The Theory of Cascade Showers in Heavy Elements

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The one-dimensional equations of the cascade theory of showers are solved, account being taken of the variation with energy of the cross section for pair production, as well, of course, as ionization and bremsstrahlung effects. This modification is necessary for the discussion of cosmic-ray showers and bursts in elements of high atomic number since, for such elements, there is a region of energy (10 Mev–200 Mev for Pb) in which the pair production process is important while the cross section usually assumed for the process is greatly in error. Introducing the usual units t of length and β of energy, both characteristic of the material traversed ($t, \beta = 240$ m, 95 Mev and 0.4 cm, 6.5 Mev for air and lead, respectively), one finds that the maximum number of particles arising in lead from an incident particle of energy $E_0 = \beta e^{\epsilon}$ varies between $E_0/11\beta$ and $E_0/10\beta$ as E_0 varies between 10^9 ev and 10^{11} ev. For a given ϵ, the maximum number of particles occurs at a distance (in units of t) slightly less for lead than for air, and for large distances the effect of the variation with energy of the pair production cross section is to increase the number of particles and γ-rays to be expected.

I

ALTHOUGH the nature of the penetrating component of cosmic radiation is at present not completely understood, the theory of the multiplication of the soft component in the atmosphere is well established. Indeed, at high altitudes, where the theory is apparently contradicted by experiment, it is, in fact, used as a basis for determining the important part played by protons as primary constituents of cosmic radiation.[1] Treating the process as taking place in one dimension, and making simplifying assumptions for the cross sections for bremsstrahlung by electrons and positrons and pair production by γ-rays, Carlson and Oppenheimer[2] and Bhabha and Heitler[3] have been able to explain the general nature of the variation of the electron-positron intensity with height in the atmosphere, and refinements of this work by Landau and Rumer,[4] Snyder[5] and Serber[6] have given the theoretical multiplication curve quite accurately. It is found that, owing to energy losses of the particles by ionization, the number of particles arising from a single electron incident on a block of material does not increase indefinitely with depth but reaches a maximum value at a depth characteristic of the incident energy and of the nature (atomic number and density) of the material. The important role played by ionization losses is, therefore, apparent. Now the energy below which such losses begin to become greater than the average energy loss of a particle by radiation is approximately $1600mc^2/Z$, where Z is the atomic number of the material traversed. It is, therefore, apparent that, for showers multiplying in elements of high atomic number, loss of energy by ionization ceases to be the dominating process at much lower energies than for showers in the atmosphere. At such energies, however, the approximate expression for the pair production cross section used in all treatments of cascade showers in the atmosphere is quite incorrect, being, in the case of lead, for instance, three or four times the value obtained from the quantum-mechanical formula. The assumed form for the cross section for bremsstrahlung, on the other hand, is in error only by a factor 1.4 at these energies. Thus the direct application to heavy elements of the theory of the multiplication of cascade showers in the atmosphere is inadmissible, for in the case of elements of high atomic number there is a wide region of energy (10 Mev–200 Mev for Pb) in which the pair production

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[1] J. R. Oppenheimer, private communication.
[2] F. Carlson and J. R. Oppenheimer, Phys. Rev. **51**, 220 (1937).
[3] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. **A159**, 432 (1937).
[4] L. Landau and G. Rumer, Proc. Roy. Soc. **A166**, 213 (1938).
[5] H. Snyder, Phys. Rev. **53**, 960 (1938).
[6] R. Serber, Phys. Rev. **54**, 317 (1938).

process is important, while the assumed form for the cross section for the process is greatly in error.

In view of recent experiments[7] in which cascade showers and bursts have been observed in an ionization chamber surrounded with lead, and because of the importance of such an experimental arrangement for the purpose of understanding the behavior of the mesotron,[8] it is of interest to apply the cascade theory to the production of showers in elements of high atomic number. In the following pages the theory of cascade showers is therefore developed, with the assumption of a cross section for the production of pairs by γ-rays which approximates closely to that calculated. As before, the whole problem is regarded as uni-dimensional and fluctuations and the Compton effect are not considered. The calculations are developed in detail for the production of showers in lead.

