(Williams) for a foil of 0.157 g/cm² and 71 percent for a foil of 0.362 g/cm². The lead perchlorate solution gives approximately the same results as the polycrystalline lead foils.

The best agreement between theory and experiment is obtained for the case of the cadmium scatterer, an element in the middle of the periodic table.

In conclusion, we believe that it is possible to state that there is evidence that the scattering for light and heavy elements is appreciably less than the theoretical values, while the middle elements appear to be in good agreement with theory. Although we have no theoretical proposals to make concerning these discrepancies, it might seem reasonable to expect the best agreement for the middle elements, because here the number of orbital electrons is large enough to justify the use of a statistical treatment of their distribution and also the atomic number is not so large that there is doubt as to the validity of the Born approximation.

Our attempts to detect an effect of crystal structure gave negative results within our rather broad limits of error. This, of course, does not preclude the possibility that a small effect exists.

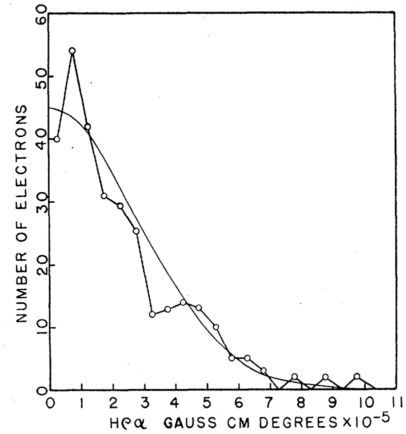

FIG. 7. Scattering of electrons by a solution of lead perchlorate.

We wish to thank Professor Goudsmit for his continued interest and for the opportunity to discuss with him various aspects of the problem. This work was made possible by a grant from the Horace H. Rackham Fund.

SEPTEMBER 1, 1941 PHYSICAL REVIEW VOLUME 60

Theory of the Magnetron. I

LÉON BRILLOUIN
University of Wisconsin, Madison, Wisconsin

(Received June 4, 1941)

A complete calculation of space charge and field repartition is given for a magnetron working under steady conditions. Electrons leaving the filament gradually acquire an angular velocity, and for distances greater than a certain length L, these electrons describe spirals around the filament. This very important length L is defined by

$$L^2 = -eI/m\omega_H^3.$$

I = current per unit of length of the filament, ω_H = Larmor's angular velocity. Under critical conditions, that is, when the magnetic field is just high enough to cut the anodic current I, the electron cloud rotates about the filament almost as a solid body with an angular velocity ω_H. A study of small oscillations with cylindrical symmetry shows that these oscillations have a proper frequency $\sqrt{2}\omega_H$, and that the magnetron is able to yield an internal ***negative resistance*** for certain frequency bands near $\sqrt{2}\omega_H$; this explains how a magnetron with one cylindrical anode can sustain continuous oscillations in an electric circuit.

1. INTRODUCTION. GENERAL OBSERVATIONS ON THE PART PLAYED BY THE MAGNETIC FIELD—LARMOR'S THEOREM

FOLLOWING Hull's original work on magnetrons, a large number of theories have been formulated; the majority, however, appear inadequate or inexact. Certain authors considered the electronic motion while neglecting the space charge; others used the space charge computed by Langmuir for a diode without a magnetic field. Such studies necessarily are highly inaccurate, since a powerful magnetic field considerably modifies the electronic paths and, consequently, the space charge. Contrariwise, authors who have endeavored to evaluate the space charge in the presence of a magnetic field do not appear to have provided practicable solutions. It therefore appears necessary to consider the problem afresh and to compute directly both the space charge and the potential distributions in the magnetron; without these essential data, the development of a coherent theory is impossible.

