Special Relativity in Refracting Media

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A reformulation of the Lorentz transformations for an observer in a refracting but non-dispersive medium is suggested. In this statement of the transformations, the speed ($\mathbf{c}=c/n$) of light signals in the medium replaces the free space speed (c) which appears in the usual form. It is shown that the Fresnel drag coefficient takes the same form in the new formulation that it did in the old. Other consequences of this assumption are discussed and, in particular, the mechanics of the photon are shown to lead to correct expressions for Snell's law and for the pressure of radiation at the boundary of the medium.

IN spite of the successes of special relativity in dealing with the problems of the velocity of light, such as the Michelson-Morley experiment, the Fresnel-Fizeau drag coefficient, and the results of Airy on aberration, the more recent attempts to base the treatment of optics on the theory have not been very successful. These attempts have followed two distinct lines. In the first, various workers[1] have developed a relativistic electrodynamics. While this work is of interest in the interpretation of many effects, it is admittedly incomplete and has led to few new optical predictions. Working along a second line, several authors[2] have attempted to deduce the laws of geometrical optics from a photon hypothesis. In general, it has been necessary in such attempts to introduce some *ad hoc* assumptions regarding the velocity or the momentum of the photon. The difficulties which have arisen seem to make it desirable to attempt another approach to the subject, in the hope that this may lead to suggestions for the development of a more complete electrodynamic theory.

In the present treatment, we shall assume that the discrete distribution of matter which comprises a medium may be replaced, to a first approximation, by a continuum characterized by phenomenological constants, such as a dielectric constant, a refractive index, etc. Under this assumption, a treatment of such optical phenomena as diffraction and scattering would be impossible. It should, however, be quite adequate for the treatment of geometrical optics and the optics of the photon. We shall further assume that the medium is refractive but non-dispersive.

Since there are no sufficiently precise experiments which involve an observer embedded in a refracting medium, we cannot say whether such an observer could detect, by *optical* experiments not involving the boundaries of the medium, his motion through the medium. It is certainly true, however, that an observer at rest with respect to the medium could not detect any absolute motion in space, since this would be a violation of the relativity postulate. Were he able to detect such motion, an observer outside the medium could also detect such motion by exchange of signals with him.

If we consider the role of the speed c in the mechanics of special relativity, we find that it acquires its special importance simply because it is the signal speed in free space, and as such it must be used in making the measurements of length and time necessary for the discussion of a mechanics. In a medium of constant refractive index n, the signal speed is $\mathbf{c}=c/n$, and it is not unreasonable to suppose that it is this speed which must replace the c of free space in the discussion of mechanics in the medium.[3] In studying the consequence of such an assumption, it will be necessary for us to refer to six different observers. Two observers, O and O', move freely

[1] For example see W. Pauli, Enc. der Math. Wiss. **5**, Part 2, 539 (1922); H. Bateman, Bull. Nat. Res. Council **4**, Part 6, 96 (1922); W. Gordon, Ann. d. Physik **72**, 421 (1923).

[2] L. Brillouin, Comptes rendus **178**, 1696 (1924); S. Serghiesco, Comptes rendus **202**, 1563 and 1761 (1936).

[3] H. Weyl has considered the possibility of this assumption, but has discarded it because of the success of the conventional theory in the deduction of the Fresnel drag coefficient [*Raum, Zeit, Materie,* fifth edition, Springer, Berlin, 1923, pp. 180–181]. We shall see later that the present assumption gives rise to the same expression for the drag coefficient.

in free space, while a third O_0 makes his measurements in free space, but is at rest with respect to the medium. Within the medium, the two observers $\mathbf{O}$ and $\mathbf{O}'$ are free to move, while the third $\mathbf{O}_0$ is fixed with respect to the medium. In discussing transformations connecting the observations of these various observers, we shall use boldface type for observations made within the medium, and italic type for those made in free space. The diacritical marking of the symbols used for the observed quantities will correspond to that of the symbols used for the observers. The complete notation, together with a statement of the relative velocities of the various observers is given in Table I. For simplicity, we shall take all velocities as parallel.

