On the Dislocation Theory of Plastic Deformation

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(Received July 3, 1941)

A detailed study has been made of the lattice imperfections which are assumed to account for plastic deformation. Calculations of the strain energy associated with these dislocations have been made in the following cases: a dislocation in a uniform shear stress, two dislocations in an infinite medium, and a dislocation near a surface. The force acting on a dislocation is found by taking the gradient of the strain energy. A force is found which tends to attract dislocations toward the surface of a specimen. It is shown that about twice as much energy is required to produce a certain amount of slip inside a solid as is required to produce the same amount of slip at the surface. The energy required to produce a dislocation is found to be several electron volts per atomic plane, the exact amount depending on where it is located and how it was produced. Finally the energy stored in a material during work hardening is calculated by assuming that the dislocations are arranged in a regular two-dimensional lattice in the material. The density of dislocations found for severely work hardened material agrees with the predictions of other investigators. Numerical values found for the energy stored during work hardening are in agreement with experiment.

INTRODUCTION

IF a single crystal is subjected to a small tensile stress it will elongate according to Hooke's law. If the component of the shear stress acting along a definite crystallographic direction exceeds a certain critical value the crystal will also undergo considerable plastic deformation. For most crystals the value of the critical shear stress is about 10^7 dynes per cm^2. The permanent deformation apparently results from a sliding of sections of the crystal over one another. The sections are bounded by certain close-packed crystallographic planes and the slip takes place along the direction mentioned above. This slip direction is usually a direction in which the atoms are closely packed. A microscopic examination of a single crystal which has been elongated by several percent shows clearly the step-like discontinuities produced at the surface when slip occurs. The sections bounded by slip planes are several microns thick at room temperatures.

In a series of articles appearing in the *Journal of Applied Physics*, Seitz and Read[1] consider the various theories which have been used to discuss plastic deformation. They conclude that the theory of "dislocations" as developed by Taylor[2] yields the most satisfactory description of plastic phenomena which has been given to date. The present paper represents a further attempt to develop and to test the validity of dislocation theory. The predictions of dislocation theory which are considered in this paper are in good agreement with experiment.

According to Taylor, slip in a solid results from the motion of a certain type of lattice imperfection through the crystal. Taylor has used these "dislocations" to discuss qualitatively the hardening which occurs when crystals are plastically deformed. Recently Mott and Nabarro[3] using dislocation theory have discussed precipitation hardening.

In order to make clear the nature of a dislocation we consider a line dislocation in a simple cubic crystal. In Fig. 1 we have indicated the positions of the atoms in a crystallographic plane which is normal to the slip plane. The axis of the dislocation is perpendicular to the plane of the figure. Thus parallel atomic planes above and below the plane of the figure will have the same distribution of atoms about the center, or axis, of the dislocation. If a dislocation moves along its slip plane across the entire specimen, then Taylor has shown that the material above the slip plane will be displaced relative to the material below the slip plane by an amount equal to the lattice constant of the material.

An examination of Fig. 1 indicates that the distortion is just that which would be produced by inserting extra half-planes of atoms into

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[1] F. Seitz and T. A. Read, J. App. Phys. **12**, 100, 170, 470 (1941).

[2] G. I. Taylor, Proc. Roy. Soc. **145**, 362 (1934).

[3] N. F. Mott and F. Nabarro, Proc. Phys. Soc. **52**, 86 (1940); F. Nabarro, Proc. Roy. Soc. **175**, 519 (1940).

an originally perfect crystal above the slip plane. Following Taylor we shall call this a positive dislocation. If the additional half-planes are inserted below instead of above the slip plane we shall call the dislocation which results a negative dislocation. Thus, if we are dealing with a positive dislocation we shall find that the material above the slip plane is compressed while the material below the slip plane has been dilated.

In the first section of this paper we shall calculate the force which acts on a dislocation when the material containing the dislocation is subjected to an external shearing force. Next, we shall consider two dislocations in an infinite medium. We shall determine the energy of the system and we shall evaluate the force acting between the dislocations. In the third section we shall discuss the case of a dislocation near a surface. Here too we shall calculate the energy of the system and the force acting on the dislocation. Finally, we shall evaluate the energy stored in a material when it is work hardened. All calculations made in this paper will use the theory of an elastic, isotropic continuum.

Dislocation in Uniform Shear Stress

According to dislocation theory an annealed specimen contains only a few line dislocations. If we apply a sufficiently large external shearing force to the specimen it will deform plastically. This deformation may take place either by the motion of existing dislocations or by the formation and motion of dislocations. In either case the force produced on an isolated dislocation by the external shearing force would be of interest. We shall, therefore, consider a line dislocation in a material which is also subjected to a uniform two-dimensional shear stress. This uniform shear stress is produced by the external force.

We shall first calculate the strain energy of the system. In order to calculate the strain energy we must know the stresses acting. It can be shown that the solution of certain two-dimensional problems of elasticity can be obtained by finding a suitable Airy stress function,[4] χ. This stress function must satisfy the differential equation

$$\frac{\partial^4\chi}{\partial x^4}+2\frac{\partial^4\chi}{\partial x^2\partial y^2}+\frac{\partial^4\chi}{\partial y^4}=0$$

and must yield stresses or displacements which satisfy the boundary conditions of the problem. The stresses are given by the following relations:

$$\sigma_x=\partial^2\chi/\partial y^2;\quad \sigma_y=\partial^2\chi/\partial x^2;\quad \tau_{xy}=-\partial^2\chi/\partial x\partial y. \tag{1}$$

The Airy stress function appropriate for this problem is[5]

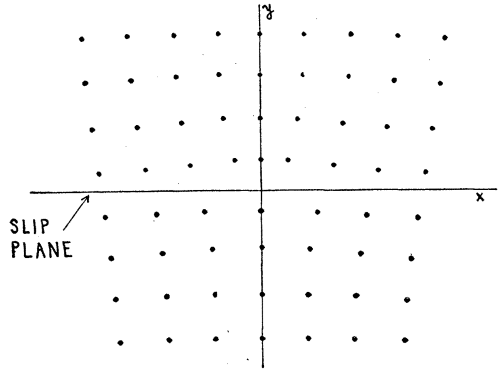

Fig. 1. Dislocation lying in a simple cubic crystal.

$$\chi=-Bxy-Dy\log(x^2+y^2)^{\frac{1}{2}}. \tag{2}$$

The first term represents the uniform shear stress; the second, as one can see from the displacements given below, represents a positive dislocation. The stresses resulting are

$$\begin{aligned}\sigma_x&=-Dy(3x^2+y^2)/(x^2+y^2)^2,\\ \sigma_y&=+Dy(x^2-y^2)/(x^2+y^2)^2,\\ \tau_{xy}&=+B+Dx(x^2-y^2)/(x^2+y^2)^2.\end{aligned} \tag{3}$$

The stresses due to the dislocation die off inversely as the first power of the distance from the center of the dislocation. The displacements can now be calculated from Hooke's law. They are

$$\begin{aligned}u&=\frac{(m-1)}{mG}D\tan^{-1}\left(\frac{y}{x}\right)+\frac{D}{2G}\frac{xy}{x^2+y^2},\\ v&=-\frac{(m-2)}{2mG}D\log(x^2+y^2)^{\frac{1}{2}}-\frac{D}{2G}\frac{x^2}{x^2+y^2}.\end{aligned} \tag{4}$$

[4] S. Timoshenko, *Theory of Elasticity* (McGraw-Hill Book Co., Inc., 1934), p. 25.