As a preliminary study to the theory of the magnetron it is interesting to state some general results about electronic paths in an electrical field repartition with central symmetry, when a magnetic field is acting on the system. Let us assume an electrical device where the potential $V(r)$ is a function of the distance r to the z axis, while the magnetic field H lies parallel to the same axis. The equations of motion follow:

$$\begin{aligned} m\ddot{x} &= -e\frac{\partial V}{\partial x}+\mu_0 e v_y H = -e\frac{\partial V}{\partial r}\frac{x}{r}+\mu_0 e\dot{y}H, \\ m\ddot{y} &= -e\frac{\partial V}{\partial y}-\mu_0 e v_x H = -e\frac{\partial V}{\partial r}\frac{y}{r}-\mu_0 e\dot{x}H, \end{aligned} \tag{1}$$

where μ_0 is the magnetic permeability in vacuum. By multiplying the first equation by $-y$ and the second by x, adding and integrating, the integral of the moment of momentum is obtained:

$$\mathfrak{J}=m(x\dot{y}-y\dot{x})+\tfrac{1}{2}\mu_0 eHr^2=c^{te}. \tag{2}$$

Let us introduce cylindrical coordinates r, θ, z around the z axis

$$\mathfrak{J}/m=r^2\dot{\theta}+\tfrac{1}{2}\mu_0(e/m)Hr^2=C;$$

hence

$$\dot{\theta}=\omega_H+C/r^2, \tag{3}$$

where

$$\omega_H=-\tfrac{1}{2}\mu_0(e/m)H \tag{4}$$

represents Larmor's angular velocity. To determine the orders of magnitude, if H be measured in gauss while all the equations are written in e.s.c.g.s. units, it is necessary to take:

$$\mu_0=c^{-2}, \quad H_{\text{e.s.c.g.s.}}=cH_{\text{gauss}},$$

c = velocity of light; on the other hand, $-e/m$

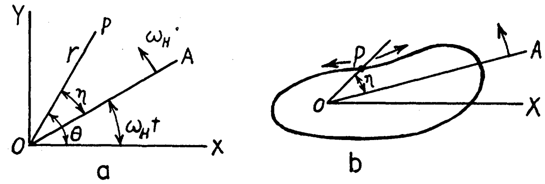

FIG. 1.

$=+5.3\times10^{17}$ e.s.c.g.s. since e is negative; then

$$\omega_H=0.884\times10^7 H_{\text{gauss}}, \tag{5}$$

which, for a field of 500 gauss, gives an angular velocity of 4.42×10^9.

If we revert to the equation of electron movement in polar coordinates r and θ, for θ, the simple result of (3) has been derived; for r,

$$m\ddot{r}=-e\frac{\partial V}{\partial r}+\mu_0 eHr\dot{\theta}+mr\dot{\theta}^2, \tag{6}$$

where the electric, Lorentz, and centrifugal forces will be recognized. If we replace $\mu_0 eH$ by the equivalent expression, $-2m\omega_H$, it is found that

$$\ddot{r}=-\frac{e}{m}\frac{\partial V}{\partial r}-2\omega_H\dot{\theta}r+r\dot{\theta}^2, \tag{7}$$

a result which may be expressed in this way:

$$\frac{\ddot{r}}{r}=-\frac{e}{mr}\frac{\partial V}{\partial r}+(\dot{\theta}-\omega_H)^2-\omega_H^2. \tag{8}$$

Comparing Eqs. (3) and (8) we notice that the angular velocity comes in both formulae with the difference $\dot{\theta}-\omega_H$ which justifies the following treatment: In the xy plane, let us use a rotating axis OA, the angular velocity of which will be equal to ω_H; then the angle η between OP and OA as shown in Fig. 1(a) is

$$\begin{aligned} \eta &= \theta-\omega_H t, \\ \dot{\eta} &= \dot{\theta}-\omega_H, \end{aligned} \tag{9}$$

and Eqs. (3) and (8) yield

$$\dot{\eta}=\frac{C}{r^2}, \quad \frac{\ddot{r}}{r}=-\frac{e}{mr}\frac{\partial V}{\partial r}+\dot{\eta}^2-\omega_H^2. \tag{10}$$

