Finite Self-Energies in Radiation Theory. Part II

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The "cutting-off method" proposed in Part I is equivalent to a field theory based on Maxwell's equations supplemented by Yukawa's equations, both fields having the same point charges as sources. The chief result is a finite self-energy $W=e^2/2r_0$ and a modified Coulomb potential $(e/r)[1-\exp(-r/r_0)]$, also derivable from a Hamiltonian in Fourier form. For accelerated motions the field theory yields a finite force of inertia $(-m\ddot{x})$ together with the universal damping term in first approximation. Small additional terms reflect the "structure" of the electron. Radiation and self-force of a vibrating electron are discussed, and the perturbation problem is formulated. The exact integration of Yukawa's field equation is given in Section 9. Our results are related to Born-Infeld's unitary field theory and Dirac's theory of the classical electron, in particular with respect to waves of velocity larger than c. The electronic mass m is the result of photons of rest mass *zero* and mesons of rest mass $M=m\cdot 2\cdot 137=274m$.

1. INTRODUCTION

IN Part I we discussed a "cutting-off method"[1] for obtaining finite self-energies of charged particles. We started from the physical consideration that the natural line breadth $\Delta\lambda=4\pi e^2/3mc^2$ allows to determine the position of an electron only within a range of order $\Delta\lambda$ since the electron does not react to the phase of an external field at its exact position but rather to the phase averaged over the range $\Delta\lambda$. More precisely, its vibrational energy is reduced, as compared with the energy without damping, by the damping factor $R=[1+(\nu\Delta\lambda/c)^2]^{-1}$. In Part I we then proposed to reduce the Fourier terms in Fermi's theory of radiation by the same factor R made invariant by Doppler correction for particles in motion. This reduction led to a finite electrostatic self-energy $W=e^2/2r_0$ and a modified Coulomb potential $(e/r)[1-\exp(-r/r_0)]$ where $r_0=\Delta\lambda/2\pi=2e^2/3mc^2$.

In Part II we start from the remark that this potential is the difference of Maxwell and Yukawa potentials as solutions of two separate sets of invariant field equations with the same point charges as sources. The special value $r_0=2e^2/3mc^2$ was chosen in Part I so that the Fourier reduction factor R_ν equals the former damping factor R. This led to an electromagnetic mass $\frac{3}{4}$ of the total mass. However, the special choice of r_0 is not essential. It would seem even more reasonable to choose the parameter $k=1/r_0$ of the field theory so as to have a *unitary field theory* in which the mass of the field equals the total mass, by virtue of the formula $W=e^2/2r_0=ke^2/2=mc^2$, or vice versa $r_0=e^2/2mc^2$. Whether this choice is correct can only be decided by experiments with extremely short waves. In view of the existence of neutral particles it has become doubtful whether one should insist on a purely electromagnetic origin of the electronic mass. On the other hand the combination of Maxwell's and Yukawa's field equations is already a deviation from the pure electromagnetic theory.

Our method offers a consistent and invariant scheme of formulae for calculating the properties of electronic charges without infinities. However, from a physical point of view it is hard to understand why the resulting potential, the resulting radiation of energy, etc., should be the *difference* rather than the sum of the two independent fields of Maxwell and Yukawa. A similar objection could be raised against Dirac's[2] *difference* of advanced and retarded potentials. We are using *retarded* potentials only, although the retarded Yukawa potential is given a negative sign. Apart from the lack of "physical understanding" of this negative sign, our method may at least be considered as an invariant formal way of avoiding infinities in classical point charges. The Fourier representation of Section 7 shall

[1] A. Landé, Phys. Rev. **60**, 121 (1941).

[2] P. A. M. Dirac, Proc. Roy. Soc. **167**, 148 (1938).

prepare the way to a future application of *quantum theory* to our classical scheme.

2. Field Equations

Already in Part I we remarked that the modified electrostatic potential resulting from the Fourier method is the difference of two separate potentials

$$V = V'' - V' = (e/r) - (e/r)\exp(-r/r_0). \quad (1)$$

V' and V'' are solutions of the differential equations

$$\Box V'' = -4\pi\rho, \quad \Box V' = -4\pi\rho + k^2 V', \quad (2)$$

to be supplemented by two equations for vector potentials

$$\Box A'' = -4\pi j/c, \quad \Box A' = -4\pi j/c + k^2 A'. \quad (2')$$

ρ and j vanish everywhere except on the world lines of point particles. The field derived from $V''A''$ is an ordinary Maxwell field. The field of meson type derived from the potential V', A' is

$$E' = -\nabla V' - \dot{A}'/c, \quad H' = \operatorname{curl} A', \qquad \operatorname{div} A' + \dot{V}'/c = 0. \quad (3)$$

Together with (2) (2') this is equivalent with the field equations

$$\begin{aligned} &\operatorname{curl} E' = -\dot{H}'/c, \quad \operatorname{div} H' = 0, \\ &\operatorname{curl} H' = \dot{E}'/c + 4\pi j/c - k^2 A', \\ &\operatorname{div} E' = 4\pi\rho - k^2 V', \quad \operatorname{div} j + \dot{\rho} = 0. \end{aligned} \quad (4)$$

Multiplying by H' and E', and subtracting we arrive at

$$-(c/4\pi)\int ds\{[E'H']_n + k^2 A_n'\} = (1/8\pi)(d/dt) \times \int dv\{E'^2 + H'^2 + k^2(V'^2 + A'^2)\} + \int E' j dv. \quad (5)$$

