odd-proton nucleus is expected to consist primarily of proton spin, as required empirically by the separation between the two groups of magnetic moments. The odd-neutron nuclei are similarly expected to have Sh made up mostly of neutron spin.

As regards orbital motion, calculations of up to six p particles and up to four d particles lead to the conclusion that attractive (primarily space-exchange) interactions favor S states, but not in configurations of many more particles, such as would be encountered in the "Hartree" model of heavy nuclei.[26] Although such calculations have not been made for the alpha-model of any heavy nucleus, the results which are available for the "Hartree" model, together with the expected low degree of degeneracy of the single-particle states in the alpha-model, lead one to expect the ^1S state of an even number of excess neutrons in the alpha-model to be well isolated below the other states, in agreement with the observed vanishing of the moments of even-even nuclei. If a single proton should be added, its coupling with the neutrons would be expected to be weak, because of the supposedly large energy difference between the ^1S and the next higher state of the neutrons.

The great advantage of the alpha-model here lies in the fact that it allows only a single proton to move freely around the framework of alphas in an odd-proton nucleus, and none in an odd-neutron nucleus. It is much more plausible that a single proton should move independently of the neutrons than that the motions of several protons should be coupled to one another but not to the neutron motion.

We have as yet considered only those states in which the framework of alphas does not rotate. If the order of the rotational states were normal, one would expect rather strong intermixture of several rotational states to form the ground state of the nucleus. In an odd-proton nucleus with a body-axis component of the proton orbital momentum characterized by the quantum number $\Lambda=1$, for example, we would have the states possessing the rotational quantum numbers $R=0$, 1, and 2 combined to form a ground state with total "orbital" quantum number $L=1$. (In keeping with molecular terminology, we should speak of K, rather than L, if the states with $R\neq 0$ were important. In diatomic molecules, one usually speaks of rotational quantum number N, rather than R, but this implies a body angular momentum normal to the body axis, which is not the case here.) The orbital gyromagnetic ratio g_L would then be a weighted average of the approximate values

$$1 \quad \text{for the state with } R=0,$$
$$(\tfrac{1}{2}+1)/2=\tfrac{3}{4} \quad \text{" " " " } R=1,$$
$$(2\times\tfrac{1}{2}-1)/2=0 \quad \text{" " " " } R=2.$$

[26] Reference 23 (a), Section III.

(If the excess neutrons were taken into account, the rotational gyromagnetic ratio $\frac{1}{2}$ here used would be replaced by a slightly smaller value, about as given by (10).) If these states were weighted about equally,[23(a)] the resultant g_L would be considerably less than the empirical value $g_L\approx\frac{7}{8}$. The extent to which the wave function with $R=2$, for example, is mixed into the ground state is expressed in perturbation theory by a coefficient of the form

$$c_2=[\int R^0\psi_p{}^1(\Sigma V_{ij})R^2\psi_p{}^{-1}d\tau]/(E_2-E_0),$$

where R^M and $\psi_p{}^m$ are the rotational and odd-proton wave functions, respectively, and E_R is the energy of the Rth rotational state. The integral in the numerator is the matrix element of the entire interaction, between the two states. Exchange has been ignored in writing it. It is of such a form that it would express the binding-type potential energy of the odd proton, but for the partial cancelations of various parts of the integration due to the orthogonality of the two wave functions involved. The cancelation might reasonably be expected to reduce the value of the integral by a factor of ten or twenty, but probably not by as much as a hundred. The integral would then probably have a value of the order of magnitude mc^2. The spacing between the rotational levels, E_2-E_0, would have the order of magnitude $mc^2/10$ if the spacing were normal. The important remark of Teller and Wheeler[25] leads us to expect the elevation of the first and second rotational states to exceed that of the eighth, so they are probably higher than mc^2. The coefficients c_1 and c_2 need be only as small as $\frac{1}{2}$ or $\frac{1}{3}$ to suppress the states $R=1$ and 2 sufficiently, since they enter quadratically in the weighted average of the magnetic moments. It is thus plausible that the coefficients are small enough to make $g_L\approx\frac{7}{8}$, in the light of the remark of Teller and Wheeler, though this would hardly be plausible without their considerations. Although the details are far from complete, we may say that the regularities cited among the magnetic moments of intermediate and heavy nuclei seem reasonably plausible on the basis of the alpha-model, and at present only on this basis.

