On the Thermal Conductivity of Dielectrics

I. POMERANCHUK

P. N. Lebedev Physical Institute, Academy of Sciences of U. S. S. R., Moscow

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As is well known, Peierls and Debye found that, if $T \gg \theta$ (T is the temperature, θ—the Debye temperature), the thermal conductivity κ of dielectrics varies inversely to the temperature. However, this turns out to be true only for dielectrics obeying the definite law of sound velocity dispersion and the definite law of the dependence of sound velocity on the direction. For all other dielectrics at $T \gg \theta$, κ is inversely proportional to T^2. This results from investigation of the free path of long elastic waves.

PEIERLS[1] and Debye[2] have found the dependence of thermal conductivity κ of dielectrics on the temperature for the cases in which T is very much greater than θ the Debye temperature. In order to clarify the problem we shall consider the situation which arises in the body with no dispersion of sound and with isotropic sound velocity.

As is known,[3] the absorption coefficient of the transversal sound (i.e., of transversal phonons) varies as ω, ω being a circular frequency. It is easily found that at $T \gg \theta$ the phonon free path l is of the form:

$$l \approx \alpha(Ms^2/KT)(K\theta/\hbar\omega), \quad (T \gg \theta). \tag{1}$$

Here M is the mass of the elementary cell, K the Boltzmann constant, s the velocity of sound, α the lattice constant. Absorption coefficient of longitudinal sound depends on the existence of the possibility of the absorption of a long wave-length longitudinal phonon by short wave-length phonons. From the conservation laws it follows, that

$$\mathbf{f}+\mathbf{F}=\mathbf{F}'; \quad \omega(\mathbf{f})+\omega(\mathbf{F})=\omega(\mathbf{f}+\mathbf{F}). \tag{2}$$

Here $\mathbf{f}$ stands for the wave vector of a long wave-length longitudinal phonon, $\mathbf{F}$ for wave vector of a Debye short wave-length phonon. We get:

$$\omega(\mathbf{f})=\mathbf{f}\frac{\partial\omega}{\partial\mathbf{F}}; \quad s_{\text{long}} f=\mathbf{f}\mathbf{V}_{\text{F}}. \tag{3}$$

s_{long} is a longitudinal sound velocity, V_{F} is the group velocity of a Debye phonon.

If, as for the body under consideration, $s_{\text{long}} \geqslant V_{\text{F}}$, the process (2) is impossible. For bodies having such a dispersion law and an anisotropic sound velocity, the absorbtion of a long wave-length longitudinal phonon is only possible if it splits into two small transversal phonons or into one small transversal and one small longitudinal phonon. The absorption coefficient turns out to be proportional to ω^4 (in distinction to ω^5, as was obtained by Slonimsky[4]). The free path l^{III} may be written, if the order of the value is only concerned, as follows:

$$l^{\text{III}} \approx \alpha(Ms^2/KT)(K\theta/\hbar\omega)^4. \tag{4}$$

Moreover, the absorption of longitudinal phonons due to collisions involving four phonons must be taken into account as well. The corresponding free path is of the form:

$$l^{\text{IV}} \approx \alpha(Ms^2/KT)^2(K\theta/\hbar\omega)^2. \tag{5}$$

Equations (4) and (5) give equal quantities at the frequency ω_0, at which

$$\hbar\omega_0 = K\theta(KT/Ms^2)^{\frac{1}{2}}, \quad (T \gg \theta). \tag{6}$$

It is evident that in these conditions the heat is transferred by small longitudinal phonons because they have enormous free paths.

The thermal conductivity κ equals:

$$\kappa \approx \int C_\omega l_\omega V_{\text{F}} \frac{\omega^2 d\omega dO}{(2\pi s)^3}, \quad dO = \sin\vartheta d\vartheta d\varphi. \tag{7}$$

[1] R. Peierls, Ann. d. Physik **3**, 1055 (1929).
[2] P. Debye, *Vorträge uber die Kinetische Theorie* etc., s. 43 (Teubner, 1914).
[3] L. Landau and G. Rumer, Sowjetunion Physik. Zeits. **11**, 18 (1937).
[4] Slonimsky, J. Exper. and Theor. Phys. (Russian) **7**, 1457 (1937).