II

Let us consider an electron of energy E_0, large compared with its rest energy, incident on or formed in a block of material, such as lead, of atomic number Z and nuclear density n. Let $\gamma(E, x)\Delta E$ denote the probable number of γ-ray quanta with energy between E and $E+\Delta E$ at a thickness x, and $N(E, x)$ the probable number of particles (electrons and positrons) at x with energy greater than E. We define $P_\gamma(E, E')\Delta E'\Delta x$ as the probability that a particle of energy E radiates a γ-ray of energy between E' and $E'+\Delta E'$, and $P_p(E', E)\Delta E\Delta x$ as the probability that a γ-ray of energy E' creates a pair of energies between E, $E'-E$ and $E+\Delta E$, $E'-E-\Delta E$, in traversing a thickness Δx. Then

$$P_\gamma(E, E')\Delta E'\Delta x=(K/E^2E')[E^2+(E-E')^2-(\tfrac{2}{3}-\lambda)E(E-E')]f(E)\Delta E'\Delta x$$

and

$$P_p(E', E)\Delta E\Delta x=(K/E'^3)[E^2+(E'-E)^2+(\tfrac{2}{3}-\lambda)E(E'-E)]g(E')\Delta E\Delta x \quad (1)$$

where

$$K=4\alpha Z^2r_0^2n\ln(191/Z^{\frac{1}{3}}), \quad (2)$$

$$\lambda=31/90\ln(191/Z^{\frac{1}{3}})\sim\tfrac{1}{10}\text{ for Pb}, \quad (3)$$

and $f(E)$ and $g(E')$ are practically constant for E, E' greater than $10^3\,mc^2$, and below this energy vary critically with E, E', respectively, and very slowly (a variation we neglect here) with E', E, respectively.

The ionization loss per unit distance may be written

$$(-\partial E/\partial x)\text{ collision}=K\beta$$

where β is an energy characteristic of the material traversed. For lead, $\beta=6.5$ Mev whereas for air $\beta=95$ Mev.

The Compton scattering of the γ-ray quanta by the atomic electrons produces two effects, both of which are relatively unimportant and will be neglected. For high energy quanta most of the energy is transferred to the electron, which continues on in the shower, producing practically the same effects as if the original γ-ray had continued unscattered. Low energy quanta and the electrons with which they collide may be scattered out of the beam, but the loss of energy per unit distance arising from this process is small compared to $K\beta$.

If we write $Kx=t$, the diffusion equations therefore become

$$\frac{\partial\gamma}{\partial t}=-\sigma\gamma g(E)-\frac{1}{E}\int_E^\infty f(E')\frac{\partial N(E')}{\partial E'}[E'^2+(E'-E)^2-(\tfrac{2}{3}-\lambda)E'(E'-E)]\frac{dE'}{E'^2}, \quad (4)$$

$$\frac{\partial N}{\partial t}=\beta\frac{\partial N}{\partial E}+2\int_E^\infty dE'\int_{E'}^\infty\frac{dE''}{E''^3}\gamma(E'')g(E'')[E'^2+(E''-E')^2+(\tfrac{2}{3}-\lambda)E'(E''-E')]$$
$$+\int_E^\infty f(E')\frac{\partial N(E')}{\partial E'}dE'\int_0^E\frac{dE''}{(E'-E'')E'^2}[E'^2+E''^2-(\tfrac{2}{3}-\lambda)E'E'']. \quad (5)^9$$

[7] M. Schein and P. S. Gill, Rev. Mod. Phys. **11**, 267 (1939).

[8] R. F. Christy and S. Kusaka, Phys. Rev. **59**, 414 (1941).

[9] In recent work by Dresden, Scott and Uhlenbeck [Phys. Rev. **59**, 112A (1941)] these diffusion equations have been solved by means of a Liouville-Neumann series with variable lower limit. As their calculations were es-

The total cross section for energy loss by radiation[10] is

$$\varphi_{\text{rad}}=\frac{1}{E}\int_0^E E'P_\gamma(E,E')dE' =K(1+\lambda/2)f(E) \tag{6}$$

from which it follows that, for $E>20\ mc^2$, $f(E)$ varies slowly between 0.75 and 1. Below $E=20\ mc^2$, as pointed out in the introduction, the energy loss by radiation in heavy elements begins to be smaller than that due to ionization. We shall, therefore, write $f(E)=1$, for it differs appreciably from this value only for energies for which the bremsstrahlung process is unimportant.