[5] This dislocation differs from the one used by Taylor. Dislocations of this type were probably first used to discuss plastic deformation by J. M. Burgers and W. G. Burgers. J. M. Burgers, Proc. K. Akad. Amst. **42**, 263, 378 (1939). We shall discuss the differences between these two types of dislocations near the end of the next section.

Here u is the displacement in the x direction and v is the displacement in the y direction. G is the shear modulus of the material, and m is the reciprocal of Poisson's ratio. If we measure the angle appearing in the equation for u from the plus x axis then the displacements are discontinuous along this axis. The material just below the positive x axis is displaced in the x direction with respect to the material just above the axis by an amount

$$\lambda=2\pi(m-1)D/mG, \qquad D=mG\lambda/2\pi(m-1). \quad (5)$$

According to Eqs. (3) and (4) the stresses and the displacements are singular at the center of the dislocation. Physically this means that an elastic continuum theory is not capable of describing the situation near the center of a dislocation. We shall, therefore, in all problems surround the axis of each dislocation by a cylinder of radius r_0. We shall choose r_0 so that the maximum strain at the surface of the cylinder is one-tenth. We shall consider only situations in which all dislocations, impurity atoms, precipitates and other singularities are separated from the dislocation in question by distances which are large compared with r_0. For such systems the stress at r_0 will be, to a good approximation, just the stress produced by the dislocation. Furthermore, the interaction energies and the forces exerted by other singularities on the dislocation will depend upon the strain energy in the material outside r_0 since the strain energy inside r_0 will not change appreciably if the dislocation is moved relative to the other singularities.

Let us calculate r_0 in the case of copper. We shall take λ of Eq. (5) to be the translation which would carry an atom of copper from a cube corner to the center of a cube face. This unit slip distance is 2.55×10^{-8} cm in copper. Using $G=4.53\times10^{11}$ dynes per cm^2 and $m=2.941$ we find from Eq. (5) that $D=2.78\times10^{3}$ dynes per cm. The shear strain is found to be the largest strain. Using Hooke's law and substituting from Eqs. (3) we find

$$\gamma_{xy}=\tau_{xy}/G=+Dx(x^2-y^2)/G(x^2+y^2)^2,$$

where γ_{xy} is the shear strain. Putting in the limiting value of one-tenth for γ_{xy} and inserting the maximum value that the expression on the right has on the surface of the cylinder we find

$$r_0=D/\gamma_{xy\,\max}\,G=6.15+10^{-8}\text{ cm for copper.} \quad (6)$$

The strain energy of a slab of material of unit thickness in the z direction is[6]

$$W=\int\int\frac{ds}{4G}\left\{\frac{(m-1)}{m}(\sigma_1+\sigma_2)^2 -2\sigma_1\sigma_2+2\tau_{12}^2\right\}, \quad (7)$$

where m is the reciprocal of Poisson's ratio and where we have assumed that the material is in a state of a plane strain. Equation (7) gives the strain energy in any orthogonal two-dimensional coordinate system. In this equation ds is the element of area and σ_1, σ_2, and τ_{12} are the stresses in the coordinate system chosen. Substituting from Eqs. (3) we find

$$W=\frac{1}{2G}\int\int dxdy\left\{B^2+\frac{D^2}{x^2+y^2} +\frac{2BDx(x^2-y^2)}{(x^2+y^2)^2}\right\}. \quad (8)$$

It is apparent from Eq. (8) that if either the uniform shear stress or the dislocation is present in the specimen it will contain strain energy. If both the uniform shear stress and the dislocation are present simultaneously, the strain energy is not simply the sum of the strain energy due to a uniform shear stress and the strain energy due to a dislocation. In addition to these self-energies we have a cross term, an interaction energy. The first two terms of the integral (8) are self-energy terms while the last term is the interaction energy in which we are interested. In order to get a sensible result one must be careful about the limits of integration. The interaction energy can be written as

$$W_{\text{int}}=\frac{2BD}{G}\left\{\int_{r_0}^{c}\int_{-c+\xi}^{-r_0}\frac{x(x^2-y^2)}{(x^2+y^2)^2}dxdy +\int_{r_0}^{c}\int_{r_0}^{c+\xi}\frac{x(x^2-y^2)}{(x^2+y^2)^2}dxdy\right\}. \quad (9)$$

In this expression C is a large distance which is

[6] E. Trefftz, *Handbuch der Physik*, Vol. VI, p. 79 (1928).

allowed to approach infinity after the integration has been carried out. The result then becomes

$$W_{\text{int}} = -2BD\xi/G. \tag{10}$$

The distance ξ essentially measures the position of the dislocation with respect to the field of uniform shear. The force acting on the dislocation is given by the negative of the derivative of this expression with respect to ξ.

$$F = +2BD/G. \tag{11}$$

The force acts in the plus x direction on this positive dislocation.

Using the value of D for a unit dislocation in copper and assuming that the uniform shear stress is 10^7 dynes per cm (i.e., the value of the critical shear stress for copper) we find the force to be 2.22×10^{-9} dyne for a segment whose length is the distance between successive atomic planes which are perpendicular to the axis of the dislocation. This force is very small compared with chemical forces.

Two Parallel Line Dislocations

In this section we shall calculate the strain energy associated with two parallel line dislocations in an isotropic, continuous medium. Quantitative values of the strain energy and of the force acting between the dislocations will be given. Finally, Taylor's treatment of dislocation theory will be discussed.

The Airy stress function for two unlike dislocations of equal strength in an infinite medium is

$$\chi = D\{y\log(x^2+y^2)^{\frac{1}{2}} - (y-y_0)\log[(x-x_0)^2+(y-y_0)^2]^{\frac{1}{2}}\}. \tag{12}$$

We have assumed that the negative dislocation has its center at the origin of our coordinate system. The positive dislocation has its center at x_0y_0. Each dislocation has associated with it a slip plane which passes through the center of the dislocation and is parallel to the xz plane (z is normal to the plane of Fig. 1). The stresses obtained from (12) are

$$\begin{aligned}
\sigma_x &= D\left\{\frac{y(3x^2+y^2)}{(x^2+y^2)^2} - \frac{(y-y_0)(3\{x-x_0\}^2+\{y-y_0\}^2)}{(\{x-x_0\}^2+\{y-y_0\}^2)^2}\right\},\\
\sigma_y &= -D\left\{\frac{y(x^2-y^2)}{(x^2+y^2)^2} - \frac{(y-y_0)(\{x-x_0\}^2-\{y-y_0\}^2)}{(\{x-x_0\}^2+\{y-y_0\}^2)^2}\right\},\\
\tau_{xy} &= -D\left\{\frac{x(x^2-y^2)}{(x^2+y^2)^2} - \frac{(x-x_0)(\{x-x_0\}^2-\{y-y_0\}^2)}{(\{x-x_0\}^2+\{y-y_0\}^2)^2}\right\}.
\end{aligned} \tag{13}$$

From Eqs. (12) and (13) one sees that if the two dislocations coincide they will neutralize one another leaving an unstrained crystal.