The second equation contains only $\dot{\eta}^2$, which means that two opposite values, $\pm C$, giving opposite angular velocities $\pm\dot{\eta}$ for each value of r

will yield similar orbits in the r, η plane. The general solution is then represented by one orbit, rotated around the center O, Fig. 1(b), with Larmor's precession ω_H, and by allowing the electron to run on this orbit (in the rotating r, η plane) with opposite velocity. This is the most general statement for Larmor's theorem. The usual case considered in atomic structures refers to problems in which the angular velocity $\dot{\eta}$ on the orbit is very much greater than Larmor's ω_H, so that the ω_H^2 term in Eq. (10) can be neglected; the rotating orbits are then similar to the unperturbed orbit obtained when the magnetic field is zero.

In the problems connected with the theory of the magnetron, such an approximation will not be allowed, and it will prove necessary to take care of the ω_H^2 term. Let us, for instance, suppose that we want to study the motions of electrons inside of a cloud of constant density, ρ. The potential distribution V is given by

$$\Delta V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -4\pi\rho \text{ e.s.c.g.s.}, \qquad (11)$$

$$V = -\pi\rho r^2,$$

where V and $\partial V/\partial r$ are zero for $r=0$. Equations (10) yield

$$\dot{\eta} = C/r^2, \quad \ddot{r}/r = (e/m)2\pi\rho + \dot{\eta}^2 - \omega_H^2. \quad (12)$$

The first condition gives the angular velocity as a function of r on *each orbit*, but different orbits in the cloud may correspond to different values of C, according to the initial velocity distributions for each special problem.

For instance, we may suppose circular orbits, each of them corresponding to a certain C value. Such circular orbits will keep the electronic density ρ of the cloud constant; $\ddot{r}$ being zero, the angular velocity is given by

$$\begin{aligned} \dot{\eta} &= \pm\left(\omega_H^2 - \frac{e}{m}2\pi\rho\right)^{\frac{1}{2}}, \\ \dot{\theta} &= \omega_H + \dot{\eta} = \omega_H \pm \left(\omega_H^2 - \frac{e}{m}2\pi\rho\right)^{\frac{1}{2}}. \end{aligned} \qquad (13)$$

In each of the two $\pm$ solutions the angular velocity is constant, independent of r, and the electronic cloud is rotating as a solid body around the Oz axis.

A constant linear velocity along the z axis may be added without any change in the preceding solutions, thus giving a possible type of electronic motions in the form of "magneto-cathodic rays."

At the limit $\rho=0$ one finds the well-known result that free electrons in a magnetic field may either stay at rest ($\dot{\theta}=0$, sign $-$) or describe circular orbits with an angular velocity twice the Larmor rotation ($\dot{\theta}=2\omega_H$, sign $+$).

Stable electronic clouds in magnetic fields are possible when their density ρ is smaller than a certain maximum value

$$0 \leqslant \rho \leqslant \rho_m, \quad \rho_m = (m/2\pi e)\omega_H^2. \qquad (14)$$

If a cloud of higher uniform density had been artificially created, it would expand according to Eq. (12) ($\ddot{r}>0$), until it reached the limit ρ_m. Clouds of density smaller than ρ_m may be stable or expand or contract to the limit ρ_m, the type of solution depending on the initial conditions of the cloud.

It will be found that electronic clouds of such types play a very important role in the theory of the magnetron.

2. Cylindrical Magnetron—Static Case

The magnetron is assumed to consist of a filament of radius a and a cylindrical anode of radius b, the magnetic field H accurately paralleling the filament (axis Oz). It is assumed that electrons without appreciable speed are emitted from the filament, and that the electric field on the filament is zero, provided the anode current is below saturation; hypotheses which have both been generally accepted since they were formulated by Langmuir. Distribution of the cylindrical space charge can thus be obtained; the potential V is solely a function of r. The equations of motion are Eqs. (1), (2), (3), and (4) of the preceding section.

