electrons were not drawn in to the target but were slowed down by the decreasing magnetic field. Fortunately the operation of the accelerator is not sensitive to the alignment of the pole faces. No difference in the output can be detected when the pole faces are placed off axis as far as a thirty-second of an inch. It is also surprising that vacuum requirements are not as severe as was expected. No rigorous outgassing is necessary and the apparatus has been run with a vacuum as poor as 10^{-5} mm Hg. The tube can be opened for changes and operated three-quarters of an hour after sealing shut.

At present, low flux densities have been used at the orbit. When these are increased, it should be possible to go to 5 million volts even with this small model. One of the promising possibilities for the induction accelerator as a research tool is that the electrons from the beam can come out through the glass walls of the doughnut after they strike the target. They should be fairly homogeneous in energy provided that the target has a high atomic number. The great increase in bremsstrahlung production with rising electron energy in addition to the concentration of this radiation in a cone of solid angle mc^2/E about the original electron direction gives the induction accelerator the possibility of providing an intense source of x-radiation for nuclear investigations. Since there is no evident limit on the energy which can be reached by induction acceleration, it may soon be possible to produce some small scale cosmic-ray phenomena in the laboratory.

I am indebted to Professor H. M. Mott-Smith and Professor R. Serber for many discussions of the theoretical aspects of this problem and to Mr. R. P. Jones for assistance in the construction of the magnet.

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Electronic Orbits in the Induction Accelerator

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The first section gives a general account of the principles of operation of the electron induction accelerator. The second section gives the more detailed analysis of the orbits of the electrons which was undertaken to serve as a guide in the design of the accelerator.

I. Introduction

The construction and operation of an induction accelerator for electrons has been discussed in the preceding paper.[1] The idea of using the principle of electromagnetic induction for the production of high energy particles has been entertained by a number of investigators,[2] but has hitherto met with little success in application. It was therefore felt necessary to carry out a more careful analysis of the orbits of electrons in changing magnetic fields for the double purpose of determining whether such a device was practicable, and to serve as a guide in its design.

The basic idea of the accelerator is a simple one. An electron started in a radially symmetric magnetic field at the proper position and with the right velocity will move in a circle of radius given by

$$p = eHr/c. \quad (1)$$

If now the magnetic flux enclosed by the orbit, ϕ, is increased, a tangential electric field will be produced at the orbit, $E_\varphi = \dot{\phi}/2\pi rc$, which will accelerate the electron. If the magnetic field is so arranged that p and H increase proportion-

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[1] Also D. W. Kerst, Phys. Rev. **58**, 841 (1940); *ibid.* **59**, 110 (1941).

[2] G. Breit and M. A. Tuve, Carnegie Institution Year Book (1927–28) No. 27, 209. R. Wideröe, Arch. f. Electrotechnik **21**, 400 (1928); E. T. S. Walton, Proc. Camb. Phil. Soc. **25**, 469–81 (1929); W. W. Jassinsky, Arch. f. Electrotechnik **30**, 500 (1936).

ately, the radius of the orbit will remain unchanged; the electron continues to move in the equilibrium orbit ($r=r_0$), but with momentum constantly increasing as H is increased.

The condition for the existence of such an equilibrium orbit is readily found. The rate of increase of momentum is

$$\dot{p}=e\dot{\phi}/2\pi r_0 c,$$

which gives

$$p=e(\phi-\phi_1)/2\pi r_0 c.$$

Using (1), with $r=r_0$, we find

$$H=(\phi-\phi_1)/2\pi r_0^2. \qquad (2)$$

Thus the change in flux through the orbit must be twice that which would obtain if the magnetic field were uniform in space. To satisfy this requirement one must have a strong central field to supply the necessary flux, and a weaker field to hold the electron in its circular orbit.

The energies which can be obtained by this means are quite high. The Illinois accelerator ($r_0=7.5$ cm, $H_{\max}=1200$ gauss) should give, according to (1), electrons of 2.2 Mev. It will be observed that, since the phase of the electron in its orbit is immaterial to the operation of the accelerator, the relativistic change of mass with velocity causes no difficulty for a machine of this type.