Here C_ω is the specific heat of a given phonon. Because small frequencies alone are of any importance, C_ω may be put equal to K. Introducing Eq. (4) into (7) we get after integration up to ω_0, this limit resulting from Eq. (6):

$$\kappa \approx \alpha K s \frac{Ms^2}{KT}\left(\frac{K\theta}{\hbar s}\right)^3 \frac{K\theta}{\hbar\omega_0}\frac{4\pi}{(2\pi)^3}$$

$$= \alpha K s \left(\frac{Ms^2}{KT}\right)^{\frac{3}{4}}\left(\frac{K\theta}{\hbar s}\right)^3 \frac{1}{2\pi^2}$$

$$= \alpha K s \left(\frac{Ms^2}{KT}\right)^{\frac{3}{4}} \frac{1}{2\pi^2\alpha^3}\left(\frac{K\theta\alpha}{\hbar s}\right)^3.$$

Let us designate $1/\alpha^3$ by n, which is the quantity of the same order as the total number of atoms (or molecules) per cc, and use the relation:

$$K\theta \sim (\hbar s/\alpha)(2\pi^2)^{\frac{1}{3}}.$$

Then

$$\kappa \sim nKs\alpha(Ms^2/KT)^{\frac{3}{4}}, \quad (T \gg \theta).$$

Thus for the bodies, for which $s_{\max} \geqslant V_F$, $s_{\max}$ being the velocity of the oscillations having the greatest velocity, the thermal conductivity at high temperatures varies as $T^{-\frac{3}{4}}$, in distinction to the results of Peierls[1] and Debye.[2]

If $s_{\max} < V_F$, the difference with respect to the absorption between "longitudinal" and "transversal" sound vanishes and the result obtained by Peierls[1] and Debye[2] remains to be true.

At low temperatures ($T \ll \theta$), according to detailed calculations, only the admixtures, the reflection from boundaries of small crystals and the collisions of triples of phonons are essential for the evaluation of κ. In distinction to the case treated by Peierls,[5] here the finite value of thermal conductivity cannot be obtained if we consider only the scattering of phonon on the atoms of admixture and the collision of triples of phonons, because in both effects the free path varies as ω^{-4}. The finite value of thermal conductivity can be obtained only if the reflection from small crystal boundaries is taken into account. The resulting expression for κ turns out to be inverse to $L^{\frac{1}{4}}$, L being the linear dimension of the crystal. Therefore, at low temperatures the thermal conductivity, as found experimentally, depends (although not strong) on the shape of the crystal. This holds even if the phonon free path for the frequency of the order $KT/\hbar$ is much less than the crystal dimensions. If the temperature is decreased and becomes less than the Debye temperature, thermal conductivity at first increases inversely to $T^{\frac{1}{4}}$, but finally it does not depend on temperature at low temperatures for which the waves with $\omega = KT/\hbar$ have the free path of the same order, as the crystal dimensions. Further decrease of temperature as is known, leads to variation of κ proportional to T^3.[6,7]

The temperature range in which κ does not change at all is especially wide for the diamond because of its high value of Debye temperature. This result explains the experimental data concerning the thermal conductivity of diamond,[8,9] according to which data κ reveals no noticeable change in the temperature range from 24 to 343°.

Detailed calculations have been published in *Journal of Physics* [**4**, 259 (1941)]. In the same publication the calculations involving the region of low temperatures are to appear shortly.

[5] R. Peierls, Ann. d. Physik **3**, 1097 (1929).

[6] W. J. de Haas and T. Biermasz, Physica **5**, 619 (1938).
[7] H. B. G. Casimir, Physica **5**, 495 (1938).
[8] Eucken, Physik. Zeits. **12**, 1055 (1911).
[9] W. J. de Haas and T. Biermasz, Physica **5**, 47 (1938).