The stress-energy-momentum tensor T' therefore has the components

$$\begin{aligned} -T_{44}' &= w' = (1/8\pi)\{E'^2 + H'^2 + k^2(V'^2 + A'^2)\}, \\ -iT_{41}' &= g' = (1/4\pi c)\{[E'H'] + k^2 V' A'\}, \end{aligned} \quad (6)$$

representing density of energy and momentum. Condensing three rows and columns to one we can write for the tensor T'

$$T' = \begin{Vmatrix} p_{mn}' & iS'/c \\ iS'/c & -w' \end{Vmatrix} + \frac{k^2}{4\pi}\begin{Vmatrix} A_m' A_n' & iA'V' \\ iA'V' & -V'^2 \end{Vmatrix} - \frac{k^2}{8\pi}\begin{Vmatrix} (A'^2 - V'^2)_{nm} & 0 \\ 0 & (A'^2 - V'^2) \end{Vmatrix}, \quad (7)$$

where p', S', w' are the usual expressions for Maxwell's tensor, Poynting's vector, and energy density in terms of E' and H'.

Corresponding equations with $k=0$, called (3') to (7'), hold for the Maxwell field E'', H''. Since the potentials V'' and V' are supposed to be subtractive the same applies to the field components $E''-E'=E$, $H''-H'=H$. But it also applies to the tensor components $T''-T'=T$. This can be learned from the fact that the last term in (5) is to be subtracted from the last term in (5') in order to give the total mechanical work $\int E\cdot j dv = \int (E''-E')j dv$.

Integration of the wave equations (2) leads to the following general solution (see Section 9) for the potential of a point charge at $xyzt$:

$$V'' - V' = ck\int\int\int\int R^{-1}J_1(kR)\rho d\xi d\eta d\zeta d\tau, \quad (8)$$

where $\rho(\xi\eta\zeta\tau)$ is the density, and R is the 4-distance

$$R^2 = (c\tau - ct)^2 - (\xi - x)^2 - (\eta - y)^2 - (\zeta - z)^2. \quad (8')$$

The integral extends over τ from $-\infty$ to t and over values $\xi\eta\zeta$ belonging to real values of R for the respective value of τ (*retarded potential*). In case of point charges e the integration over $\xi\eta\zeta$ yields a factor e so that (8) simplifies to a sum over the point charges:

$$V'' - V' = \sum eck\int R^{-1}J_1(kR)d\tau = \sum eck\int_0^\infty R^{-1}J_1(kR)(-\partial\tau/\partial R)dR. \quad (9)$$

Replacing ρ by j/c and j by $e\dot{\xi}$ for point charges one obtains the vector potential

$$A'' - A' = \sum ek\int_0^\infty R^{-1}J_1(kR) \times \dot{\xi}(\tau)(-\partial\tau/\partial R)dR. \quad (9')$$

The proof of the fundamental formula (9) is given in Section 9.

V' and V'' can always be supplemented by solutions of the homogeneous Maxwell and Yukawa equations. The latter are solved by plane waves λ whose frequency is

$$\nu^*=[\nu_0^2+(c/\lambda)^2]^{\frac{1}{2}} \quad \text{where} \quad \nu_0=kc/2\pi,$$

so that the phase velocity $c^*=\nu^*\lambda$ is larger than c:

$$c^*=[c^2+(\nu_0\lambda)^2]^{\frac{1}{2}}.$$

A similar velocity larger than c occurred also in Dirac's theory of the classical point electron[2] and does not interfere with invariance. ν_0 is the minimum frequency of Yukawa waves.

3. Applications. Uniform Motion

(I) As our first example we consider an electron at rest at the zero point $\xi\eta\zeta=0$. At $t=0$ and at the distance $r^2=x^2+y^2+z^2$ we have from (8')

$$R^2=(c\tau)^2-r^2, \quad \partial c\tau/\partial R=-R(R^2+r^2)^{-\frac{1}{2}},$$

the root with minus sign since $\tau<t$. According to (9) we obtain

$$\begin{aligned} V''-V'&=ek\int_0^\infty J_1(kR)(R^2+r^2)^{-\frac{1}{2}}dR \\ &=(e/r)[1-\exp(-kr)]. \end{aligned} \tag{10}$$

In particular for $r=0$ we have $(V''-V')_0=ek$, that is, the modified Coulomb potential of Part I and the starting point of Part II. Instead of k we may write $1/r_0$.

The electrostatic self-energy of an electron at rest is

$$W_{\text{rest}}=(1/8\pi)\int dv\{E''^2-(E'^2+k^2V'^2)\},$$

where

$$E''=e/r^2, \quad V'=(e/r)\exp(-kr), \quad E'=-\partial V'/\partial r.$$

Integration gives

$$W_{\text{rest}}=e^2k/2. \tag{10'}$$

The same result could have been obtained by the simpler formula

$$W_{\text{rest}}=\tfrac{1}{2}e(V''-V')_0=e^2k/2.$$

The electromagnetic rest mass is therefore

$$m=e^2k/2c^2, \text{ vice versa } k=2mc^2/e^2, \tag{10''}$$

k determines the ratio e^2/m, where the "mass m" is defined as the factor of c^2 in the self-energy, and the "charge e" is defined as the factor of $1/r$ in the mutual energy. The corresponding meson mass is

$$M=h\nu_0/c^2=kh/2\pi=2m(hc/2\pi c^2)=m\cdot 2\cdot 137.$$

(II) If the electron moves uniformly with velocity $v_x/c=\beta$ we have

$$\xi=\beta c\tau, \quad \eta=\zeta=0,$$

$$R^2=(c\tau-ct)^2-(\beta c\tau-x)^2-y^2-z^2,$$

$$\partial c\tau/\partial R=-R(1-\beta^2)^{-\frac{1}{2}}(R^2+a^2)^{-\frac{1}{2}},$$

where

$$a^2=(1-\beta^2)^{-1}[r^2-2\beta ctx+\beta^2(c^2t^2+x^2-r^2)],$$

so that

$$\begin{aligned} V''-V'&=ek(1-\beta^2)^{-\frac{1}{2}}\int_0^\infty J_1(kR)(R^2+a^2)^{-\frac{1}{2}}dR \\ &=e[a(1-\beta^2)^{\frac{1}{2}}]^{-1}[1-\exp(-ak)]. \end{aligned} \tag{11}$$