However, the very fact that such high velocities are attained brings with it new difficulties which at first sight seem rather formidable. Since, during most of the time the electron is being accelerated, its velocity is very nearly the velocity of light, its path length in the machine is very great. The magnet of the Illinois machine was activated by 600-cycle a.c., and the acceleration took place during a quarter-cycle. In this time the electrons traveled nearly 100 kilometers and made about 200,000 revolutions. It is therefore essential, to obtain any appreciable beam intensity and to overcome the effects of scattering by the residual gas and of space charge, that there be strong focusing forces to hold the electron in its equilibrium orbit.

The focusing requirements impose additional restrictions on the form of the magnetic field. Consider first the radial focusing, that is, the motion in the plane of the orbit. The equilibrium Eq. (1) is just the condition for the balancing of the magnetic and centrifugal forces. If the electron is displaced from the equilibrium radius, an unbalanced force will act, and, since the centrifugal force is proportional to $1/r$, this force will be directed towards the equilibrium radius provided the magnetic field falls off less rapidly than $1/r$. The electron, when displaced, will then oscillate around the equilibrium orbit. We shall see that this is a damped oscillation;[3] its amplitude is proportional to $H^{-\frac{1}{2}}$. The focusing

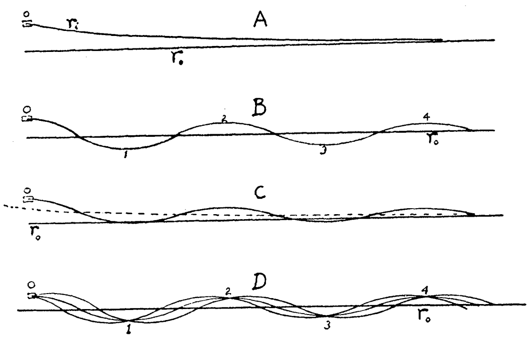

FIG. 1. The figures represent the developed paths of electrons. r_0 is the position of the equilibrium orbit, and r_i is the position of the instantaneous circle. The injector is at 0. (A) Path of an electron injected tangentially on its instantaneous circle. It approaches r_0 without oscillation. (B) Path of an electron with an instantaneous circle coincident with the equilibrium orbit. Oscillation occurs about r_0. (C) Path of an electron whose instantaneous circle does not pass through 0 at the time of injection but is between 0 and r_0. The oscillation is about r_i while r_i approaches r_0. (D) A real beam from the injector showing image formation at 1, 2, 3 and 4. The instantaneous circle is coincident with the equilibrium orbit.

is thus most effective at small velocities, where scattering and space charge effects are most important. Figure 1(B) shows such a damped oscillation about the equilibrium radius. There will also be vertical focusing, forces tending to drive the electron back to the plane of the orbit, provided the magnetic field strength is decreasing with increasing r. The restoring force here is due to the curvature of the magnetic field near the median plane; the focusing action is similar to the magnetic focusing effective in

[3] This can be shown in the following way. The energy of oscillation is $E=J\nu_r$, where J is the action variable, ν_r the frequency of oscillation. The amplitude of oscillation a is given by $E=\frac{1}{2}M(2\pi\nu_r)^2a^2$, with M the transverse mass. Thus $a^2=J/2\pi^2M\nu_r$. The frequency of rotation of the electron in its orbit is $\nu=p/2\pi rM$, so, writing $n_i=\nu_r/\nu$, $a^2=Jr/\pi n_i p$. It follows from the adiabatic invariance of the action variable that $a\sim(r/n_ip)^{\frac{1}{2}}\sim(1/n_iH)^{\frac{1}{2}}$. For $H\sim 1/r^n$, n_i turns out to be a constant.

the cyclotron. The vertical oscillations are similarly damped. Thus both radial and vertical focusing can be obtained, for a magnetic field which in the region of the orbit is of the form $H \sim 1/r^n$, within the limits $0<n<1$.

The injection of electrons into the machine is a problem which also has its difficulties. It is obviously not possible to inject the electrons at the position of the equilibrium orbit. Nor is it possible to shoot them into the magnetic field from an external source as was attempted by Wideröe; for, unless the magnetic field were increasing extremely rapidly, the orbits would be nearly symmetric around the point where the radial velocity vanished: the electrons would shoot right out again. Thus the electrons must be injected at a point within the magnetic field, either inside or outside the equilibrium orbit (in the following we shall suppose it outside).