Substituting the stresses given in Eqs. (13) into expression (7) we obtain the strain energy for an infinite slab of material which is of unit thickness in the z direction:

$$\begin{aligned}
W = \frac{D^2}{2G}\int\int dxdy\Bigg\{&\frac{2(m-1)}{m}\left[\frac{y^2}{r_1^4}+\frac{(y-y_0)^2}{r_2^4}-\frac{2y(y-y_0)}{r_1^2r_2^2}\right]+\frac{x^2-y^2}{r_1^4}+\frac{\{x-x_0\}^2-\{y-y_0\}^2}{r_2^4}\\
&-\frac{2x(x-x_0)(x^2-y^2)(\{x-x_0\}^2-\{y-y_0\}^2)}{r_1^4r_2^4}-\frac{y(y-y_0)(x^2-y^2)(3\{x-x_0\}^2+\{y-y_0\}^2)}{r_1^4r_2^4}\\
&-\frac{y(y-y_0)(3x^2+y^2)(\{x-x_0\}^2-\{y-y_0\}^2)}{r_1^4r_2^4}\Bigg\},
\end{aligned} \tag{14}$$

where

$$r_1 = (x^2+y^2)^{\frac{1}{2}}, \quad r_2 = [(x-x_0)^2+(y-y_0)^2]^{\frac{1}{2}}.$$

Let us surround the axis of each dislocation by a cylinder of radius r_0 and proceed with the integration of Eq. (14) over the portions of the slab which lie outside these cylinders. The terms in Eq. (14) can be divided into two classes. First, there are the self-energy terms which contain r_1 or r_2. Second, there are the interaction terms which contain both r_1 and r_2. The self-energy terms can be integrated immediately and yield

$$W_{\text{self}}=\frac{2\pi D^2(m-1)}{mG}\log\frac{L}{r_0}, \tag{15}$$

where L is essentially the radius of our cylindrical specimen which has the two dislocations near its center. We introduce new Cartesian coordinates into the interaction terms taking the origin midway between the two dislocations. The new coordinates ζ and η are given by

$$x=\zeta\cos\phi-\eta\sin\phi+\tfrac{1}{2}R\cos\phi,\quad y=\zeta\sin\phi+\eta\cos\phi+\tfrac{1}{2}R\sin\phi, \tag{16}$$

where

$$R=(x_0^2+y_0^2)^{\frac{1}{2}},\quad \tan\phi=y_0/x_0.$$

The interaction terms can then be written:

$$W_{\text{int}}=-\frac{D^2}{2G}\int\int d\zeta d\eta\left\{\frac{4(m-1)}{m}\left[\frac{\zeta^2\sin^2\phi+\eta^2\cos^2\phi-\frac{1}{4}R^2\sin^2\phi}{\zeta^4+2\zeta^2(\eta^2-\frac{1}{4}R^2)+(\eta^2+\frac{1}{4}R^2)^2}\right]\right.$$
$$+\frac{\zeta^6\cos2\phi+\zeta^4(\eta^2\cos2\phi-\frac{3}{4}R^2\cos2\phi)+\zeta^2(-\eta^4\cos2\phi-\eta^2R^2[4+2\cos2\phi]+[3R^4\cos2\phi/16])}{[\zeta^4+2\zeta^2(\eta^2-\frac{1}{4}R^2)+(\eta^2+\frac{1}{4}R^2)^2]^2}$$
$$\left.+\frac{(-\eta^6\cos2\phi+\eta^4\frac{1}{4}R^2[-4+2\cos2\phi]+\eta^2\frac{1}{16}R^4[4+\cos2\phi]-[R^6\cos2\phi/64])}{[\zeta^4+2\zeta^2(\eta^2-\frac{1}{4}R^2)+(\eta^2+\frac{1}{4}R^2)^2]^2}\right\}. \tag{17}$$

In Eq. (17) we have omitted terms that are odd in ζ or η since these would obviously not contribute to the integral. If we now integrate Eq. (17) over the region outside the cylinders we obtain the interaction energy. Adding this interaction energy to the self-energy we find that the total, elastic, strain energy of the positive and negative dislocation in an infinite medium is

$$W=\frac{mG\lambda^2}{2\pi(m-1)}\log\frac{R}{2r_0}-\frac{m^2G\lambda^2}{4\pi(m-1)^2}[\cos2\phi-\tfrac{1}{2}]. \tag{18}$$

Expression (18) is approximate since we have neglected small terms of order $(m^2G\lambda^2/4\pi(m-1)^2)\times(r_0^2/R^2)$. The force acting on the positive dislocation at x_0y_0 has the components

$$F_R=-\frac{mG\lambda^2}{2\pi(m-1)}\frac{1}{R},$$
$$F_\phi=-\frac{m^2G\lambda^2}{2\pi(m-1)^2}\frac{\sin2\phi}{R}. \tag{19}$$

Numerical values of the strain energy and the force acting between dislocations have been calculated for various distances of separation in copper. The results are given in Figs. 2, 3, and 4. In Fig. 2 we have given the strain energy in a slab whose thickness is the distance between successive atomic planes which are perpendicular to the axis of the dislocation. In Figs. 3 and 4 we have given the force acting on a dislocation segment whose length is equal to the thickness of the slab just mentioned. In Fig. 4 we have given that component of the force between two dislocations which acts along the slip direction. We have assumed that the slip planes of the two dislocations being considered are separated by by 10^{-6} cm.

We are now in a position to discuss some of the differences between the dislocation theory developed by Taylor and the dislocation theory used in the present paper. Imperfections of the type shown in Fig. 1 can be obtained in elastic

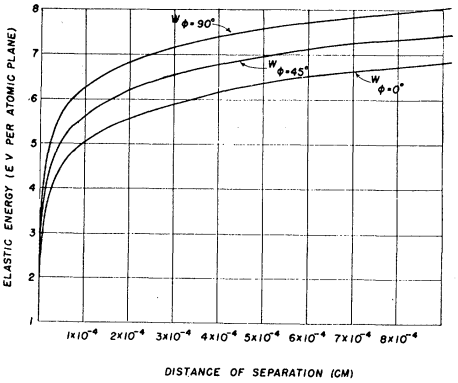

FIG. 2. Strain energy for various distances of separation in copper.