It can easily be shown geometrically that

$$a(1-\beta^2)^{\frac{1}{2}}=d\cdot(1-\beta\cos\alpha)$$

where d is the distance between $xyzt$ and the position ξ of the particle at the retarded time τ so that $d=c(t-\tau)$, whereas α is the angle between d and the velocity. Therefore the first part of (11) agrees with the Lienard-Wiechert potential of a point charge V'', whereas the second part represents the Yukawa potential V'. The vector potential is

$$A_x''-A_x'=(V''-V')(v_x/c).$$

In particular, on the moving electron itself ($a=0$) we have

$$\begin{aligned} V_0&=(V''-V')_0=ek(1-\beta^2)^{-\frac{1}{2}}, \\ A_0&=(A''-A')_0=ek\beta(1-\beta^2)^{-\frac{1}{2}}. \end{aligned} \tag{12}$$

Energy and momentum of the field of the moving electron are therefore

$$\begin{aligned} W&=eV_0/2=mc^2(1-\beta^2)^{-\frac{1}{2}}, \\ G&=eA_0/2c=mv(1-\beta^2)^{-\frac{1}{2}}, \end{aligned} \tag{12'}$$

agreeing with relativity. An explicit proof of (12′) is given in (29) Section 8. When two like (unlike) point charges approach to the distance *zero* the finite work $e^2k=2mc^2$ is spent (gained).

4. Acceleration. Self-Force

Next we consider an electron moving along the x-axis with coordinates

$$\xi=u\tau+\tfrac{1}{2}f\tau^2+\tfrac{1}{6}g\tau^3, \quad \eta=\zeta=0. \tag{13}$$

The meaning of u, f, g is seen from the values at $\tau=0$:

$$u=(d\xi/d\tau)_0, \quad f=(d^2\xi/d\tau^2)_0, \quad g=(d^3\xi/d\tau^3)_0.$$

At an external point $r^2=x^2+y^2+z^2$ at time t we have from (8′)

$$R^2=(c\tau-ct)^2-(u\tau+\tfrac{1}{2}f\tau^2+\tfrac{1}{6}g\tau^3-x)^2-y^2-z^2.$$

For the field on the particle itself at $\tau=0$ consider t and r as small so that r^2 can be neglected altogether. Also neglecting second orders of u, f, g we obtain approximately

$$c\tau=ct-R+xR^{-1}(u\tau+\tfrac{1}{2}f\tau^2+\tfrac{1}{6}g\tau^3).$$

In the same approximation we can on the right replace τ by $(t-R/c)$, hence

$$\begin{aligned} cd\tau/dR=&-1-xR^{-2}\{u(t-R/c)\\ &+\tfrac{1}{2}f(t-R/c)^2+\tfrac{1}{6}g(t-R/c)^3\}\\ &-xR^{-1}c^{-1}\{u+f(t-R/c)+\tfrac{1}{2}g(t-R/c)^2\}. \end{aligned} \tag{14}$$

There are two special cases of (14). If $t=0$ and r is small we obtain

$$cd\tau/dR=-1+\tfrac{1}{2}xf/c-\tfrac{1}{3}xgR/c^3. \tag{14′}$$

If t is small and $r=0$ hence $x=0$ we have

$$cd\tau/dR=-1. \tag{14″}$$

We are now prepared to calculate V near the electron at $t=0$ for small r. Substituting (14′) in (9) we obtain

$$\begin{aligned} V''-V'&=(1-\tfrac{1}{2}xf/c^2)ek\int_0^\infty J_1(kR)R^{-1}dR\\ &\quad+(xge/3c^3)\int_0^\infty J_1(kR)d(kR)\\ &=(1-\tfrac{1}{2}xf/c^2)ek+xge/3c^3, \end{aligned} \tag{15}$$

since both integrals are unity. Therefore

$$-\partial V/\partial x=\tfrac{1}{2}fek/c^2-eg/3c^3.$$

Substituting (14″) and $\xi=u+f\tau+\frac{1}{2}g\tau^2$ in (9′) we arrive at

$$\begin{aligned} A''-A'&=(ek/c)\int_0^\infty[u+f(t-R/c)\\ &\quad+\tfrac{1}{2}g(t-R/c)^2]J_1(kR)R^{-1}dR, \end{aligned}$$

$$\begin{aligned} -(\partial A/\partial ct)_{t=0}&=(ek/c)\int_0^\infty(-f+gR/c)J_1(kR)R^{-1}dR\\ &=-ekf/c+ge/c^2. \end{aligned}$$

The magnetic field H vanishes on the electron so that the self-force $F=eE=-e\partial V/\partial x-e\partial A/\partial ct$ at $t=0$ becomes

$$\begin{aligned} F&=-(e^2k/2c^2)\cdot f+(2e^2/3c^3)\cdot g,\\ &=-m(d^2\xi/d\tau^2)+(2e^2/3c^3)(d^3\xi/d\tau^3). \end{aligned} \tag{16}$$

where m is the mass due to the two fields, see (10″). If the whole mass of the electron is due to the fields, the various field centers (point charges) move *according to the field equations alone* in such a way that "the total field force is zero" on every single particle.