The total cross section for pair production is given by

$$\varphi_{\text{pair}}=\int_0^{E'} P_p(E',E)dE =\sigma Kg(E') \text{ where } \sigma=(7/9)-(\lambda/6). \tag{7}$$

For $E'=20\ mc^2$, $g(E')=0.33$ and even for $E'=100\,mc^2$, $g(E')\sim 0.7$. It is just the effect of the variation of $g(E')$ with E' that it is our purpose to investigate. Since ionization losses are the dominating processes for air showers for energies up to 200 mc^2 it is patent that for such showers this variation may be neglected.

We therefore fit the calculated function φ_{pair} as given by Heitler[11] by means of the empirical formula

$$g(E')=\delta_1+\delta_2\ln E'-\delta(\ln E')^2. \tag{8}$$

The coefficients δ_1, δ_2 are given in terms of δ by the condition that, for $E'=E_0>10^3\ mc^2$, $g(E')=1$ and $dg(E')/dE'=0$. Thus we ensure that for $E'\sim E_0$ Eq. (8) yields the correct value and energy dependence for φ_{pair}. We then have

$$g(E')=1-\delta\ln(E_0/E)^2. \tag{9}$$

In the limit $\delta\to 0$ it is clear that $g(E')\to 1$ and the equations go over continuously into those solved by Serber.[6]

sentially for showers in air, the variation with energy of the pair production cross section was not considered, although their method could be extended to include this refinement. I am indebted to Professor G. Uhlenbeck for informing me of this work. I am also indebted to Dr. H. J. Bhabha for informing me that he and Chakrabarthy have recently obtained a solution of the cascade problem for air, correct to within 5 percent. The method adopted by these authors could also be extended to the present problem.

[10] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, 1936), p. 172.

[11] W. Heitler, reference 10, p. 201.

The coefficient δ is chosen to give the best fit of (9) to the theoretical curve of reference 11. It is practically independent of the material traversed, but is critically dependent on E_0. In fitting (9) to the calculated curve, it is important to obtain a correct fit for high energies and an approximate fit for low energies, since for most of the shower the average energy of the particles is greater than 100 mc^2. To do this it is necessary to take different forms for the dependence of δ on E_0 for different orders of magnitude of E_0. One finds that

$$\begin{aligned}\delta&=0.30/(\ln E_0-4.50)^2 \text{ for } E_0\sim 10^3\ mc^2,\\ &=0.21/(\ln E_0-5.30)^2 \text{ for } E_0\sim 10^4\ mc^2,\\ &=0.14/(\ln E_0-6.24)^2 \text{ for } E_0\sim 10^5\ mc^2.\end{aligned} \tag{10}$$

This choice of the coefficients makes $g(E')$ a slowly varying function of E_0, the initial energy. While from the point of view of the physical interpretation of $g(E')$ this is meaningless, from the mathematical viewpoint this trick introduces an enormous simplification into the right-hand side of the fundamental Eq. (12). Moreover, the theoretical curve is represented quite accurately by (9), with δ given by (10), for $20\ mc^2<E\leqslant E_0\lesssim 10^5\ mc^2$ and extremely accurately for $E_0/50<E\leqslant E_0$.

As usual, we first solve the equations neglecting the inhomogeneous terms in β arising from the effect of ionization. The advantage of the choice of a form (9) for $g(E')$ involving the energy dependence logarithmically, now becomes apparent, for in a sense these logarithmic terms do not destroy the homogeneity of the equations. This is seen most easily by writing

$$\begin{aligned}\gamma(E,t)&=\frac{e^{-\sigma t}}{E}\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\left(\frac{E_0}{E}\right)^y f(y,t)dy,\\ N(E,t)&=e^{-\sigma t}\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\left(\frac{E_0}{E}\right)^y g(y,t)dy\end{aligned} \tag{11}$$

from which it follows that

$$\ln(E/E_0)\gamma(E,t) =\frac{e^{-\sigma t}}{E}\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\left(\frac{E_0}{E}\right)^y \frac{\partial f(y,t)}{\partial y}dy,$$

$$(\ln(E/E_0))^2\gamma(E,t) =\frac{e^{-\sigma t}}{E}\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\left(\frac{E_0}{E}\right)^y \frac{\partial^2 f(y,t)}{\partial y^2}dy.$$