If now the velocity of the observer O' with respect to O is U, the velocity addition theorem will take the usual form:

$$v'=(v-U)/(1-Uv/c^2). \qquad (1)$$

The corresponding theorem for any two observers $\mathbf{O}$ and $\mathbf{O}'$ within the medium will then be

$$\mathbf{v}'=(\mathbf{v}-\mathbf{U})/(1-\mathbf{U}\mathbf{v}/\mathbf{c}^2). \qquad (1')$$

There is one very important consequence of this formula. If $\mathbf{O}$ observes a signal of velocity $\mathbf{v}=\mathbf{c}=c/n$, then $\mathbf{O}'$ will observe the same velocity, i.e., $\mathbf{v}'=\mathbf{c}=c/n$. This indicates that an observer in a continuous medium cannot, by measurements of the velocity of light alone, determine his velocity with respect to the medium. It, of course, follows *a fortiori* that an observer in the medium will be unable to observe any absolute motion in space.

At first glance, it would seem that Eq. (1′) was in contradiction with the experimental results on the Fresnel drag coefficient, but it is to be noted that in every experiment which verifies this coefficient, the observer is located *outside* the medium. In order to discuss experiments of this kind, we must therefore obtain transformations connecting the results of observers in two different media. So far, however, we have made no assumption connecting observations within the medium with those made outside. The simplest assumption of this nature which we can make is that measuring rods and clocks are unaffected by transfer between two observers O_0 and $\mathbf{O}_0$, both of whom are at rest

TABLE I. *Notation. Observers O_0 and $\mathbf{O}_0$ are at rest with respect to the medium.*

	In Free Space				In Medium		
Observer	O'	O	O_0		$\mathbf{O}_0$	$\mathbf{O}$	$\mathbf{O}'$
Speed of Light		c				$\mathbf{c}=c/n$	
x coordinate	x'	x	x_0	=	$\mathbf{x}_0$	$\mathbf{x}$	$\mathbf{x}'$
t coordinate	t'	t	t_0	=	$\mathbf{t}_0$	$\mathbf{t}$	$\mathbf{t}'$
Velocity	v'	v	v_0	=	$\mathbf{v}_0$	$\mathbf{v}$	$\mathbf{v}'$
Velocity of O'	0	U	—		—	—	—
" " O	$-U$	0	U_0		—	$\mathbf{V}$	—
" " O_0	—	$-U_0$	0		0	—	—
" " $\mathbf{O}_0$	—	—	0		0	$-\mathbf{U}_0$	—
" " $\mathbf{O}$	—	V	—		$\mathbf{U}_0$	0	$-\mathbf{U}$
" " $\mathbf{O}'$	—	—	—		—	$\mathbf{U}$	0

with respect to the medium. Thus the kinematic observations of these two observers will be identical and we shall have $x_0=\mathbf{x}_0$, $t_0=\mathbf{t}_0$, and $v_0=\mathbf{v}_0$. We must now establish the velocity addition theorem for the two observers O and $\mathbf{O}$. To do this we make use of the proper observers O_0 and $\mathbf{O}_0$.

If the observer $\mathbf{O}$ observes a particle of velocity $\mathbf{v}$, the observer $\mathbf{O}_0$ will observe a velocity $\mathbf{v}_0$ given by

$$\mathbf{v}_0=(\mathbf{v}+\mathbf{U}_0)/(1+\mathbf{v}\mathbf{U}_0/\mathbf{c}^2),$$

in which $\mathbf{U}_0$ is the velocity of $\mathbf{O}$ with respect to the medium. In accordance with the above assumption, however, $\mathbf{v}_0=v_0$ and we can find, for the velocity v observed by an observer O, the formula:

$$v=\frac{\mathbf{v}(1-U_0\mathbf{U}_0/\mathbf{c}^2)+(\mathbf{U}_0-U_0)}{(1-U_0\mathbf{U}_0/c^2)+\mathbf{v}(\mathbf{U}_0/\mathbf{c}^2-U_0/c^2)}, \qquad (2)$$

in which U_0 is the velocity of O with respect to the medium. Similarly, by interchanging boldface and italic type, it follows from symmetry that

$$\mathbf{v}=\frac{v(1-U_0\mathbf{U}_0/c^2)+(U_0-\mathbf{U}_0)}{(1-U_0\mathbf{U}_0/\mathbf{c}^2)+v(U_0/c^2-\mathbf{U}_0/\mathbf{c}^2)}. \qquad (2')$$

From these formulae, we can obtain directly the observed relative velocities of the two systems. The observer O finds for the velocity of the system $\mathbf{O}$ a value V given by

$$V=(\mathbf{U}_0-U_0)/(1-U_0\mathbf{U}_0/c^2), \qquad (3)$$

and with an interchange of observers we find

$$\mathbf{V}=(U_0-\mathbf{U}_0)/(1-U_0\mathbf{U}_0/\mathbf{c}^2). \qquad (3')$$

There is thus no simple way of expressing the formulae (2) and (2′) in terms of the observed relative velocities of the two systems.