Charges with different masses and radii could be accounted for by extending the sums (9) (9′) over terms with different individual values of k.

The self-force (16) at $t=0$ is composed of retarded contributions of the whole path (13) during $\tau<0$. But only the immediate past matters in producing the two terms of (16). In case of a more general path

$$\xi=u\tau+\tfrac{1}{2}f\tau^2+\tfrac{1}{6}g\tau^3+\tfrac{1}{24}h\tau^4+\cdots \quad \text{for } \tau<0$$

it turns out that the higher terms contribute also with their remote past to the self-force at $t=0$. In connection with the fact that the velocity $\dot{\xi}$ on such a path would be larger than c in the remote past, each higher term furnishes an infinite contribution to F. Even if an infinite series in τ should represent a physically possible path $\xi(\tau)$ (vibrating electron) an integration by terms is not possible.

It would be an interesting problem to find an accelerated path such that the self-force vanishes at all times. As long as the acceleration is small this path will coincide with that derived by Dirac[2] from the difference of advanced and retarded Maxwell potentials.

5. Vibrating Electron. Radiation

An electronic point shall vibrate about the zero point with coordinates

$$\xi = a \sin(\omega\tau), \quad \eta = 0 = \zeta.$$

The amplitude a shall be small compared with the wave-length $2\pi c/\omega$ and with the electronic radius $r_0 = 1/k = c/\omega_0$ so that a^2 can always be neglected. At distance $r^2 = x^2+y^2+z^2$ we have (8′)

$$R^2 = c^2(\tau - t)^2 + 2ax \sin(\omega\tau) - r^2,$$

hence in the same approximation

$$c\tau = ct - (R^2+r^2)^{\frac{1}{2}} + (R^2+r^2)^{-\frac{1}{2}} ax \sin(\omega\tau)$$

with the argument $\omega\tau = \omega t - (R^2+r^2)^{\frac{1}{2}}\omega/c$ when neglecting a^2. Thus

$$\begin{aligned} -R^{-1}\partial(c\tau)/\partial R = {} & (R^2+r^2)^{-\frac{1}{2}} \\ & + (ax\omega/c)(R^2+r^2)^{-1} \cos(\omega\tau) \\ & + ax(R^2+r^2)^{-\frac{3}{2}} \sin(\omega\tau) \end{aligned} \tag{17}$$

consisting of three terms. For large r (wave zone) the second term alone contributes to V, and the first term alone to A. With $d\xi/d\tau = a\omega \cos(\omega\tau)$ we obtain from (9) (9′)

$$\begin{aligned} V'' - V' &= (eax\omega/c)\int_0^\infty J_1(kR)(R^2+r^2)^{-1} \\ &\quad \times \cos[\omega t - (R^2+r^2)^{\frac{1}{2}}\omega/c]\, d(kR), \\ A'' - A' &= (ea\omega/c)\int_0^\infty J_1(kR)(R^2+r^2)^{-\frac{1}{2}} \\ &\quad \times \cos[\omega t - (R^2+r^2)^{\frac{1}{2}}\omega/c]\, d(kR). \end{aligned} \tag{18}$$

In the limit of small ω one can replace (R^2+r^2) by r^2 not only for small R but also for large R since the integrals give negligible contributions for large R if ω is small. In this case (18) simplifies to

$$\begin{aligned} V &= V'' - V' = (ea\omega/c)(x/r^2)\cos[\omega(t-r/c)], \\ A &= A'' - A' = (ea\omega/c)(1/r)\cos[\omega(t-r/c)], \end{aligned} \tag{18′}$$

that are the well-known Lorentz potentials. But for larger ω one has to use the complete integrals (18); we have not been able to evaluate them. From what is learned in (20′) about the self-force of a vibrating electron the Lorentz potentials (18′) are valid up to $\omega = \omega_0$, and a decrease of the emitted potential will occur only for $\omega > \omega_0$ in connection with the fact that Yukawa waves have a minimum frequency $\omega_0 = kc$ (Section 2). This result is confirmed by the Fourier method, Section 8, III.

6. Vibrating Electron. Self-Force

In order to find the self-force of a vibrating electron at time t it is convenient to have the electron in the zero point $r = 0$ at time t. We therefore consider the position

$$\xi = a \sin(\omega\tau) - a \cdot s, \quad \eta = \zeta = 0,$$

where the parameter s will later be equated to $\sin(\omega t)$. With small r and neglection of r^2 we now have

$$c\tau = ct - R + xaR^{-1}[\sin(\omega\tau) - s].$$

On the right we can replace τ by $(t - R/c)$ if a is small. Then

$$\begin{aligned} \partial c\tau/\partial R = {} & -1 - xaR^{-2}[\sin(\omega t - \omega R/c) - s] \\ & - (xa\omega/Rc)\cos(\omega t - \omega R/c). \end{aligned}$$

Substituting in (9) (9′) and neglecting a^2 we obtain

$$\begin{aligned} \partial V/\partial x &= eka\int_0^\infty J_1(kR)\{R^{-3}[\sin(\omega t - \omega R/c) - s] \\ &\quad + (\omega/R^2c)\cos(\omega t - \omega R/c)\}\, dR, \\ \partial A/\partial ct &= -eka(\omega/c)^2\int_0^\infty J_1(kR)R^{-1} \\ &\quad \times \sin(\omega t - \omega R/c)\, dR. \end{aligned}$$