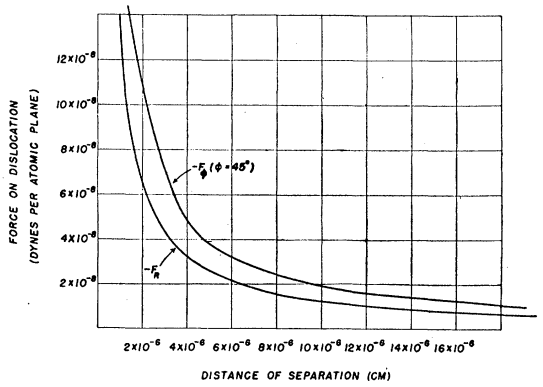

FIG. 3. Force acting on dislocation segment.

theory if one uses the following stress function:

$$\chi = Ay \log(x^2+y^2)^{\frac{1}{2}} + Bx \tan^{-1}(y/x). \quad (20)$$

The strength and nature of the dislocation represented by the above function are determined by the values assigned to the constants A and B. If we put $A=B=-\lambda G/\pi$ then we are dealing with a positive Taylor dislocation whose slip distance is λ. If we take $A=-mG\lambda/2\pi(m-1)$ and $B=0$ we are dealing with a positive Burgers dislocation whose slip distance is also λ.

We have not used Taylor dislocations for the following reason. Suppose we have a block which is finite in the x and y directions and which contains a dislocation. Let us take the surface of the block to be free from external forces. Then it can be shown that the boundary conditions which apply to this problem can only be satisfied if one takes the dislocation to be a Burgers dislocation.

In any dislocation theory it is important to decide how a dislocation will move along its slip plane under the influence of various stress fields which may be present in the material. Taylor treats the problem in the following way. Consider a positive dislocation located at the origin of a coordinate system. The xz plane will be taken to be the slip plane and the x axis the slip direction. Taylor, Burgers, and Mott and Nabarro assume that this positive dislocation will move in the plus x direction if the total external shear stress acting at the origin in that direction is greater than zero. If the above assumption is valid it would be most reasonable to assume further that the force tending to move a dislocation in a given direction is proportional to the external shear stress which acts in that direction at the center of the dislocation. This last assumption can be shown to be in error in the following way. If the assumption is used to calculate the force acting between two Burgers dislocations the result found does not agree in form with the force obtained by calculating the strain energy of the system. If one considers Taylor dislocations in various stress fields then the assumption gives the correct form for the forces, but the constant of proportionality seems to depend upon the particular stress system being considered. For example, one finds that

$$2F_A/\tau_A = F_B/\tau_B. \quad (21)$$

Here F_A is the force acting on a positive Taylor dislocation in a uniform shear stress; τ_A is the value of the uniform shear stress at the center of the dislocation. F_B is the force acting between a positive Taylor dislocation and an equal negative Taylor dislocation; τ_B is the shear stress produced at the center of the positive dislocation by the nearby negative dislocation. These forces are obtained by calculating the strain energy of the system considered.

In spite of the discussion which we have just given it can be seen that the results of Taylor's theory are qualitatively correct. This is true because Taylor assumed an interaction between two dislocations which is essentially of the correct form (i.e. the force dies off inversely as the first power of the distance between the dislocations). The error resulting from the use of the wrong

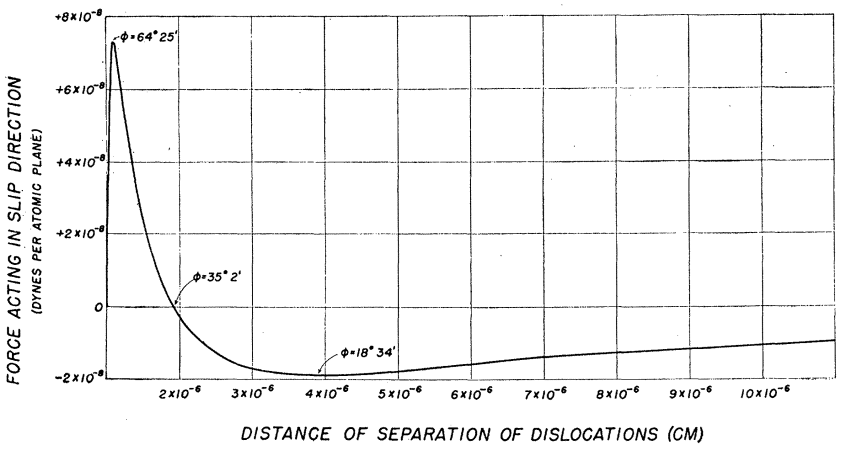

FIG. 4. Force between two dislocations in direction of slip.

type of dislocation causes Taylor to overestimate the critical shear stress by a factor of about $4(m-1)/m$. For copper this factor has the value 2.64. Our discussion indicates that Taylor's method of determining when a dislocation will move may also be in error. If this is true then Eq. (21) allows us to estimate the error made. Such an error would cause Taylor to underestimate the critical shear stress by a factor of about two. Thus Taylor's results are not in error by more than a factor three.

A DISLOCATION IN A CYLINDER

There are two ways in which dislocations could be formed. First, dislocation pairs could be created inside the specimen. Each dislocation pair would consist of a positive and a negative dislocation. Secondly, single dislocations may be formed at the surface of the material. In order to consider these possibilities in detail we shall calculate the strain energy and the force acting on a dislocation near a surface. Finally we shall discuss briefly the annealing process in metals.

Let us consider a dislocation in a circular cylinder of radius ρ. We shall suppose that the dislocation is not in general at the center of the cylinder. We shall take the axis of the dislocation parallel to the axis of the cylinder. The symmetry of this problem is that of two eccentric cylinders. Problems of this type can be conveniently treated using bipolar coordinates. G. B. Jeffery[7] has used this orthogonal system of coordinates to treat two-dimensional elastic problems and we shall use many of his results. The curvilinear coordinates are defined by the relation

$$\alpha+i\beta\equiv\log\left\{\frac{x+i(y+c)}{x+i(y-c)}\right\}, \tag{22}$$

where c is a real positive constant and x and y are Cartesian coordinates. Solving for x and y we find

$$x=c\sin\beta/(\cosh\alpha-\cos\beta),$$
$$y=c\sinh\alpha/(\cosh\alpha-\cos\beta). \tag{23}$$

The points $0,+c$ and $0,-c$ are the two poles of the system. It is clear that if we keep β fixed and change α by an amount $d\alpha$ we shall traverse a distance

$$d\alpha/h=cd\alpha/(\cosh\alpha-\cos\beta)$$

similarly if we keep α fixed and change β by $d\beta$ we shall move through the distance $d\beta/h$. Figure 5 shows the two orthogonal families of circles that form this curvilinear coordinate net.