The analysis of the paths of the electrons is greatly simplified by the fact that the magnetic field increases little during one revolution of the electron. The adiabatic theorem can therefore be applied: it states that under these circumstances the motion is very nearly the same at each instant as if the magnetic field were held fixed at its instantaneous value. Consider now an electron injected tangentially with a given momentum p, at radius r. As the magnetic field is increased from zero there will come a time when p, H, and r satisfy the relation (1). An electron injected at this time would, if the magnetic field remained constant, travel in a circle. The adiabatic theorem tells us that at any later time, even though the field is increasing, the position of the electron will still be given by (1); at each instant the electron is moving on its "instantaneous circle." Since the injection radius is larger than the radius of the equilibrium orbit, the flux within the instantaneous circle is less than that necessary to hold the electron at r. Momentum is gained less rapidly than necessary to keep the radius of the orbit constant, and the instantaneous circle shrinks towards the equilibrium orbit. The actual orbit of the electron will thus be a spiral which approaches the equilibrium orbit asymptotically as shown in Fig. 1(A).

This contraction of the instantaneous orbit in itself provides relatively little clearance for the injection electrodes. The contraction is fastest for small velocities, but it is inadvisable to use too small injection energies because of relatively large space charge forces and scattering. In the University of Illinois accelerator (at injection energy 180 v, injection radius 9 cm, $r_0=7.5$ cm) the instantaneous circle shrinks 1 mm during the first revolution of the electron. However, electrons injected not quite tangentially, or a little too early or too late, so that at the time of injection their instantaneous circle lies outside or inside the injection radius, will, as the discussion of focusing shows, oscillate with decreasing amplitude about the appropriate spiral orbit. This situation is shown by Fig. 1(C). The period of these oscillations is different from the period of revolution, so that the electron will, in general, execute several decreasing oscillations before the maximum displacement occurs near the starting electrodes. There is thus a considerable range of starting angles and times which allow capture into the equilibrium orbit, even though the electrons are injected at constant energy. In Fig. 1(D) the paths of divergent rays from an injector are shown. The formation of an image occurs at intervals of half an oscillation. In addition to these magnetic effects, the space charge spreading of the beam can also play a role in causing electrons to clear the injector.

After the electrons have been accelerated a variety of devices can be employed to bring them to strike a target, all of them based, of course, on destroying the relation (2) at the appropriate time. A simple method was used in the Illinois accelerator to accomplish this. A small amount of iron dust in an insulating cement made up a part of the magnetic circuit which fed flux through the center of the orbit. These dust particles saturate before the iron core does. When saturation sets in, p increases less rapidly than the magnetic field at the orbit. The electrons then spiral inward until they hit the target.

II. The Electronic Orbits

The position of the electron will be specified by the cylindrical coordinates (r, φ, z). We shall first consider the motion in the median plane

($z=0$), in which the magnetic field has only a z component, $H_z=H(r,t)$. This field can be derived from a vector potential, $A_\varphi=A(r,t)$, by the relation

$$H_z=r^{-1}\partial(rA)/\partial r=A/r+\partial A/\partial r. \quad (3)$$

There is a concomitant electric field $E_\varphi=-\dot{A}/c$.

The Hamiltonian function is

$$\mathcal{H}=c(m^2c^2+p_r^2+p^2)^{\frac{1}{2}}, \quad p=p_\varphi/r+eA/c.$$

Here p_φ is the canonical angular momentum, defined by the relation

$$p=mr\dot{\varphi}/(1-v^2/c^2)^{\frac{1}{2}},$$

and is a constant of the motion.

For any p_φ, the radius r_i of the instantaneous circle is determined by the condition $\dot{p}_r=0$, or, since $\dot{p}_r=c^2pp'/\mathcal{H}$, by

$$p'_i=-(p_\varphi/r^2-eA'/c)_i=0. \quad (4)$$

We use a prime to denote partial differentiation with respect to r. The subscript i indicates that the quantity is to be evaluated at $r=r_i$; the subscript 0 will be used for $r=r_0$. Since an electron moving on the instantaneous circle has a momentum p_i, we see, using (3), that (4) is another expression of the relation (1).