This interpretation of the distribution of magnetic moments stands in extreme contrast with the interpretation

TABLE IV. *Nuclear quadrupole moments.**

	I	μ/μ_N	$Q/(10^{-24}\text{ cm}^2)$
$_1H^2$	1	2.79	−0.003
$_{29}Cu^{63}$	3/2	2.43	−0.1
$_{29}Cu^{65}$	3/2	2.54	−0.1
$_{31}Ga^{69}$	3/2	2.11	0.2
$_{31}Ga^{71}$	3/2	2.69	0.13
$_{33}As^{75}$	3/2	1.6	0.3
$_{36}Kr^{83}$	9/2	−1.0	0.15
$_{49}In^{115}$	9/2	5.43	0.84
$_{53}I^{127}$	5/2	2.8	0.8(?)
$_{54}Xe^{131}$	3/2	0.7	0±0.1
$_{63}Eu^{151}$	5/2	3.4	1.2
$_{63}Eu^{153}$	5/2	1.5	2.5
$_{70}Yb^{173}$	5/2	−0.7	3.9
$_{71}Lu^{175}$	7/2	2.6	5.9
$_{71}Lu^{176}$	⩾7	3.8	7
$_{75}Re^{185}$	5/2	3.3	2.8
$_{75}Re^{187}$	5/2	3.3	2.6
$_{80}Hg^{201}$	3/2	−0.6	0.5
$_{83}Bi^{208}$	9/2	3.6	−0.4

* The sources of the data in Table IV are: H: Reference 17. Cu: Schüler and Schmidt, Zeits. f. Physik **111**, 165 (1938). Ga: Schüler and Korsching, Zeits. f. Physik **103**, 434 (1936). As: Schüler and Marketu, Zeits. f. Physik **102**, 703 (1936). Kr; Xe: H. Korsching, Zeits. f. Physik **109**, 349 (1938). In: D. R. Hamilton, Phys. Rev. **56**, 30 (1939). I: S. Murakawa, Zeits. f. Physik **112**, 234 (1939); compare Th. Schmidt, Zeits. f. Physik **112**, 199 (1939). Eu: Schüler and Schmidt, Zeits. f. Physik **94**, 457 (1935); H. Casimir, Physica **2**, 713 (1935). Yb: Schüler, Roig, and Korsching, Zeits. f. Physik **111**, 165 (1938). Cp: H. Gollnow, Zeits. f. Physik **103**, 443 (1936); Schüler and Gollnow, Zeits. f. Physik **113**, 1 (1939). Re: Schüler and Korsching, Zeits. f. Physik **105**, 168 (1937). Hg: Schüler and Schmidt, Zeits. f. Physik **98**, 239 (1935). Bi: Schüler and Schmidt, Zeits. f. Physik **99**, 797 (1936).

proposed by Margenau and Wigner.[24] They do not recognize as significant the apparent division into two groups in each of Figs. 3 and 4, but they do attempt to interpret the fact that the observed magnetic moments extend to higher values in Fig. 3 and to lower values in Fig. 4 than one would expect on the basis of equation (10) and the assumption $S=\frac{1}{2}$. This they attribute to the frequent strong admixture of states with $S=\frac{3}{2}$, which obviously may have much larger or smaller magnetic moments, depending on the orientation of S relative to L, than have states with $S=\frac{1}{2}$. One would also expect on this basis the rather symmetrical occurrence of lower values in Fig. 3 and higher values in Fig. 4, which are not observed. Thus the trend of the observed magnetic moments to one side of the "expected" region, as well as the apparent division into two groups, is not explained by their formally very elegant treatment of the problem.

Third, we come to a striking regularity among observed nuclear electric quadrupole moments, Q, which are listed in Table IV. As far as they have been observed, the *large quadrupole moments are all positive* and are found in the region of the periodic table between atomic numbers sixty and eighty, rising to a maximum near atomic number seventy, in the rare earths. The observed quadrupole moments show no distinction between odd-proton and odd-neutron nuclei, except that the few negative quadrupole moments occur only in odd-proton nuclei. The existence of a quadrupole moment