We now put $s = \sin(\omega t)$. In the self-force

$$F = -e\partial V/\partial x - e\partial A/\partial ct,$$

we separate terms with factors $\sin(\omega t)$ and $\cos(\omega t)$. Writing

$$-\omega^2 a \sin(\omega t) = d^2\xi/d\tau^2, \quad -\omega^3 a \cos(\omega t) = d^3\xi/d\tau^3,$$

and using the abbreviations $kR = u$, $\omega R/c = qu$ with

$$e^2k/2c^2 = m, \quad ck = c/r_0 = \omega_0, \quad \omega/\omega_0 = q, \tag{19}$$

we obtain

$$F=-m(d^2\xi/d\tau^2)f(q)+(2e^2/3c^3)(d^3\xi/d\tau^3)g(q),$$

$$f(q)=2\int_0^\infty J_1(u)u^{-1}\{\cos(qu)-(qu)^{-1}\sin(qu)+(qu)^{-2}[1-\cos(qu)]\}du,$$

$$g(q)=\tfrac{3}{2}\int_0^\infty J_1(u)\{(qu)^{-1}\sin(qu)+(qu)^{-3}[(qu)\cos(qu)-\sin(qu)]\}du. \tag{20}$$

For small $q=\omega/\omega_0$ the factors f and g reduce to *unity* so that F has the classical form (16). In general F can be found with help of auxiliary formulae in two cases $0\leqslant q\leqslant 1$ and $q\geqslant 1$, respectively:

$$\int_0^\infty J_1(u)\frac{\sin(qu)}{(qu)}du=\begin{cases}1\\ 1-(1-q^{-2})^{\frac{1}{2}},\end{cases}$$

$$\int_0^\infty J_1(u)\frac{\sin(qu)}{(qu)}\frac{du}{u}=\begin{cases}\frac{1}{2}[(1-q^2)^{\frac{1}{2}}+q^{-1}\sin^{-1}q]\\ (\pi/4)q^{-1},\end{cases}$$

$$\int_0^\infty J_1(u)\cos(qu)\frac{du}{u}=\begin{cases}(1-q^2)^{\frac{1}{2}}\\ 0.\end{cases}$$

Also

$$(qu)^{-2}[1-\cos(qu)]=(qu)^{-1}\int_0^1\sin(pqu)dp,$$

$$(qu)^{-2}[(qu)\cos(qu)-\sin(qu)]=-\int_0^1\sin(pqu)pdp.$$

With these formulae we obtain in (20)

$$f(q)=\begin{cases}(1/3q^2)[1-(1-q^2)^{\frac{3}{2}}]\\ \qquad+q^{-2}[(1-q^2)^{\frac{1}{2}}-1]+(1-q^2)^{\frac{1}{2}}\\ \qquad-(2/3q^2)\end{cases}$$

$$g(q)=\begin{cases}1\\ 1-(1-q^{-2})^{\frac{3}{2}},\end{cases} \tag{20'}$$

for $0\leqslant q\leqslant 1$ and $q\geqslant 1$, respectively. For large frequencies $q\gg 1$ both f and g tend toward zero as though inertia and damping were fading out with increasing ω. The amplitude a shall always be small compared with c/ω. If we should consider frequencies $\omega\sim\omega_0$ and yet amplitudes a not small compared with the electronic radius $r_0=\omega_0/c$ the result would be quite different. The dependence of the "scattering cross section" on the amplitude for frequencies of order ω_0 could be used for checking the classical theory if this effect were not completely overshadowed by quantum effects just at these high frequencies.

It is quite significant that the decrease of the factor $g(q)$ begins only at the characteristic frequency ω_0 itself. Since the damping term with $(2e^2/3c^3)\cdot g(q)$ is the counterpart of long distance radiation, this means that up to $\omega=\omega_0$ only Maxwell radiation is emitted, in agreement with the fact (Section 2) that Yukawa waves have a minimum frequency $\omega_0=kc$. We learn from (20') that for $\omega>\omega_0$ the ratio of Maxwell to Yukawa radiated energy is 1 to $(1-q^{-2})^{\frac{3}{2}}$. The same result can be obtained directly by the Fourier method.

7. Fourier Method. Hamiltonian

For particles at rest the field theory is equivalent with the Fourier reduction method of Part I. For particles in motion the field theory gives results differing from the Doppler correction of Part I, but consistent with relativity.

Similar to Fermi we expand the Yukawa potential $V'(r,t)$ into a Fourier series of standing waves of direction α_s in a large space Ω (for details compare with Fermi[3]):

$$V'(r,t)=c(8\pi/\Omega)^{\frac{1}{2}}\textstyle\sum_s Q_s'(t)\cos\Gamma_{sr},\qquad \Gamma_{sr}=(\omega_s/c)(\alpha_s\cdot r)+\text{phase}. \tag{21}$$

Substitution into Eq. (2) gives the following differential equation for Q_s'

$$d^2Q_s'/dt^2+(\omega_s^2+k^2c^2)Q_s'=c(8\pi/\Omega)^{\frac{1}{2}}\textstyle\sum_i e_i\cos\Gamma_{si} \tag{21'}$$

[3] E. Fermi, Rev. Mod. Phys. **4**, 87 (1932).

in case of point charges. For uniform motions where r_i in Γ_{si} changes with time (21) is solved by

$$Q'_s(t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\cos\Gamma_{si}[\omega_s^2(1-\beta_i^2\cos^2\vartheta_{si})+\omega_0^2]^{-1}. \tag{22}$$

ϑ_{si} is the angle between the wave normal α_s and the direction of β_i, and ω_0 is written for kc. The dependence of t is contained in r_i in Γ_{si}. Corresponding Fourier series for the Maxwell potential V'' lead to

$$Q''(t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\cos\Gamma_{si}[\omega_s^2(1-\beta_i^2\cos^2\vartheta_{si})]^{-1}. \tag{22'}$$

(22) and (22′) can be augmented by solutions of the homogeneous differential equations. The Fourier coefficient $Q_s=Q''_s-Q'_s$ of the resulting potential $V=V''-V'$ can be written

$$Q_s(t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\cos\Gamma_{si}\omega_s^{-2}(1-\beta_i^2\cos^2\vartheta_{si})^{-1}R_{si} \tag{23}$$

and contains the "reduction factor"

$$R_{si}=[1+(1-\beta_i^2\cos^2\vartheta_{si})\omega_s^2/\omega_0^2]^{-1}. \tag{24}$$

For $\beta_i=0$ this is the factor used in Part I for particles at rest.