Using the relation (9) and omitting, at first, the terms in β, one finds that Eqs. (4) and (5) become

$$\ddot{f}-(\sigma-A(y))\dot{f}-B(y)C(y)f=\rho\left[\dot{f}''-\left(\sigma-A(y)+\frac{B(y)C(y)}{\sigma}\right)f''\right], \quad (12)$$

$$\dot{f}-\rho f''=yC(y)g. \quad (13)$$

in which a dot and a prime denote differentiation with respect to t and y, respectively, $\rho=\delta\sigma$, and

$$\begin{aligned} A(y)&=(4/3+\lambda)(\psi(y)+\gamma)-(5/6+\lambda)\\ &\quad+(1+3\lambda)/3(y+1)+1/(y+2),\\ B(y)&=2/(y+1)-(8+6\lambda)/3(y+2)\\ &\quad+(8+6\lambda)/3(y+3),\\ C(y)&=(4+3\lambda)/3y-(4+3\lambda)/3(y+1)\\ &\quad+1/(y+2). \end{aligned} \quad (14)$$

Since λ is a small slowly varying function of Z, the functions $A(y)$, $B(y)$, $C(y)$ and σ are practically independent of Z. For convenience in comparing the results for lead showers with those already calculated for air, we shall neglect this slight variation, introducing thereby an error of less than 1 percent.

Consider the equation obtained from (12) by neglecting the right-hand side, which depends on the small parameter ρ. The solution of this equation corresponding to the initial condition of one electron of energy E_0 and no γ-rays incident at $t=0$ is

$$f_0(t,y)=[C(y)/\mu_1-\mu_2][\exp(\mu_1 t)-\exp(\mu_2 t)] \quad (15)$$

where μ_1 is the positive root and μ_2 the negative root of the equation

$$\mu^2-(\sigma-A(y))\mu-B(y)C(y)=0. \quad (16)$$

Inserting (15) on the right-hand side of (12), we find that, because of the appearance of terms involving the second derivative of f_0 with respect to y, the exact solution $f_1(t, y)$ of the equation involves terms of order ρt, ρt^2, ρt^3, the ρt^3 terms cancelling because of (16). More precisely,

$$f_1(t,y)=\sum_{r=1,2}(-1)^{r-1}\sum_{n=0}^{3}\alpha_{rn}t^n\exp(\mu_r t) \quad (17)$$

where

$$\begin{aligned} \alpha_{r0}&=(-1)^{r-1}C(y)/D_r,\\ \alpha_{r1}&=\{\rho[\alpha_{r0}''\theta_r+\mu_r''\alpha_{r0}+2\mu_r'\alpha_{r0}']-2\alpha_{r2}\}/D_r,\\ \alpha_{r2}&=\{\rho[\theta_r(u_r''\alpha_{r0}+2\mu_r'\alpha_{r0}')+2\mu_r'^2\alpha_{r0}]-6\alpha_{r3}\}/2D_r,\\ \alpha_{r3}&=\rho\mu_r'^2\alpha_{r0}\theta_r/3D_r,\\ D_r&=2\mu_r-\sigma+A=(-1)^{r-1}(\mu_1-\mu_2),\\ \theta_r&=\mu_r-\sigma+A(y)-B(y)C(y)/\sigma. \end{aligned} \quad (18)$$

Inserting (17) into the right-hand side of (12) we obtain the second approximation

$$f_2(t,y)=\sum_{r=1,2}(-1)^{r-1}\sum_{n=0}^{6}\beta_{rn}t^n\exp(\mu_r t), \quad (19)$$

involving terms of order $(\rho t^3)^2$, and in general the leading term of the nth approximation obtained in this manner is of order $(\rho t^3)^n$. The significance of this will be discussed after we have taken into account the effects of ionization. To do this we follow exactly the calculations of Serber. To obtain an exact solution of the equations, even in the form of a contour integral, is a difficult procedure, but to a first approximation we have

$$N(t,E)=-\frac{e^{-\sigma t}}{4\pi^2}\int_C\frac{dy}{yC(y)}\left(\frac{E}{E_0}\right)^{-y}\int_S ds\frac{\Gamma(-s)\Gamma(y+s)}{\Gamma(y)}\left(\frac{E}{\beta}\right)^{-s}\sum_{r=1,2}K_r(y,s)(-1)^{r-1}\times\sum_{n=0}^{3}\left[\frac{\partial}{\partial t}(\alpha_{rn}t^n\exp[\mu_r t])-\rho\frac{\partial^2}{\partial y^2}\alpha_{r0}\exp(\mu_r t)\right], \quad (20)$$

where

$$\{\mu_r(A(y+s)-A(y))-[B(y+s)C(y+s)-B(y)C(y)]\}K_r(y,s)=s\mu_r K_r(y,s-1). \quad (21)$$