The static case, which will first be considered, is characterized by the fact that V does not depend on time.

The constant C of (3) is determined by the fact that electrons are emitted without speed from the filament, so that $\dot{\theta}$ is zero for $r=a$:

$$\dot{\theta} = \omega_H(1 - a^2/r^2). \qquad (15)$$

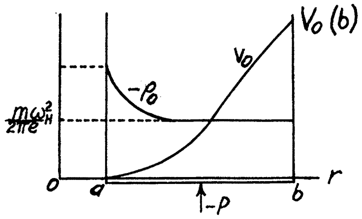

FIG. 2. Critical conditions, $I=0$.

The Lorentz force, due to the magnetic field H, does not influence the velocity since it is perpendicular to the latter. Energy conservation accordingly may be written:

$$\tfrac{1}{2}mv^2+eV=\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+eV=C'$$

and this new constant is zero if

$$V(a)=0 \text{ on the filament;} \tag{16}$$

hence

$$\dot{r}^2+\omega_H^2r^2(1-a^2/r^2)^2+(2e/m)V=0. \tag{17}$$

The equations of movement, therefore, may be integrated directly from (15) and (17) without determining the potential distribution as a function of r.

In (17) one result is at once evident: since the velocity of rotation $\dot{\theta}$ is determined by the magnetic field, the kinetic energy at the distance r cannot be less than:

$$\frac{m}{2}r^2\dot{\theta}^2=\frac{m}{2}\omega_H^2r^2\left(1-\frac{a^2}{r^2}\right)^2.$$

Hence, if the potential $V(r)$ does not suffice to give the electron greater kinetic energy, the radial velocity $\dot{r}$ is annulled and the current is interrupted. Consequently

$$\dot{r}=0, \quad -(2e/m)V_0=\omega_H^2r^2(1-a^2/r^2)^2. \tag{18}$$

This is Hull's *limiting value of the potential* V_0 at the distance r, where the anode current is just cut off by the magnetic field. This limiting value $V_0(r)$ is thus defined without the necessity of determining the distribution of the space charge or the potentials between the anode and cathode —a remarkable fact which discloses the possibility of finding the correct value of the critical potential $V_0(r)$ without considering the space charge.

Let us now come back to Eq. (7) in polar coordinates. Making use of Eq. (15) we find

$$\begin{aligned} m\ddot{r}&=-e\partial V/\partial r-mr\dot{\theta}(2\omega_H-\dot{\theta})\\ &=-e\partial V/\partial r-m\omega_H^2r(1-a^4/r^4)\\ &=-(\partial/\partial r)[eV+\tfrac{1}{2}m\omega_H^2(r^2-a^4/r^2)]. \end{aligned} \tag{19}$$

The radial acceleration $\ddot{r}$ is accordingly governed by an apparent potential function $P(r)$:

$$\begin{aligned} m\ddot{r}&=-e(\partial P/\partial r),\\ P&=V-V_0=V+(m/2e)\omega_H^2(r-a^2/r)^2. \end{aligned} \tag{20}$$

In the function $P(r)$ a constant $-m\omega_H^2a^2$ has been added in order to reduce $P(a)$ to zero on the filament $r=a$; thus the same function is made to appear as in the energy equation (17). A fact worthy of emphasis is the following:

With the apparent potential $P(r)$, the radial movement of the electron may be studied by means of Eq. (20) without considering the rotation ω around the filament. These general results are valid in all cases either with or without space charge.