If now the observer **O** observes a light pulse of velocity $\mathbf{c}=c/n$, then the velocity v of this pulse with respect to O will be obtained by making the substitution $\mathbf{v}=\mathbf{c}$ in Eq. (2). We then obtain

$$v=(\mathbf{c}-U_0)/(1-\mathbf{c}U_0/c^2),$$

which may be rewritten in the form

$$v=(c/n-U_0)/(1-U_0/nc) \\ =c/n-U_0(1-1/n^2)+\cdots. \quad (4)$$

to give the usual form for the Fresnel drag coefficient in special relativity for a medium whose velocity is $-U_0$ with respect to the observer.

It is interesting to note here that there should be an inverse drag effect on light traveling in free space as seen by an observer moving in the medium. An observer **O** should find a velocity $\mathbf{v}$ for a light signal in vacuum [cf. Eq. (2′)] given by

$$\mathbf{v}=(c-\mathbf{U}_0)/(1-c\mathbf{U}_0/\mathbf{c}^2),$$

which may be rewritten in the form

$$\mathbf{v}=(c-\mathbf{U}_0)/(1-n^2\mathbf{U}_0/c) \\ =c-\mathbf{U}_0(1-n^2)+\cdots. \quad (4')$$

We notice that this velocity depends only on the velocity of **O** with respect to the medium, just as, in the previous case, the observed velocity depended only on the velocity of the free space observer with respect to the medium.

If $U_0=\mathbf{U}_0$, the velocities of the observers O and **O** with respect to the medium are the same, and the transformation (2) takes the form:

$$v=\mathbf{v}(1-U_0^2/c^2)/[(1-U_0^2/c^2) \\ +U_0\mathbf{v}(c^{-2}-\mathbf{c}^{-2})], \quad (5)$$

with an inverse transformation obtained by an interchange of type face. We notice that in the present case $V=\mathbf{V}=0$, i.e., both observers agree that they are at rest with respect to one another.

From now on we shall be concerned only with the observations of the proper observers O_0 and $\mathbf{O}_0$. We shall therefore drop the use of the subscript zero, except as it is required in the normal usage for the rest mass and rest energy, and in the labeling of the observers.

We shall be particularly interested in the expressions for the energy and momentum of a particle as observed within the medium. These follow from an argument exactly analogous to that used in the conventional treatment.[4] For the proper observer $\mathbf{O}_0$ we then find that

$$\mathbf{E}=\mathbf{m}\mathbf{c}^2=\mathbf{m}_0\mathbf{c}^2/(1-\mathbf{v}^2/\mathbf{c}^2)^{\frac{1}{2}} \quad (6)$$

and

$$\mathbf{p}=\mathbf{m}\mathbf{v}=\mathbf{m}_0\mathbf{v}/(1-\mathbf{v}^2/\mathbf{c}^2)^{\frac{1}{2}}. \quad (7)$$

We thus obtain, for the ratio of energy to momentum in the medium, the expression

$$\mathbf{E}/\mathbf{p}=\mathbf{c}^2/\mathbf{v}. \quad (8)$$

If we expand the expressions (6) and (7) in terms of the proper rest mass $\mathbf{m}_0$, we find that

$$\mathbf{E}=\mathbf{m}_0\mathbf{c}^2+\tfrac{1}{2}\mathbf{m}_0\mathbf{v}^2+3\mathbf{m}_0\mathbf{v}^4/8\mathbf{c}^2+\cdots, \quad (6')$$

and that

$$\mathbf{p}=\mathbf{m}_0\mathbf{v}+\tfrac{1}{2}\mathbf{m}_0\mathbf{v}^3/\mathbf{c}^2+\cdots. \quad (7')$$

Thus the Newtonian terms are left unaffected by the proposed assumption. The most serious difference between the present and the conventional treatment is that the rest energy in the medium is now $\mathbf{E}_0=\mathbf{m}_0\mathbf{c}^2=\mathbf{m}_0c^2/n^2$. At first glance this result may seem strange, as it indicates that a particle passing from one medium to another must change either its rest mass or its energy. If we remember, however, that the energy necessary to build up an electric charge configuration is inversely proportional to the dielectric constant of the medium surrounding it (i.e., in this case to n^2), the new result seems in complete agreement with classical electromagnetic theory.