Corresponding calculations can be applied to the vector potential $A=A'-A''$:

$$A(r,t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_s[\alpha_s\chi_s(t)+A_sq_s(t)]\sin\Gamma_{sr}$$

with α_s=longitudinal, A_s=transversal unit vector. (3′) is equivalent to $\omega_s\chi'_s+\dot{Q}'_s=0$. The Fourier coefficients $\chi_s=\chi''_s-\chi'_s$ and $q_s=q''_s-q'_s$ are

$$\begin{matrix}\chi_s\\ q_s\end{matrix}(t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\beta_i\begin{matrix}\cos\\ \sin\end{matrix}(\vartheta_{si})\sin\Gamma_{si}\omega_s^{-2}(1-\beta_i^2\cos^2\vartheta_{si})^{-1}R_{si} \tag{25}$$

with the same factor R_{si}. In all cases one may add solutions of the homogeneous equation (21′) etc.

The *Hamiltonian* of the resulting field is $H=H''-H'$, *viz.*

$$\begin{aligned}H=&\sum_i m_ic^2+\sum_i(\dot{r}_i(p_i^{\text{kin}}+p_i^{\text{pot}}))+c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\sum_s(Q''_s-Q'_s)\cos\Gamma_{si}\\ &-c(8\pi/\Omega)^{\frac{1}{2}}\sum_i(e_i/c)\sum_s(\dot{r}_i,\alpha_s(\chi''_s-\chi'_s)+A_s(q''_s-q'_s))\sin\Gamma_{si}\\ &+\tfrac{1}{2}\sum_s[(p''^2_s+\xi''^2_s-P''^2_s)+\omega_s^2(q''^2_s+\chi''^2_s-Q''^2_s)]\\ &-\tfrac{1}{2}\sum_s[(p'^2_s+\xi'^2_s-P'^2)+(\omega_s^2+\omega_0^2)(q'^2_s+\chi'^2_s-Q'^2_s)]\end{aligned} \tag{26}$$

with coordinates $Q'_sQ''_sq'_sq''_s\chi'_s\chi''_s$ and momenta $P'_sP''_sp'_sp''_s\xi'_s\xi''_s$ and with $\omega_0=kc$. (26) is verified by the canonical equations

$$\dot{Q}'_s=\partial H/\partial P'_s=-P'_s,\quad \dot{P}'_s=-\partial H/\partial Q'_s, \tag{26'}$$

etc., that agree with the differential equations (21) etc., for Q'_s, Q''_s etc., (compare with Fermi, reference 3, p. 129).

The first sum in (26) is the *mechanical* rest energy, the second sum contains the *mechanical* kinetic energy. In a unitary theory with $k=2mc^2/e^2$ these two terms are to be omitted. The sum over $(\dot{r}_ip_i^{\text{pot}})$ is identical with the whole second line of (26) so that these sums cancel one another. The energy reduces to the two last lines of (26).

8. APPLICATIONS OF THE FOURIER METHOD

(I) First we consider a system of electrons *at rest.* Here (22) simplifies to

$$Q_s'=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\cos\Gamma_{si}(\overset{2}{\omega_s}+\overset{2}{\omega_0})^{-1},$$
$$\dot{Q}_s'=P_s'=p_s'=\xi_s'=q_s'=\chi_s'=0$$

and similar formulae for Q_s'' etc., without $\overset{2}{\omega_0}$. The second line in (26) turns out to be half as large as, and of opposite sign as the third line, so that H reduces to

$$H=\tfrac{1}{2}c^2(8\pi/\Omega)\sum_i\sum_j e_ie_j\sum_s\cos\Gamma_{si}\cos\Gamma_{sj}\times[(\overset{2}{\omega_s})^{-1}-(\overset{2}{\omega_s}+\overset{2}{\omega_0})^{-1}],$$

obviously a difference of Maxwell and Yukawa terms. Instead we may write

$$H=\tfrac{1}{2}c(8\pi/\Omega)\sum_i\sum_j e_ie_j\sum_s\cos\Gamma_{si}\cos\Gamma_{sj}\times\omega_s^{-2}[1+(\omega_s/\omega_0)^2]^{-1},\quad(27)$$

that is, Fermi's expression "reduced" by a factor $R=[1+(\omega_s/\omega_0)^2]^{-1}$ as in Part I. Replacing the summation $\sum_s$ by an integration with Jeans' factor $(\Omega/2\pi^2c^3)\overset{2}{\omega_s}d\omega_s$ one arrives (see Part I) at

$$H=\sum_i(e_i\omega_0/2c)+\sum_i\sum_j(e_ie_j/r)[1-\exp(-r\omega_0/c)]\quad(27')$$

in agreement with (10) (10′), representing finite self-energies and modified Coulomb energies.