$$Q=\sum_{\text{protons}}(3z^2-r^2)_{\text{Av}}$$

indicates a deviation from a spherically symmetrical average charge distribution within the nucleus. A positive value indicates a distribution extended along the axis of the nuclear spin $I\hbar$ (prolate or cigar-shaped nucleus) and a negative value implies a flattening toward the plane normal to this axis (oblate or doorknob-shaped nucleus). A deviation from the spherically symmetrical shape is easily provided, for most nuclei, by the alpha-model. The problem of the energetically favorable packing of spherical alphas has been discussed in this connection by Wefelmeier,[27] who has shown that the most elongated framework of alphas is expected in the neighborhood of atomic number 71, just where the largest quadrupole moments are observed. The problem of the orientation of the alpha-framework relative to the total angular momentum, $I\hbar$, has been discussed by Fano,[28] who showed that it is at least plausible that the orbital angular momentum of a single proton or neutron should be oriented along the long axis of an elongated nucleus.[21(b)] If S is only $\frac{1}{2}$, as is strongly indicated by the regularities among the magnetic moments discussed above, the average direction of the orbital angular momentum, $L\hbar$, is the direction of I. The natural explanations of the magnetic-moment regularities and of the quadrupole moments are thus closely related and entirely compatible.

An alternative mechanism to provide an elongated nucleus on the basis of the "droplet model" has been proposed by Weizsäcker, treating the competition between the deforming tendency of the Coulomb force and the restoring tendency of the surface tension.[29] Considering only ellipsoidal deformations, he found a small range of stability, but Bohr and Wheeler[5(b)] have shown that the charged droplet is unstable in this range relative to deformations of the form

[27] W. Wefelmeier, Naturwiss. **25**, 525 (1937); Zeits. f. Physik **107**, 332 (1937).
[28] U. Fano, Naturwiss. **25**, 602 (1937).
[29] C. F. v. Weizsäcker, Naturwiss. **27**, 133, 277 (1939).

$r(\theta) = c(3\cos^2\theta - 1)$, and the droplet is only stable in the spherical form.

The principal assumption involved in this treatment is that the body energy is proportional simply to the volume of the nucleus, and otherwise independent of the shape (which affects the surface-tension term and the Coulomb term only). One might profitably modify the body term in the droplet model to favor those shapes which allow the compact packing of alphas. But the existence of large positive quadrupole moments in the region of the rare earths suggests that the alpha-model, rather than the droplet model, should be taken as the first approximation, leaving the less important surface-tension and Coulomb effects to be introduced possibly as corrections.

The fact that negative quadrupole moments occur only in the odd-proton nuclei is also in nice accord with the assumption that one particle is chiefly responsible for the orbital angular momentum. A single proton moving in any state but an *s* state has a negative quadrupole moment, and this may determine the sign of the nuclear quadrupole moment, if the framework of alphas is nearly spherically symmetrical. The nucleus of $_{26}$Fe is most favorably made up of an essentially spherical structure of thirteen alphas,[25] and the two isotopes of $_{29}$Cu possessing negative quadrupole moments follow this closely in the periodic table.

The extra alpha in $_{29}$Cu beyond the thirteen which make up a spherical structure presents a difficulty of detail. It would be interesting to know whether the element which is simpler in this respect, $_{27}$Co, has a larger negative quadrupole moment than Cu, as the simplest picture leads one to expect, but this is difficult to observe because of the complex atomic configurations involved. The other nucleus having negative quadrupole moment, $_{83}Bi^{208}$, follows mass number 76 (38 alphas again form a spherical structure[29]) rather remotely, with the positive moment of $_{80}Hg^{201}$ appearing in between. This presents further difficulty of detail, and makes doubtful the expectation that the quadrupole moment of Co should be negative. It has been suggested that the stability of a framework of alphas might require one or several alphas to be dissolved in the "mortar" which helps to bind the remaining alphas together.[29]

It would also be interesting to determine whether the quadrupole moments of Tl, Cd, Sn, and Xe^{129} are actually very small. According to our interpretation of the magnetic moments, they possess, in first approximation, an odd particle in an *s* state. The nuclear spin would then not be coupled to the asymmetry of the alpha-framework in the same way as has been supposed for other nuclei, and the quadrupole moment would vanish in this approximation. A higher-order coupling, perhaps by tensor forces, could, however, easily introduce a quadrupole moment.