Let us suppose that the dislocation is at the pole which lies at $0,+c$. The surface of the material will be represented by the circle on which $\sinh\alpha=c/\rho$. If the surface of the cylinder is to be free from external forces the boundary conditions which the stresses must fulfill are

$$\sigma_\alpha=0, \qquad \tau_{\alpha\beta}=0 \tag{24}$$

on the cylinder having $\sinh\alpha=c/\rho$. We shall try a stress function which we believe to be appropriate for this problem. We put

$$\chi=\frac{D\sin\beta}{h}\{\alpha+\tfrac{1}{2}\exp[-2(\alpha-\alpha_1)]\}, \tag{25}$$

[7] G. B. Jeffery, Trans. Roy. Soc. **221**, 265 (1920). See also E. Coker and L. Filon, *Photoelasticity* (Cambridge University Press, 1931).

where α_1 is defined by $\sinh\alpha_1=c/\rho$. If the first term of this expression is transformed into Cartesian coordinates we find that it represents a positive dislocation at $0,+c$ and a negative, "image" dislocation at $0,-c$. This stress function is a linear combination of terms which Jeffery has shown satisfy Airy's equation. Jeffery has also given relations which enable one to calculate the stresses in bipolar coordinates once the stress function is known. Using these we find

$$\sigma_\alpha=-\frac{D\sinh\alpha\sin\beta}{c}\{1-\exp[-2(\alpha-\alpha_1)]\},$$

$$\sigma_\beta=+\frac{D\sin\beta}{c}\{2(\cosh\alpha-\cos\beta)\exp[-2(\alpha-\alpha_1)]-\sinh\alpha[1-\exp[-2(\alpha-\alpha_1)]]\}, \tag{26}$$

$$\tau_{\alpha\beta}=-\frac{D\cos\beta}{c}(\cosh\alpha-\cos\beta)\{1-\exp[-2(\alpha-\alpha_1)]\}.$$

It is at once evident that these stresses satisfy the boundary conditions given above.

We can now calculate the strain energy. Substituting the stresses into Eq. (7) we obtain

$$W=\frac{D^2}{2G}\int_{\alpha_1}^{\alpha_0}\int_{-\pi}^{\pi}d\alpha d\beta\Big[2(m-1)\sin^2\beta\exp[-4(\alpha-\alpha_1)]+m\cos^2\beta\{1-\exp[-2(\alpha-\alpha_1)]\}^2$$
$$-\frac{2(m-2)\sinh\alpha\exp[-2(\alpha-\alpha_1)]\{1-\exp[-2(\alpha-\alpha_1)]\}\sin^2\beta}{\cosh\alpha-\cos\beta}$$
$$+\frac{(m-2)\sinh^2\alpha\sin^2\beta\{1-\exp[-2(\alpha-\alpha_1)]\}^2}{(\cosh\alpha-\cos\beta)^2}\Big].$$

In this equation we do not integrate α to plus infinity because, just as in our previous problems, we find that elastic theory is not able to describe the distortion of the material near the center of the dislocation. Instead we put $\sinh\alpha_0=c/r_0$. Carrying out the integrations we find

$$W=\frac{mG\lambda^2}{4\pi(m-1)^2}\Big[(m-1)\alpha+\tfrac{1}{2}(m-2)\sinh\alpha\{\sinh\alpha-\cosh\alpha\}+\left(\frac{3m-4}{2}\right)\exp[-2(\alpha-\alpha_1)]$$
$$-\left(\frac{3m-4}{4}\right)\exp[-4(\alpha-\alpha_1)]-\left(\frac{m-2}{2}\right)\exp[-4\alpha+2\alpha_1]+\left(\frac{m-2}{4}\right)\exp[-6\alpha+4\alpha_1]\Big]_{\alpha_1}^{\alpha_0}. \tag{27}$$

Let us rewrite this equation in a more useful form for the special case of a dislocation inside the plane face of an infinite block of material. We find that, to a good approximation,

$$W=\frac{mG\lambda^2}{4\pi(m-1)}\log\frac{2R}{r_0}+\text{surface terms}, \tag{28}$$

where R is the distance of the dislocation from the surface of the cylinder. The first term is the leading term in this expression. This term represents the strain energy in the block due to the stresses produced by the dislocation and the "image" dislocation which is a distance R outside the plane face. The use of an "image" dose not

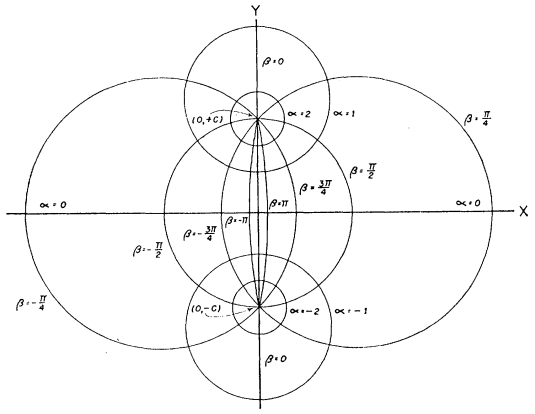

FIG. 5. Bipolar coordinate system.

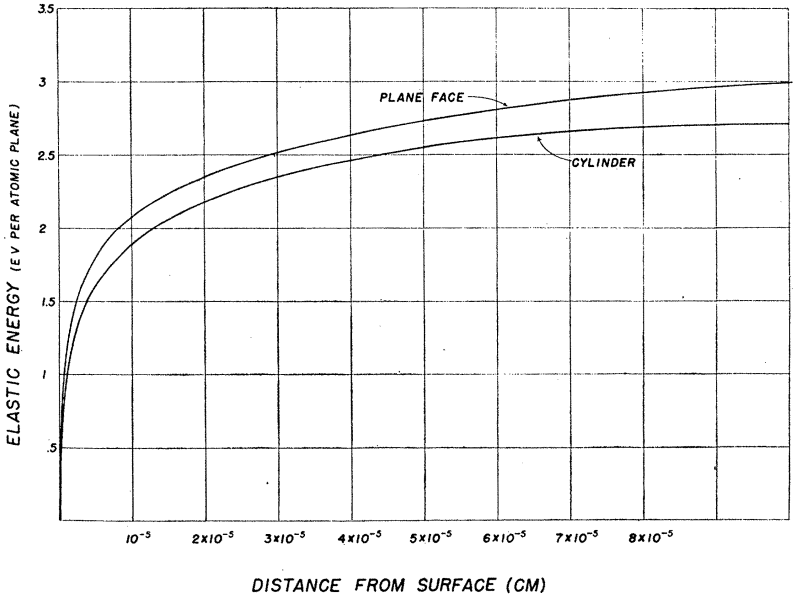

FIG. 6. Strain energy as a function of distance from surface.