(II) Our second example shall be *one* electron in uniform motion with velocity $\dot{r}_i=\beta_ic$. Here we have to substitute the complete expressions (22) into the last two lines of (26). In addition we have

$$P_s'=-\partial H/\partial P_s'=-\dot{Q}_s'$$
$$=c(8\pi/\Omega)^{\frac{1}{2}}e_i\sin\Gamma_{si}\omega_s\beta_i\cos\vartheta_{si}\times[\overset{2}{\omega_s}(1-\overset{2}{\beta_i}\cos^2\vartheta_{si})+\overset{2}{\omega_0}]^{-1}$$

and corresponding expressions for the other momenta. First we sum over all directions ϑ_{si} by integrating with factor $\frac{1}{2}d(\cos\vartheta)$ from -1 to 1, and replace $\sin^2\Gamma_{si}$ and $\cos^2\Gamma_{si}$ by $\frac{1}{2}$ because of irregular phases. The second line of (26) then gives

$$c^2(8\pi/\Omega)(e^2/2)\sum_s\{\omega_s^{-2}\beta_i^{-1}\operatorname{tgh}^{-1}\beta_i-(\omega_s\beta_i)^{-1}(\overset{2}{\omega_s}+\overset{2}{\omega_0})^{-\frac{1}{2}}\operatorname{tgh}^{-1}(\omega_s\beta_i(\overset{2}{\omega_s}+\overset{2}{\omega_0})^{-\frac{1}{2}})\},\quad(28)$$

obviously a difference of Maxwell and Yukawa terms. The latter are small compared with the former for $\omega_s\ll\omega_0$. The last line of (26) gives

$$-\tfrac{1}{2}c^2(8\pi/\Omega)(e^2/2)\times\sum_s\{(\overset{2}{\omega_s})^{-1}-[\overset{2}{\omega_s}+(\overset{2}{\omega_0}/(1-\overset{2}{\beta_i}))]^{-1}\}.\quad(28')$$

The tgh^{-1} in (28) can be expanded in powers of β_i and then integrated term by term with Jeans' factor $(\Omega/2\pi^2c^3)\overset{2}{\omega_s}d\omega_s$. The result is a series that condenses to the simple expression $(e^2\omega_0/c)(1-\overset{2}{\beta_i})^{-\frac{1}{2}}$. On the other hand (28′) can be integrated directly with the result $-\frac{1}{2}(e^2\omega_0/c)(1-\overset{2}{\beta_i})^{-\frac{1}{2}}$ just half as large as (28). The balance is

$$H=(e^2\omega_0/2c)(1-\overset{2}{\beta_i})^{-\frac{1}{2}}=mc^2(1-\overset{2}{\beta_i})^{-\frac{1}{2}},\quad(29)$$

agreeing with (12′) and verifying the invariance of the method in Fourier form.

(III) The Fourier method can also be applied to the *radiation* emitted by a vibrating electron similar to Heitler's method.[4] The result is quoted at the end of Section 6.

Our invariant "cutting-off method" consists in using the Hamiltonian $H=H''-H'$ rather than H'' alone.

The *perturbation problem* of the classical Hamiltonian is this. The electrons may first move with prescribed *constant* velocities $\dot{r}_i$. The corresponding "electronic" field coordinates $Q_s'(t)$, etc., are given in (22), etc., augmented by solutions of the homogeneous equations (21′) representing the initial "pure field." Between t_1 and t_2 the electrons shall move on prescribed accelerated path $[r_i(t)$ in Γ_{si} on the right of (21′) prescribed$]$. The equation of motion (21′) together with the initial value $Q_s'(t_1)$ determines the final value $Q_s'(t_2)$. Its continuation for $t>t_2$ can again be separated into an "electronic field" of type (22) and a "pure field" if the electrons move with constant velocities after t_2. Such a separation is not possible during the time of radiation $t_1<t<t_2$. The prescribed path ought to be such that the total force (no matter whether it can be separated into external and self-force) vanishes on every electron.

[4] W. Heitler, *Quantum Theory of Radiation* (Oxford, 1936), p. 54.

9. Appendix. Exact Integration of Yukawa's Equation

The exact solution (V'', A'') of the Maxwell equations produced by charges ρ, j is well known. The exact solution of the combined Maxwell-Yukawa problem shall be given here. The exact solution of the Yukawa problem alone might be of value for the discussion of meson problems, irrespective of the present field theory.

In order to solve the differential equation

$$\nabla^2 V' - \partial^2 V'/\partial c^2 t^2 = -4\pi\rho + k^2 V' \qquad (30)$$

we use the Fourier transformation

$$u'(A, B, C; t) = \int\int\int V'(x, y, z; t) \times \exp[-2i\pi(Ax+By+Cz)]dxdydz,$$

$$\sigma(A, B, C; t) = \int\int\int \rho(x, y, z; t) \times \exp[-2i\pi(Ax+By+Cz)]dxdydz.$$

Their inversion substituted in (30) yields for u' the equation

$$-4\pi^2(A^2+B^2+C^2)u' - \partial^2 u'/\partial c^2 t^2 = -4\pi\sigma + k^2 u'. \qquad (31)$$

With the abbreviation

$$p^2 = c^2[k^2 + 4\pi^2(A^2+B^2+C^2)], \qquad (32)$$
$$q^2 = c^2 4\pi^2(A^2+B^2+C^2),$$

(31) reduces to

$$p^2 u' + \partial^2 u'/\partial t^2 = 4\pi c^2\sigma.$$

The solution for vanishing u' and $\partial u'/\partial t$ at $t=-\infty$ is

$$u'(A, B, C, t) = \int_{-\infty}^{t} \sin[p(t-\tau)]p^{-1}4\pi c^2 \times \sigma(A, B, C; \tau)d\tau. \qquad (33)$$