3. Static Space Charge: Critical Potential

To obtain a potential distribution in a static state, it is necessary to introduce the space charge $\rho(r)$, a function of the radius r and independent of time. From the cylindrical symmetry of the system we have:

$$\Delta V=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)=-4\pi\rho, \quad I=2\pi r\rho v_r. \tag{21}$$

The current I, per unit length of filament, is a constant independent of r; v_r is taken from (17) and the following is obtained:

$$\begin{aligned} \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)&=-\frac{2I}{v_r}\\ &=-\frac{2I}{\left[-\dfrac{2e}{m}V-\omega_H^2\left(r-\dfrac{a^2}{r}\right)^2\right]^{\frac{1}{2}}}. \end{aligned} \tag{22}$$

The conditions are the following:

$$V(a)=0 \text{ on the filament.}$$

$$(\partial V/\partial r)_{r=a}=0 \text{ no saturation.}$$

Since the non-linear Eq. (22) would require detailed discussion, it is preferable to start with the simpler cases where *the current I is zero*. Two methods are available, according to Eq. (21):

(A) $$\rho\neq 0,\quad v_r=0,$$

which yields

$$V_0(r)=-\frac{m}{2e}\omega_H{}^2\left(r-\frac{a^2}{r}\right)^2,\quad P(r)=0. \qquad (23)$$

Expression (23) completely satisfies the two limiting conditions of (22).

(B) $$\rho=0,\quad v_r\neq 0;$$

hence,

$$V=B\log r+C, \qquad (24)$$

which corresponds to a definitely electrostatic potential without space charge.

Let us now examine the distributions of the charges and potential in a magnetron when the potential of the anode ($r=b$) is rather low so that no current flows.

The limiting case corresponding to the maximum possible potential on the plate, with no current flowing, is obtained by taking

$$V_0(b)=-(m/2e)\omega_H{}^2(b-a^2/b)^2 \qquad (18)$$

as shown in preceding section, Eq. (18); *it is the critical potential.* The density is not zero, and space charges are present throughout the medium between the filament and anode; Eq. (21) gives their value:

$$\rho_0(r)=(m\omega_H{}^2/2\pi e)(1+a^4/r^4). \qquad (25)$$

The radial velocity is zero throughout; the electrons follow circular trajectories, centered on the filament. The Lorentz and centrifugal forces are in exact equilibrium with the electrostatic force in Eq. (19), as is also evident from (20), since the distribution (18), (25) completely annuls $P(r)$.

In the case of a potential $V_0(b)$ below the critical potential, there is obtained a charge distribution (25) extending from the filament ($r=a$) up to a certain cylinder $r=b'$. This particular form of distribution then stops abruptly; the potential distribution continues in the form of a logarithmic field at the cylinder b', and the two constants B and C may be determined by the conditions:

$$-\frac{m}{2e}\omega_H{}^2\left(b'-\frac{a^2}{b'}\right)^2=B\log b'+C$$

$$-\frac{m}{e}\omega_H{}^2\left(b'-\frac{a^4}{b'^3}\right)=\frac{B}{b'}. \qquad (26)$$

When the potential $V(b)$ of the anode is progressively lowered, the space charge is correspondingly restricted to a decreasing cylinder b' around the filament; and, when $V(b)$ is zero, the space charge disappears.

Figure 2 illustrates the case of critical potential $V_0(b)$; the different curves represent the space charge density ρ, the electrostatic potential $V(r)$ and the apparent radial potential $P(r)$.

Figure 3 shows the potential distribution, $V(r)$, for anode potentials lower than the critical value. The apparent potential energy of the electron is eP or $-\epsilon P$, ϵ designating the absolute value of an electron charge. Hence,

$$P=V-V_0$$

according to (20). The behavior of this function is easily described and is important to recognize inasmuch as the apparent energy P governs the electronic radial movements.

In the critical state (Fig. 2), $-P$ is identically zero. The electrons of the cloud forming the space charge have no radial velocity.

When the magnetron is below its critical state (Fig. 3), the apparent potential $-P$ rises between b' and b in front of the anode. The critical distribution shown in Fig. 1 seems to be that described by Hull[1] in a brief note, for which he claims to have found good experimental evidence from space charge density measurements.

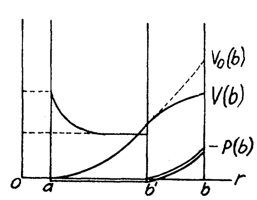

FIG. 3. Potential distribution, $V(r)$ for anode potentials lower than the critical value.