completely satisfy the boundary conditions at the plane face and the additional stresses used produce the "surface terms" in the strain energy. These "surface terms" are almost independent of the distance of the dislocation from the surface except at small distances. Because of this the first term increases in importance as the dislocation moves away from the surface. If we consider a very large copper block which has a dislocation 5×10^{-7} cm inside a plane face we find that the surface terms are 26 percent of the image term. If the dislocation is 10^{-4} cm from the surface, the surface terms amount to 8 percent of the image term. The force acting on the dislocation is obtained by taking the derivative of the strain energy with respect to the distance from the surface. The result for the case of a dislocation inside a plane surface is

$$F=-\frac{mG\lambda^2}{4\pi(m-1)}\frac{1}{R}+\text{surface terms.} \quad (29)$$

The curves drawn in Figs. 6 and 7 give the strain energy and the force acting on a dislocation as a function of its distance from the surface in a copper block. We have considered two cases, one in which we have a dislocation near the plane face of a semi-infinite block, another in which we have a dislocation in a cylinder of radius 10^{-4} cm. In the second case the strain energy reaches a maximum when the dislocation is at center of the cylinder. The strain energy curve is symmetrical about this maximum. Instead of plotting the elastic energy of a slab of unit thickness, we have given the elastic energy in a slab whose thickness is the distance between successive atomic planes which are perpendicular to the axis of the dislocation.

We can now decide whether it is energetically easier to form dislocations at surfaces or in pairs inside the specimen. In order to make a comparison we shall suppose that slip of one atomic unit has occurred in both cases along a distance R in the slip direction. For a dislocation near a surface, R is the distance from the center of the dislocation to the surface. In the case of two dislocations inside an infinite block, R is the distance between the dislocations. By comparing the curves of Fig. 6 with that of Fig. 2 one finds that, in general, it requires about twice as much energy to produce a pair of dislocations as it takes to form a single dislocation near the surface. This does not necessarily mean that most of the dislocations are formed at the surface of the specimen. One must also remember that there is, in general, a much larger volume in which dislocations can be formed inside the material than there is near the surface.

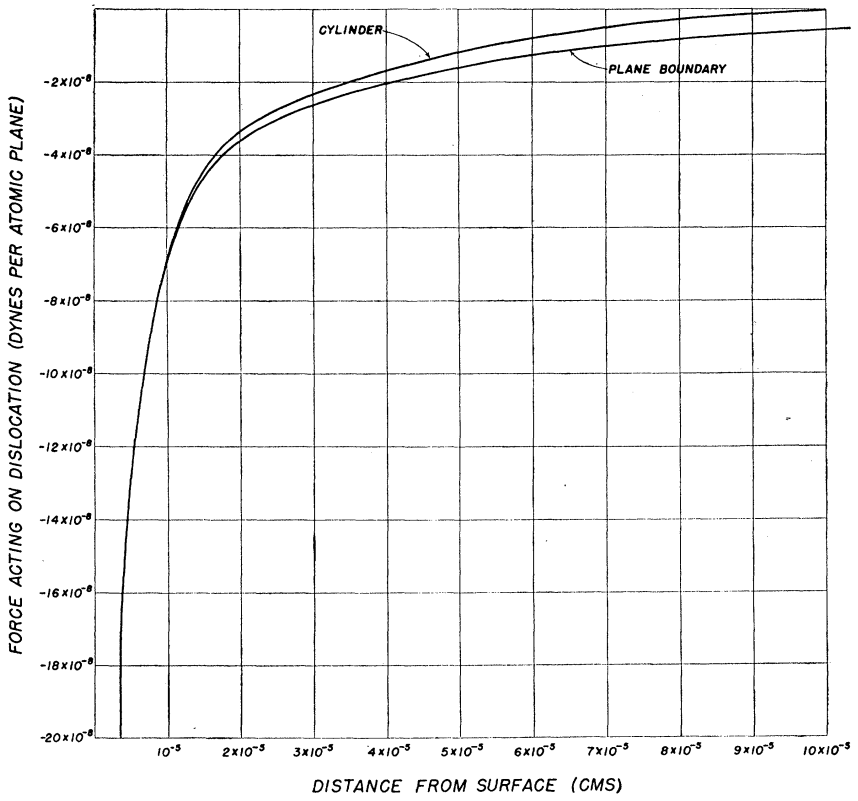

FIG. 7. Force on dislocation as a function of distance from surface.

If dislocation theory is correct, then the forces investigated in this section and in the previous one will be of great importance for any theory of annealing. Taylor has shown that a substance containing a large number of dislocations is hard. However, if we have a specimen which initially contains a large number of dislocations per cm^2, then the image force acting toward the surface will cause dislocations to move out of the specimen. If we have positive and negative dislocations inside the specimen which are on the same slip plane, they can attract and annihilate one another. Unfortunately, the experimental data in this particular field are not sufficient for a quantitative test of the theory. Further data on resoftening temperatures for single crystals, on the effect of small amounts of soluble impurities on the resoftening temperatures of single crystals, and studies of the effect of the size of the single crystal on resoftening would be valuable. Measurements of the internal friction as a function of the time for material having a decrement of about 10^{-4} on small single crystals of various sizes would also be of interest.

THE ENERGY STORED DURING WORK HARDENING

Experimentally it is found that if a metal is severely cold worked a certain amount of energy is stored in the metal. This energy can be released by heating the material. Taylor and Quinney[8] have made measurements on the amount of energy stored in this way. In this section we should like to see if we can check their results using dislocation theory.

We shall assume that a work hardened material contains many positive and negative dislocations which are arranged in a two-dimensional lattice. We shall first calculate the total energy of a dislocation pair in the rectangular lattice shown in Fig. 8. The elastic energy required to form a dislocation pair whose distance

[8] G. I. Taylor and H. Quinney, Proc. Roy. Soc. **143**, 307 (1934); **163**, 157 (1937).

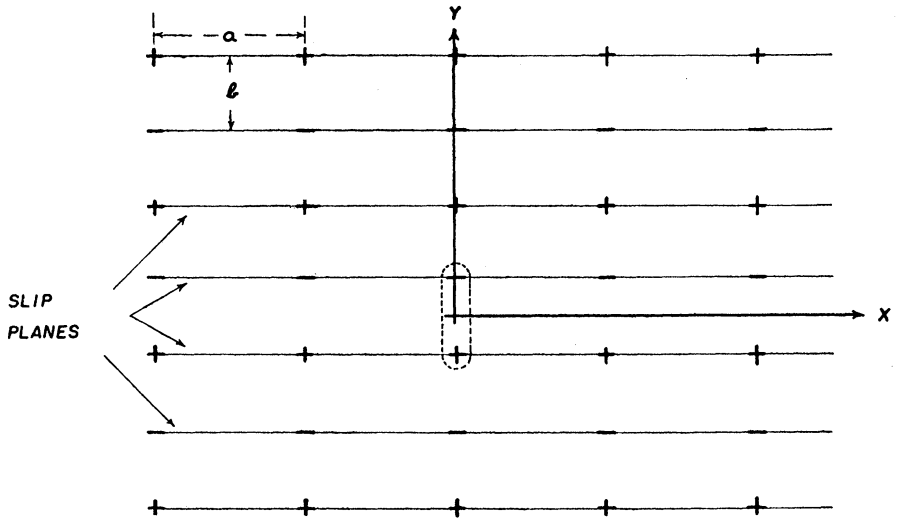

FIG. 8. Rectangular lattice and slip planes.

of separation is R is given by Eq. (18) as

$$W=\frac{mG\lambda^2}{2\pi(m-1)}\log\frac{R}{2r_0}$$
$$-\frac{m^2G\lambda^2}{4\pi(m-1)^2}[\cos 2\phi-\tfrac{1}{2}]. \quad (30)$$

The energy of a dislocation pair in a lattice is the energy required to form an isolated pair minus the binding energy of the pair in the lattice. The interaction of two dislocations is given by a logarithmic potential such as is shown above. The above relation includes not only the interaction between the two dislocations, but also the self-energy of the dislocations.