We now have the necessary equations for the consideration of geometrical optics. It is commonly recognized that the result of a measurement of the velocity of light yields the group velocity, but there remains the question of whether this quantity is identical with the velocity of the photon in a refractive medium. If we limit ourselves to true velocity determinations, such as those of Fizeau, Foucault, and Michelson, rather than interference experiments, there seems to be no doubt that the measured velocity in a non-dispersive medium is that of the photon, since the detecting device in

[4] See for example: R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford, 1934), Chapter III.

the measurement is one which responds to photon energy. Consequently, we shall suppose that the photon velocity in a refractive medium is given by $\mathbf{c}=c/n$. Substitution of this value into Eqs. (6) and (7) shows that the rest mass of the photon must be zero for a proper observer in the medium. Equation (8) then becomes, for the photon:

$$\mathbf{E}/\mathbf{p}=\mathbf{c}=c/n. \tag{9}$$

When a photon passes from one medium to another, it is almost certain that its energy remains unchanged. We may reach this conclusion either from the consideration of the constancy of frequency of a wave in passing from one medium to another, or from the photoelectric effect. In the former case the energy must remain constant if the Einstein frequency relation is to hold. In the latter case, the actual absorption of the photon takes place within the medium, and it is very hard to see how the Einstein photoelectric equation could hold if the photon energy changed at the boundary. Consequently, for a given photon in vacuum, we have

$$E/p=c;$$

and for the same photon in the medium, we have

$$\mathbf{E}/\mathbf{p}=\mathbf{c}=c/n;$$

with the condition that $E=\mathbf{E}$. It then follows that

$$\mathbf{p}=np. \tag{10}$$

This equation indicates that the momentum of the photon increases as it passes from free space into the refractive medium. If we suppose that the change of momentum is normal to the surface, as is indicated by the lack of a tangential light pressure; and if we let I and R, respectively, be the angles of incidence and refraction; we then have

$$\mathbf{p}/p=n=\sin I/\sin R,$$

so that Snell's law follows directly from the photon mechanics.

Use of the component of the momentum perpendicular to the surface enables us to calculate the radiation pressure on the interface. If N photons per second pass unit area of the surface, and none are reflected, the pressure will be

$$P=(Nh\nu/c)(\cos I-n\cos R),$$

which reduces for normal incidence to

$$P=(Nh\nu/c)(1-n).$$

While Eq. (10) eliminates the difficulty which has been present in earlier particle treatments of optics, namely that the momentum must increase when a photon passes into a more refractive medium, it seems strange that it ascribes to the photon a momentum which is inversely proportional to its velocity. This, of course, follows only because of the constancy of the energy, which requires that the effective mass of the photon increase by a factor n^2 in passing the boundary.

If the usual de Broglie relationships are derived for an observer in the medium, the relation between the velocity of a particle $\mathbf{G}$ and the phase velocity of the associated wave $\mathbf{W}$ becomes

$$\mathbf{GW}=\mathbf{c}^2=c^2/n^2. \tag{11}$$

In the case of the non-dispersive medium this equation reduces to a trivial identity for the photon and the electromagnetic wave.

The treatment of dispersive media would be of considerable interest, but it presents many difficulties. It seems clear, however, that the mechanics of an observer embedded in a dispersive medium may depend on the exact nature of the signal which he uses in making his observations and in synchronizing his clocks.

We present this paper, with an alternative formulation of the special relativity postulates, with considerable hesitation. The difficulties of performing direct experiments to test its validity seem insurmountable, since such experiments would involve precise measurements with the apparatus moving in a refracting medium. The only justification for the proposed system seems to be the experimental determination of the ratio $\mathbf{p}/p$ given by the interpretation of Snell's law on a photon hypothesis.