The condition for motion in the equilibrium circle ($r_i=r_0$, a constant independent of time) is then

$$p_\varphi=0, \quad A'_0=0, \quad (5)$$

since p_φ/r_0^2 is independent of time, and we take A of the form[4] $A(r,t)=f(r)h(t)$.

It can readily be shown that the condition $A'_0=0$ is equivalent to (2). We have

$$\phi=\int\mathbf{H}\cdot d\boldsymbol{\sigma}=\int \operatorname{curl}\mathbf{A}\cdot d\boldsymbol{\sigma}=\int\mathbf{A}\cdot d\mathbf{s}=2\pi r_0A_0,$$

and from (3), $H_0=A_0/r_0$. Thus on the equilibrium circle, $\phi_0=2\pi r_0^2H_0$.

It may be remarked that for a magnetic field *everywhere* of the form $H=b/r^n$ (and thus lacking the central field necessary to satisfy the flux condition (2)) the vector potential is $A=b/(2-n)r^{n-1}$, and (4) gives $er_i^2H_i/c=(2-n)p_\varphi/(1-n)$, or, from (1), $p_ir_i=\text{constant}$.

[4] The case in which there is a time independent flux ϕ_1 through the center of the orbit is also of interest. It can be represented by $A(r,t)=f(r)h(t)+\phi_1/2\pi r$, which necessitates only replacing p_φ, in (5), by $p_\varphi+e\phi_1/2\pi c$.

In such a field the electron spirals rapidly towards the center as its momentum increases.

Radial focusing

In order to take advantage of the adiabatic theorem, we write $r=r_i+x$, and suppose $x\ll r_i$. If we introduce x and $p_x=p_r$ as new canonical variables, and in the new Hamiltonian function, $\mathcal{H}_1=\mathcal{H}-p_x\dot{r}_i$, retain only terms not higher than quadratic in x and p_x we obtain

$$\mathcal{H}_1=\epsilon+\tfrac{1}{2}c^2p_x^2/\epsilon+\tfrac{1}{2}p_ip''_ix^2/\epsilon-p_x\dot{r}_i. \quad (6)$$

Here $\epsilon=c(m^2c^2+p_i^2)^{\frac{1}{2}}$ is the energy of an electron moving on the instantaneous circle.

The equations of motion are

$$\dot{x}=c^2p_x/\epsilon-\dot{r}_i,$$
$$\dot{p}_x=-c^2p_ip''_ix/\epsilon, \quad (7)$$

which, by eliminating p_x, give

$$\ddot{x}+\dot{\epsilon}\dot{x}/\epsilon+c^4p_ip''_ix/\epsilon^2=-\ddot{r}_i-\dot{\epsilon}\dot{r}_i/\epsilon. \quad (8)$$

The forcing terms on the right represent the nonadiabatic corrections to the motion.

If we write $x=(mc^2/\epsilon)^{\frac{1}{2}}u$, and omit the forcing terms for the moment, (8) becomes

$$\ddot{u}+\omega_r^2u=0,$$
$$\omega_r^2=c^4p_ip''_i/\epsilon^2-\tfrac{1}{2}\ddot{\epsilon}/\epsilon+\tfrac{1}{4}\dot{\epsilon}^2/\epsilon^2\sim c^4p_ip''_i/\epsilon^2, \quad (9)$$

since the second and third terms of ω_r^2 are smaller than the first by a factor of order $(cp_m/\epsilon)^2f^2/\nu^2$, where f is the frequency of the magnetic field, ν the frequency of revolution of the electron, and p_m is the maximum momentum attained by the electron. If we write

$$p''_i=n_i^2p_i/r_i^2, \quad (10)$$

Eq. (9) takes the simple form

$$\omega_r=n_iv/r_i.$$

The frequency of radial oscillation is thus just n_i times the frequency of rotation of the electron in its orbit. In general n_i will be a slowly varying function of r, but for $H\sim 1/r^n$, n_i is independent of r, $n_i=(1-n)^{\frac{1}{2}}$. We also see that for focusing oscillations to occur we must have $p''_i>0$, which, on the equilibrium circle, reduces to $A''_0>0$, in virtue of (5). The equilibrium circle is thus the point of minimum electric field.