Let us calculate the interaction energy between the dislocation pair located at the origin and the pair located at $x=0$, $y=2nb$ (see Fig. 8). We shall assume that the members of each pair are separated by a distance b. The energy which the system acquires because of the presence of the positive dislocation at $0,-b/2$ in the field of the pair at 0, $2nb$ is

$$W_+=\frac{mG\lambda^2}{2\pi(m-1)}\log\left(\frac{2nb+b}{2nb}\right).$$

The energy which the system acquires because of the presence of the negative dislocation near the origin in the field of the pair at 0, $2nb$ is

$$W_-=\frac{mG\lambda^2}{2\pi(m-1)}\log\left(\frac{2nb-b}{2nb}\right).$$

Thus the interaction energy of the two pair of dislocations is

$$W_1=-\frac{mG\lambda^2}{2\pi(m-1)}\log\left(\frac{4n^2}{4n^2-1}\right). \quad (31)$$

It is evident that W_1 vanishes when n approaches infinity. W_1 is therefore an interaction energy, not a self-energy. Using this expression we can calculate the interaction energy between the pair at the origin and all other pairs in the plane $x=0$. We find

$$W_2=-\frac{mG\lambda^2}{\pi(m-1)}\sum_{n=1}^{\infty}\log\left(\frac{4n^2}{4n^2-1}\right). \quad (32)$$

We should like to calculate the interaction energy between the pair at the origin and the dislocations in the plane for which $x=+\tau a$. In making this calculation we shall use Eq. (30) to obtain the interaction energy of two dislocations. In using Eq. (30) for any two dislocations of the lattice we shall assume that ϕ remains constant although R may change. When the lattice of dislocations is considered we may there-

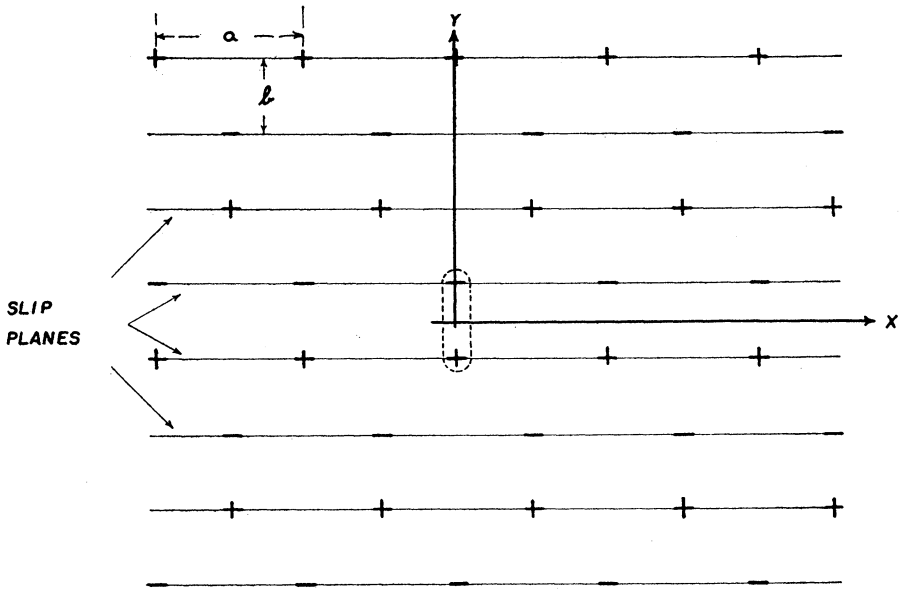

FIG. 9. Dislocations in rectangular lattice.

fore vary the distance of separation of the dislocations, but not the angles of the lattice. The resulting interaction potential has exactly the same form as the electrostatic interaction of two long, parallel, uniformly charged rods. Madelung[9] has calculated the potential due to a plane lattice of positively charged lines. Using his result we find that the interaction energy between the positive dislocation at $x=0$, $y=-\frac{1}{2}b$ and the positive dislocations in the plane at $x=+\tau a$ is

$$W_3=\frac{mG\lambda^2}{2\pi(m-1)}\sum_{l=1}^{\infty}\frac{e^{-\pi l\tau a/b}}{l}\cos\left\{\frac{\pi l(-\frac{1}{2}b+\frac{1}{2}b)}{b}\right\}-\frac{mG\lambda^2\tau a}{(m-1)b}+K,$$

where K is a constant. Similarly the interaction energy between the negative dislocations in the plane $x=+\tau a$ and the positive dislocation being considered is:

$$W_4=-\frac{mG\lambda^2}{2\pi(m-1)}\sum_{l=1}^{\infty}\frac{e^{-\pi l\tau a/b}}{l}\cos\left\{\frac{\pi l(-\frac{1}{2}b-\frac{1}{2}b)}{b}\right\}+\frac{mG\lambda^2\tau a}{(m-1)b}-K.$$

The total interaction energy of the positive dislocation with dislocations in the plane $x=\tau a$ is

$$W_5=\frac{mG\lambda^2}{2\pi(m-1)}\sum_{\substack{l=1\\ l\text{ is odd}}}^{\infty}\frac{2e^{-\pi l\tau a/b}}{l}. \tag{33}$$

With this result we find the total interaction energy of the positive dislocation with dislocations in all planes except the plane having $x=0$ is

$$W_6=\sum_{\tau=1}^{\infty}2W_5=\frac{2mG\lambda^2}{\pi(m-1)}\sum_{\tau=1}^{\infty}\sum_{\substack{l=1\\ l\text{ is odd}}}^{\infty}\frac{e^{-\pi l\tau a/b}}{l}. \tag{34}$$

The sum over l can be carried out and we find that Eq. (34) becomes

$$W_6=\frac{mG\lambda^2}{\pi(m-1)}\sum_{\tau=1}^{\infty}\log\left(\frac{1+e^{-\pi\tau a/b}}{1-e^{-\pi\tau a/b}}\right). \tag{35}$$

[9] E. Madelung, Physik. Zeits. **19**, 527 (1918).