Among the lighter nuclei, a number of individual magnetic moments have been calculated in first order both by the alpha-model and by the "Hartree" model.[2(d), 30] Aside from the case of Li^7, the two models have been about equally successful in accounting very roughly for the magnetic moments (or equally unsuccessful, according to one's point of view). The observed magnetic moments are in general smaller in magnitude than the theoretical results. This is in agreement with the expectation that higher-order admixture of excited states would tend to reduce the magnetic moment, toward the average magnetic moment of all the possible excited states. An example of such admixture has been discussed in greater detail[31] in the case of Li^7. The extent of the division of nuclear magnetic moments into two groups, in Figs. 3 and 4, suggests the particular type of admixture which may be described as diluting the spin angular momentum Sh with neutron spin in the odd-proton case, or with proton spin in the odd-neutron case, still preserving the meaning of the quantum numbers L and S. This type of deviation from the first-order result also suffices to reconcile the experimental and theoretical results (with either model) for the individual light nuclei (still excepting Li^7). The magnetic moment of Li^7 definitely favors the alpha-model of that nucleus,[31, 32] although even with this model the experimental value slightly exceeds the theoretical—a discrepancy in the sense not anticipated by the above discussion of admixture.[30(a)] In this and similar light nuclei, the alpha-model involves exchange between an alpha and a triton.[4, 30] The model involving this concept is expected to apply to Li^7 better than to the other appropriate light nuclei, because the binding of the triton to the rest of the nucleus is considerably less than the internal binding energy of the triton in the case of Li^7

[30] (a) D. R. Inglis, Phys. Rev. **55**, 329 (1939); (b) **56**, 1175 (1939); (c) R. G. Sachs, Phys. Rev. **55**, 825 (1939).
[31] D. R. Inglis, Phys. Rev. **53**, 880 (1938).
[32] H. A. Bethe, Phys. Rev. **53**, 842 (1938).

only—in other cases the triton would be greatly distorted.[30(b)] The alpha-model with a triton is more sensitive to higher-order corrections and less apt to be successful than is the ordinary alpha-model, and the fairly satisfactory success in Li^7 helps to make plausible the application of the ordinary alpha-model to intermediate and heavy nuclei.

The data which we have discussed, though evasive in detail, present an impression of coherence in broad outline, which is shattered only by the astounding nuclear spin and magnetic moment of K^{40} recently announced by Zacharias:[33] $I=4$, $\mu=-1.29\mu_N$. Whether or not L and S are good quantum numbers, the ground state may be described in terms of one or several (LS) coupling states, which must have preponderantly negative magnetic moment. We consider the states $^1G_4$, $^3FGH_4$, $^5DFGHI_4$, etc., as possibilities. The orbital gyromagnetic ratio g_L is practically always positive (or zero for pure neutron motion) so $^1G_4$ is unsatisfactory. In a triplet state, the spin gyromagnetic ratio g_S would, if we judge by the lighter odd-odd nuclei, be positive and not greater than the deuteron value, 0.85. Of the triplet states, only $^3H_4$ has the spin "pointing backwards" to give a negative contribution, and for it

$$\mu(^3H_4)=(\tfrac{4}{5})(6g_L-g_S)\mu_N,$$

which is positive for any reasonable value of g_L. In a quintet state it is much more likely that the spin gyromagnetic ratio should be strongly negative, making a negative magnetic moment with the spin "pointing forward," as in the state $^5D_4$, for which we have

$$\mu(^5D_4)=2(g_L+g_S)\mu_N.$$

In either the alpha-model or the "Hartree" model of K^{40}, there are three neutrons and one proton left over to contribute to the magnetic moment in first order, and one expects that S, if it is as large as 2, would be made up of three neutron spins and one proton spin, making $g_S=(2.78-3\times1.92)/2=-1.49$ in first order. If g_L were about $\frac{1}{4}$, as one would expect from the numbers of particles participating, $\mu(^5D_4)$ would be about $-2.5\mu_N$, a sufficiently large negative value to give rise, with admixture of other states, to the negative value observed. The state $^5F_4$ would have the magnetic moment $(13g_L+7g_S)\mu_N/5\approx-1.5\mu_N$, which is also negative and sufficiently large. Those of the other quintet states are not. We conclude that the ground state of K^{40} is preponderantly $^5D_4$ and perhaps $^5F_4$. If the ground state is thus essentially a quintet state, three neutron spins are parallel to one another (and to one proton spin). All other nuclear moments, especially the vanishing nuclear spins of even-even nuclei, demand the assumption that like-particle spins tend to orient themselves anti-parallel to one another. This result is also expected of simple attractive forces and the Pauli principle. The reason for an exception in the case of K^{40} is not at present apparent. The heavier odd-odd nucleus $_{71}Lu^{176}$ listed in Table III presents no such problem: it may well be preponderantly a triplet, as are the lighter odd-odd nuclei, and probably $^3I_7$, having two odd particles each with the orbital quantum number 3, like the one in the neighboring nucleus $_{71}Lu^{175}$.

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[33] J. R. Zacharias, Phys. Rev. **60**, 168 (1941).