and

$$K_r(y,0)=1.$$

As before, the main contribution to the shower arises from the terms corresponding to the positive root of Eq. (18), $r=1$. The total number of electrons and positrons at a depth t in the shower, as estimated from (20) by the method of steepest descents, is, therefore,

$$N(\epsilon,t)=N_0(\epsilon,t)\left[1+\frac{t}{\mu_1\alpha_{10}}(\mu_1\alpha_{11}+2\alpha_{12}-2\mu_1'\alpha_{10}'-\mu''\alpha_{10})+\frac{t^2}{\mu_1\alpha_{10}}(\mu_1\alpha_{12}+3\alpha_{13}-\rho\alpha_{10}\mu_1'^2)+t^3\frac{\alpha_{13}}{\alpha_{10}}\right] \quad (22)$$

where

$$t=(1-\epsilon y)/\mu_1'y, \qquad \epsilon=\ln(E_0/\beta),$$

and $N_0(\epsilon, t)$ is the number of particles at t, calculated by Serber neglecting the variation with energy of the pair production cross section.[12]

Expression (22) has been obtained from the first approximation Eq. (17) involving terms of order ρt^3. In a similar manner an expression for $N(\epsilon, t)$ could be obtained from the nth approximation and it is found to involve terms of order $(\rho t^3)^n$. For large thicknesses t, therefore, the expansion has little meaning. At the point $y=1$, however, which is near the maximum of the shower, θ—and hence α_{13}—vanishes. The leading term in the nth approximation is then of order $(\rho t^2)^n$, a quantity which is small compared to unity and which decreases as ϵ increases. The successive approximations, therefore, form a rapidly convergent series, and for $E_0 \gtrsim 10^4\ mc^2$ the maximum of the shower is given in terms of $(N_0)_{\max}$ to within 5 percent. For $E_0 \sim 10^3\ mc^2$ it is necessary to consider the second approximation, which for this energy differs from the first approximation by 8 percent.

From (22), and estimating the second approximation which is too cumbersome to write down in detail, one finds that

$$N_{\max} \approx (N_0)_{\max}[1-0.57\delta(\epsilon-1)(\epsilon+0.18) + 0.1\delta^2(\epsilon-1)^4] \quad (23)$$

with δ given by (10), and

$$(N_0)_{\max}=0.4E_0/\beta(1+1.6(\epsilon-1))^{\frac{1}{2}}.$$

It is clear that the maximum value of N occurs for a value of t and hence of y, slightly less than that which makes N_0 a maximum. The difference is very small, however, and the error involved in taking $N_{\max}=(N)_{y=1}$ is of order 1 percent.

For a given value of ϵ, the value of δ given by (10) is much smaller for air than for lead. The corresponding correction to $N_{\max}$ for air is, therefore, much smaller, being of order 7 percent and decreasing with increasing ϵ. With present calculations of the multiplication of cascade showers in air, which are possibly in error by as much as 20 percent, this refinement is therefore negligible.

For all thicknesses the number of γ-ray quanta predicted from the above calculations will be greater than that given by the application of the usual cascade theory, but for $t<3t_{\max}$ the number of particles predicted is less. Owing to the uncertainty of these and previous calculations, however, we give numerical values only for the maximum of the particle multiplication curve in lead (Table I), calculated from (23) for various values of the initial energy E_0. The maximum number of particles is seen to vary between $E_0/11\beta$ and $E_0/10\beta$ as E_0 varies from 10^9 ev to 10^{11} ev, i.e., between $E_0/8E_c$ and $E_0/7E_c$ where $E_c=\beta/\ln 2$ is the critical energy, at which the average rates of energy loss by ionization and radiation are equal.

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TABLE I. *The maximum number of particles arising in lead and in the atmosphere from a single electron of energy E_0 is given for various values of the ratio E_0/β, where β is an energy characteristic of the material traversed.* ($\beta=6.5$ *Mev for* Pb, *95 Mev for air.*)

E_0/β	155	1550	15500
$(E_0)_{\text{Pb}}$	10^9 ev	10^{10} ev	10^{11} ev
$(N_{\max})_{\text{Pb}}$	13.5	148	1550
$(E_0)_{\text{air}}$	1.5×10^{10} ev	1.5×10^{11} ev	1.5×10^{12} ev
$(N_{0\max})_{\text{air}}$	21.9	200	1940

[12] Reference 6, Eq. (7).