If we choose $t=0$ when $A=0$, we require the solution of (9) only for $\omega_rt\gg 1$, since this condition is already met at the time of injection of the electron. The asymptotic form of the solution for $\omega_r t \gg 1$ is

$$u = a(c/r_0\omega_r)^{\frac{1}{2}} \sin\left[\int_0^t \omega_r dt + \gamma\right],$$

or

$$x = a(mcr_i/n_i p_i r_0)^{\frac{1}{2}} \sin\left[\int_0^t \omega_r dt + \gamma\right], \quad (11)$$

with a and γ arbitrary constants. The amplitude of the radial oscillations is thus damped by a factor $(p_i/r_i)^{-\frac{1}{2}} = (eH_i/c)^{-\frac{1}{2}}$.

Injection of the electron

An electron injected on its instantaneous circle, will, in the adiabatic limit, continue to move on this circle as it shrinks down to the equilibrium orbit. The rate of contraction of the instantaneous circle can be obtained by differentiating (4),

$$\dot{r}_i = -e\dot{A}'_i/cp''_i = -\dot{A}'_i/(A''_i + 2A'_i/r_i).$$

For $r_i - r_0$ small this reduces, in virtue of (5), to $\dot{r}_i = -(r_i - r_0)\dot{A}''_0/A''_0$. The shift of the instantaneous circle toward the equilibrium orbit produced by a change δH of the magnetic field is thus given by

$$\delta r_i/(r_i - r_0) = -\delta H/H. \quad (12)$$

An electron injected not on the instantaneous circle, but in its neighborhood, will oscillate about the instantaneous circle; its motion is described by (11). Since the instantaneous circle is simultaneously approaching the equilibrium orbit, the resultant motion is that shown in Fig. 1(C).

Both the damping of the oscillation about the instantaneous circle and the shrinking of the instantaneous circle toward the equilibrium orbit are used to cause the electrons to miss the injector on successive revolutions. In terms of the voltage gain per revolution δV, and the injection voltage V, the damping is expressed by

$$\delta a/a = -\tfrac{1}{4}\delta V/V, \quad (13)$$

where $\delta a/a$ is the fractional decrease in the amplitude of oscillation. Similarly the shrinking described by (12) is

$$\delta r_i/(r_i - r_0) = -\tfrac{1}{2}\delta V/V. \quad (14)$$

We may verify that the wandering from the instantaneous circle caused by the nonadiabatic correction terms in (8) is in fact very small. The term $\dot{\epsilon}\ddot{r}_i/\epsilon$ is of order $\ddot{r}_i v^2/c^2$, and will be neglected. Let t_I be the time of injection. The asymptotic solution of (8) for $\omega_r t \gg 1$ can be written down with the aid of the solutions (11). It is

$$x = -\int_{t_I}^{t} \left\{\frac{n_i(s)p_i(s)r_i(t)}{n_i(t)p_i(t)r_i(s)}\right\}^{\frac{1}{2}} \frac{\ddot{r}_i(s)}{\omega_r(s)} \sin\left[\int_s^t \omega_r dt'\right] ds$$

$$= \left\{\frac{n_i(t_I)p_i(t_I)r_i(t)}{n_i(t)p_i(t)r_i(t_I)}\right\}^{\frac{1}{2}} \frac{\ddot{r}_i(t_I)}{\omega_r(t_I)^2} \times \cos\left[\int_{t_I}^t \omega_r dt'\right] - \frac{\ddot{r}_i(t)}{\omega_r(t)^2}.$$

The maximum displacement is

$$x_m = -2\ddot{r}_i(t_I)/\omega_r(t_I)^2.$$

In the interesting case, near the beginning of the acceleration when the field is increasing linearly with time, differentiation of (12) gives $\ddot{r}_i = -2\dot{r}_i\dot{H}/H$. If δH and δr_i are the increments of H and r_i during one revolution of the electron, we then have

$$x_m = \delta r_i \delta H/\pi^2 n_i^2 H,$$

all quantities being evaluated at $t = t_I$.