Reversing the Fourier transformation we obtain

$$V' = \int\int\int dAdBdC \exp[2i\pi(Ax+By+Cz)] \times \int_{-\infty}^{t} d\tau \sin[p(t-\tau)]p^{-1}4\pi c^2 \times \int\int\int d\xi d\eta d\zeta \rho(\xi, \eta, \zeta; \tau) \times \exp[-2i\pi(A\xi+B\eta+C\zeta)]. \qquad (34)$$

Changing the order of integration we arrive at

$$V'(xyzt) = 4\pi c^2 \int\int\int\int_{-\infty}^{t} \rho(\xi\eta\zeta\tau) \times K'(x-\xi, \cdots, t-\tau)d\xi d\eta d\zeta d\tau \qquad (35)$$

with the kernel function

$$K'(x, y, z, t) = \int\int\int \sin\eta pt)p^{-1} \times \exp[+2i\pi(Ax+By+Cz)]dAdBdC. \qquad (36)$$

The same considerations applied to the Maxwell field (with $k=0$) result in a corresponding formula for V'' with kernel K'' in which p is replaced by q of (32). The resulting potential is

$$V'' - V' = 4\pi c^2 \int\int\int\int_{-\infty}^{t} \rho(\xi\eta\zeta\tau) \times (K''-K')d\xi d\eta d\zeta d\tau. \qquad (37)$$

The kernel $K''-K'$ can be evaluated in polar coordinates. With

$$x^2+y^2+z^2=r^2, \quad A^2+B^2+C^2=D^2$$

we obtain

$$K''-K' = 2\int_0^{\infty} \sin(2\pi rD)r^{-1} \times [\sin(pt)p^{-1} - \sin(qt)q^{-1}]DdD, \qquad (38)$$

in which $p^2=c^2(k^2+4\pi^2D^2)$ and $q^2=c^2 4\pi^2 D^2$. Writing

$$p = ck\cosh\theta, \quad q = ck\sinh\theta, \quad 2\pi D = k\sinh\theta,$$

we have

$$K''-K' = (k^2/4\pi^2)2\int_0^{\infty} \sin(kr\sinh\theta)r^{-1} \times [\sin(ckt\cosh\theta)(ck\cosh\theta)^{-1} - \sin(ckt\sinh\theta)(ck\sinh\theta)^{-1}] \times \sinh\theta\cosh\theta d\theta. \qquad (39)$$

(39) has two forms according to $ct>r$, or $ct<r$. If $ct>r$ we write

$$ckt = \lambda\cosh\alpha, \quad kr = \lambda\sinh\alpha, \quad \lambda^2 = k^2(c^2t^2 - r^2).$$

Associating and dissociating sines and cosines we arrive after some transformations at the simple value of (39):

$$K''-K'=(k^2/4\pi c\lambda)J_1(\lambda).$$

If $ct<r$ we write

$$ckt=\lambda' \sinh\alpha, \quad kr=\lambda' \cosh\alpha,$$
$$\lambda'^2=k^2(r^2-c^2t^2)=-\lambda^2.$$

Here λ is imaginary, and the result is

$$K''-K'=0.$$

(37) thus yields the exact solution

$$V''-V'=ck^2\int_{-\infty}^{t}\int\int\int J_1(\lambda)\lambda^{-1}\rho(\xi\eta\zeta\tau)d\tau d\xi d\eta d\zeta, \tag{40}$$

where

$$\lambda^2=k^2[(ct-c\tau)^2-(x-\xi)^2-(y-\eta)^2-(z-\zeta)^2].$$

The $\xi\eta\zeta$-integral is extended over values that make λ real.

Subtracting $V''-V'$ from the well-known Maxwell solution V'' with the same charges as sources one obtains the exact solution of the Yukawa equation.

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Intensities in Perturbations

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Quantitative formulae for the intensities of lines from a pair of interacting states to a common lower state are applied to perturbations in band spectra. It appears that rotational and vibrational perturbations show quite a different behavior. In the former there is a direct superposition of intensities of the two interacting states whereas in the latter interference effects give an entirely different picture. The general formulae are applied to a few cases in the H_2 spectrum.

THE so-called perturbations in molecular spectra are a special case of interaction between a pair of quantum-mechanical states. So far the energy shifts due to the perturbations have been well studied but very little definite information is available in the literature about the intensities except some general qualitative statements. The perturbations are often of great importance for the interpretation of complex spectra and the intensities may be just as helpful as the frequency shifts.

In the present paper the general theory of the intensities in perturbations is given which follows directly from the general quantum-mechanical treatment of interaction of states, and is then applied to a few special cases.

§1. General Theory

Assume that ψ_1 and ψ_2 are the wave functions of two states satisfying

$$H^0\psi_n=W_n{}^0\psi_n, \quad n=1, 2,$$

and that ψ_a and ψ_b are the corresponding wave functions of the perturbed problem[1]

$$(H^0+H^1)\psi_m=(W_1{}^0+E_m)\psi_m, \quad m=a, b.$$

We assume further that only the interaction between ψ_1 and ψ_2 is appreciable. This is an assumption very well realized in most molecular perturbations,[2] and means that the problem can be treated without further approximations. We shall drop this restriction in certain cases later on in this paper. We can express the neglect of the other interactions by assuming all matrix elements of the perturbation matrix to be zero[3]

[1] Throughout this paper indices 1 and 2 will refer to the unperturbed and a and b to the perturbed states. ψ_a corresponds to ψ_1 and ψ_b to ψ_2 in the sense that ψ_1 furnishes the largest contribution to ψ_a, etc.

[2] We leave the influence of the electron spin out of consideration. The results are then strictly applicable only to singlets and to those multiple terms where the multiplet separation is small compared to the distance of the interacting states. The considerations can easily be expanded to the more general case.

[3] This does not necessarily mean that the other matrix elements are actually small compared to H'_{12}. Even if they