TABLE I. *Data used in calculations.*

	NaCl	Cu	Al	Ni	Fe
Slip direction	(110)	(110)	(110)	(110)	(111)
Modulus of rigidity $\times 10^{-11}$ (dynes/cm^2)	1.872	4.53	2.665	7.69	8.114
Poisson's ratio	0.2053	0.340	0.343	0.309	0.280
r_0(A)	7.969	6.15	6.90	5.71	5.48
λ(A)	3.979	2.552	2.855	2.480	2.478
Density (g/cm^3)	2.165	8.93	2.699	8.80	7.865

TABLE II. *Results of calculations.*

	Cu	Al	Ni	Fe
Spacing (a) (Rectangular lattice)	34.72	32.38	35.75	29.99
Spacing (a) (Diagonal lattice)	29.24	27.26	30.18	25.03
Energy stored (cal./g) (Rectangular lattice)	0.518	1.013	0.851	0.963
Energy stored (cal./g) (Diagonal lattice)	0.507	1.130	0.832	0.942
Energy stored (cal./g) (Experimental value)	0.5	1.1	0.78	1.2

We now note that the interaction energy of the negative dislocation at $x=0$, $y=+\frac{1}{2}b$ with dislocations in all planes except the plane having $x=0$ is equal to the interaction energy of the positive dislocation as given in Eq. (35). Thus using Eqs. (30), (32), and (35) the total energy required to form a pair of dislocations in a rectangular lattice is found to be:

$$W_{\text{pair}}=\frac{mG\lambda^2}{2\pi(m-1)}\left[\log\frac{b}{2r_0}+\frac{3m}{4(m-1)}-2\sum_{n=1}^{\infty}\log\left(\frac{4n^2}{4n^2-1}\right)+4\sum_{r=1}^{\infty}\log\left(\frac{1+e^{-\pi ra/b}}{1-e^{-\pi ra/b}}\right)\right]. \tag{36}$$

This is not yet complete since we must add to this twice the energy required to distort the material inside a cylinder of radius r_0 around the center of a dislocation. This cannot be calculated from elastic theory. However, H. B. Huntington[10] has made a calculation of this energy using atomic concepts. He found the energy inside a cylinder of radius r_0 to be about 0.8 ev per atomic plane in the case of sodium chloride. We can estimate the value of this energy in other materials in the following way. It is assumed that the strain energy inside the cylinder of radius r_0 is proportional to the strain energy outside the cylinder in the case of an isolated dislocation. The constant of proportionality is taken to be the same for all materials and is determined by using Huntington's value for the energy inside the cylinder in the case of sodium chloride. For a dislocation at the center of a cylinder of material of radius 10^{-4} cm, there is about four times as much strain energy outside r_0 as there is inside.

It will be noticed that the rectangular lattice of dislocations is not stable. The lattice planes parallel to the $x=0$ plane repel one another and the lattice tends to spread out in the x direction. This is not a really serious difficulty for two reasons. First, there exists a small periodic potential of atomic origin which probably prevents dislocations from moving along the slip planes unless some external stress is present. Second, the actual lattice of dislocations cannot be very stable because resoftening takes place in some materials at room temperature. Extensive resoftening can be brought about in a single crystal of aluminum by heating it to 600°C. The lattice cannot decrease its energy by making b smaller because dislocations can only move along slip planes, not perpendicular to them. Thus b is determined by the way in which the dislocations are formed.

We have also made calculations for a lattice which is more stable than the rectangular lattice. The arrangement of dislocations in this lattice is shown in Fig. 9. The calculation of the energy required to produce a pair of dislocations in such a lattice is performed in exactly the same manner as the calculation for the rectangular lattice. The energy of a pair of dislocations in this case is

$$W_{\text{pair}}=\frac{mG\lambda^2}{2\pi(m-1)}\left[\log\frac{b}{2r_0}+\frac{3m}{4(m-1)}-2\sum_{\substack{n=2\\ n\text{ even}}}^{\infty}\log\left(\frac{4n^2}{4n^2-1}\right)+2\sum_{r=1}^{\infty}\log\left(\frac{1+e^{-\pi ra/b}}{1-e^{-\pi ra/b}}\right)-2\sum_{\substack{s=1\\ s\text{ odd}}}^{\infty}\log\left(\frac{1+e^{-\pi sa/4b}}{1-e^{-\pi sa/4b}}\right)\right]. \tag{37}$$

[10] H. B. Huntington, Phys. Rev. **59**, 942A (1941).

The above equation gives a pair energy which is least when a and b are small. A decrease in a is presumably prevented by the periodic potential mentioned above.

Calculations of the energy stored in various materials during severe work hardening have been made using Eqs. (36) and (37) together with values for the energy inside r_0. The experimental information used in these calculations is collected in Table I and the results are given in Table II. We have assumed in these calculations that a equals b. The calculations are made in two ways. First, one can determine that separation distance a which will give agreement with the experimental value of the energy stored. It is most convenient here to use as the unit of length the distance between nearest neighbors measured in the slip direction. The unit of length will thus be different for each substance considered. The results of these calculations are given in the first two rows of Table II. These results indicate that, in the most thoroughly work hardened state, dislocations are separated by about the same number of atomic spacings in different materials. A second way to make the calculation is to assume that dislocations are separated by the same number of atomic distances in different work hardened materials. The energy stored during work hardening can then be calculated and checked against the experimental value. The results of this calculation are given in the third and fourth rows of Table II. The experimental values are given in the fifth row of the table. The dislocations were assumed to be separated by 34 atomic distances in the rectangular lattice and 29 atomic distances in the diagonal lattice. These results are in remarkably good agreement with experiment except in the case of iron. The large error found for iron may be due to the fact that this metal has a different structure and a different set of slip planes. We have assumed that the axes of all line dislocations of the lattice are parallel. This is probably not the case in iron.

It might be well to point out that the results obtained with the rectangular dislocation lattice do not differ markedly from those obtained with the diagonal lattice. It is probable that neither of the lattices considered here exists in a work hardened material. We are, therefore, fortunate in finding that the results of the calculation do not depend much on the particular dislocation lattice used. The energy stored during work hardening is determined mainly by the elastic constants of the material, its density, and the value of the unit slip distance. We do not yet understand why dislocations are separated by the same number of atoms in different work hardened materials.

The separation distances of dislocations in severely work hardened materials are about 10^{-6} cm according to our calculations. For example, for copper in the case of the rectangular lattice $a=0.89\times10^{-6}$ cm; in the case of the diagonal lattice $a=0.74\times10^{-6}$ cm. This general result agrees with values for the density of dislocations in highly work hardened materials found by using other information.[11]

The author is much indebted to Professor F. Seitz for suggesting several of these problems and for many valuable discussions during the progress of the research.

[11] W. F. Brown Jr., Phys. Rev. **59**, 528 (1941); F. Seitz and T. A. Read, J. App. Phys. **12**, 178 (1941).