Vertical focusing

Near the median plane the magnetic field has a radial component determined by $\partial H_r/\partial z = \partial H_z/\partial r$. The vector potential then takes the form

$$A_\varphi = A(r, t) - \tfrac{1}{2}A_1(r, t)z^2,$$

where

$$A_1(r, t) = \partial H_z/\partial r\,|_{z=0} = (c/e)[(rp)'/r]'_i.$$

For small deviations from the median plane, the Hamiltonian (6) must be supplemented by the terms

$$\mathcal{H}_z = \tfrac{1}{2}c^2 p_z^2/\epsilon - \tfrac{1}{2}c^2 p_i[(rp)'/r]'_i z^2/\epsilon.$$

The equations of motion are

$$\dot{z} = c^2 p_z/\epsilon, \quad \dot{p}_z = c^2 p_i[(rp)'/r]'_i z/\epsilon,$$

which differ from (7) only in the absence of a forcing term, and in the replacement of p''_i by

$-[(rp)'/r]'_i$. The solutions are thus the same as (11), but with ω_r and n_i replaced by $\omega_A = m_i v/r_i$ and m_i, defined by

$$-[(rp)'/r]'_i = m_i^2 p_i / r_i^2. \quad (15)$$

Choice of the field shape

It has been shown that there must be a position of minimum electric field (i.e., $A'_0=0$, $A''_0>0$) for radial stability, and that the field must decrease with increasing distance from the axis for axial stability of the electrons. If the field in the region of the electron's orbit is of the form

$$H_z = H_0(t)(r_0/r)^n,$$

the vector potential is given by

$$A = \frac{H_0(t)}{2-n}\left[\frac{r_0^n}{r^{n-1}} + (1-n)\frac{r_0^2}{r}\right].$$

The second term in the bracket gives an electric field, but no magnetic field at the position of the orbit; it represents the effect of the strong central magnetic field required to satisfy the flux condition (2).

Evaluating (10) and (15) for this potential we find

$$n_i^2 = (1-n), \quad m_i^2 = n.$$

The conditions for radial and vertical focusing, $n_i^2>0$, $m_i^2>0$, thus demand $0<n<1$, as we have already seen.

An electron deflected from its instantaneous circle by a small angle scattering will execute oscillations of amplitude proportional to the transverse velocity imparted by the scattering divided by the initial frequency of oscillation. The ratio of radial to vertical amplitude will thus be

$$x/z = [n/(1-n)]^{\frac{1}{2}}.$$

Since in the Illinois accelerator the vertical clearance was smaller than the horizontal, it was desired to have vertical focusing stronger than horizontal focusing. The pole pieces were accordingly shaped to give a field with $n=\frac{2}{3}$.

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The Resonance Energy of the Thorium Capture Process

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IN a paper dealing with the capture cross section for thermal neutrons in thorium[1] it was pointed out that the pure capture process in thorium has a resonance character with a large contribution from thermal neutrons.[2] The following experiments were carried out in order to determine the resonance energy of this process.

As the neutron source available was only 100 mg Ra+Be, it was not possible to determine the resonance energy in the usual way, i.e., by measuring the absorption coefficient of boron for the resonance neutrons with thorium 233 as detector. We therefore made use of a less direct method by measuring the absorption in thorium for various neutron resonance groups picked out by suitable detectors. If there are found neutron groups having absorption coefficients whose ratio is not inversely proportional to the ratio of the respective velocities, one can expect that a resonance group of the thorium capture process is lying within the range of the energies under investigation.

Since the large contribution of thermal neutrons to the capture process in Th suggested that the resonance energy might be rather low, Au ($E_r=3.5$ ev),[3] In ($E_r=0.9$ ev)[4] and Rh, which has the same resonance energy as In were used as

[1] L. Meitner, Nature **45**, 422 (1940).
[2] L. Meitner, O. Hahn and F. Strassmann, Zeits. f. Physik **109**, 538 (1938).
[3] O. R. Frisch, Kgl. Danske Vid. Selsk. Math.-Phys. Medd. **14**, Nr. 5.
[4] C. P. Baker and R. F. Bacher, Phys. Rev. **57**, 1076 (1940).