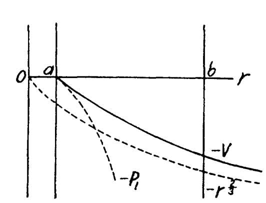

FIG. 4. Values from Langmuir's solution for the diode.

[1] A. W. Hull, Phys. Rev. **23**, 112A (1924).

4. Direct-Current Condition without Saturation

Since the fundamental equation (22) has just been considered for the various cases where the current is nil, it is now desirable to determine the direct-current conditions. Equation (22) will be transcribed, with the unknown function taken as the apparent potential $P(r)$ defined in (20) instead of the electrostatic potential $V(r)$.

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)=\frac{\partial}{\partial r}r\frac{\partial}{\partial r}(P+V_0)=-I\left(-\frac{2m}{e}\right)^{\frac{1}{2}}\frac{1}{P^{\frac{1}{2}}}.$$

If we replace V_0 by its value (20) we have

$$P^{\frac{1}{2}}\left[-\frac{2m\omega_H{}^2}{e}\left(r+\frac{a^4}{r^3}\right)+\frac{\partial}{\partial r}r\frac{\partial P}{\partial r}\right]=-I\left(-\frac{2m}{e}\right)^{\frac{1}{2}}. \quad (27)$$

This equation is rigorous. The solution $P(r)$ must be such that conditions (22) will be satisfied on the filament.

$$P(a)=0, \quad \partial P/\partial r=0, \quad r=a$$

on the filament (27).

The solution of (27) can only be derived approximately; two distinct regions must be considered as limiting cases:

A. Proximity to filament

The second member remains constant and, on the filament, P is nil; the term in brackets must, therefore, be infinite. The required solution is of the type:

$$P=A(r-a)^n,$$

which gives

$$A^{\frac{1}{2}}(r-a)^{n/2}\left[-\frac{2m\omega_H{}^2}{e}2a+nA(r-a)^{n-1}+aAn(n-1)(r-a)^{n-2}\right]=-I\left(-\frac{2m}{e}\right)^{\frac{1}{2}}.$$

If we take $n=4/3$, the two first terms are zero; in view of the factor $(r-a)^{n/2}$ the last term remains finite and it is found that:

$$P=\tfrac{1}{2}\left(-\frac{m}{e}\right)^{\frac{1}{3}}\left(\frac{9I}{a}(r-a)^2\right)^{\frac{2}{3}} \quad r-a\ll 1. \quad (28)$$

This solution is valid only in the immediate vicinity of the cathode, while $r-a$ remains very small. At increasing distances from the cathode, account must be taken of the fact that $\partial P/\partial r$ is no longer zero; and the magnetic term in ω_H must be dealt with. Let us first determine what happens at a short distance from the filament where the magnetic term still remains very small; the problem of the ordinary diode is re-encountered.

The diode without a magnetic field has been dealt with by Langmuir;[2] the potentials V and P are identical and Eq. (27) reduces to

$$P^{\frac{1}{2}}\frac{\partial}{\partial r}r\frac{\partial P}{\partial r}=-I\left(-\frac{2m}{e}\right)^{\frac{1}{2}}. \quad (29)$$

For $r\gg a$, far from the filament, a solution is found in $r^{\frac{2}{3}}$ and the complete solution may be written:

$$P_L=\tfrac{1}{2}(-m/e)^{\frac{1}{3}}(9I\beta^2 r)^{\frac{2}{3}}, \quad (30)$$

where β^2 is a function of r/a, calculated by Langmuir:

r/a	1	1.25	1.5	1.75	2	2.5	3
β^2	0	0.045	0.116	0.2	0.275	0.405	0.512

4	5	6	7	8	9	10	15	∞
0.665	0.775	0.818	0.867	0.902	0.925	0.94	0.978	1

Equation (30) presents the same structure as (28) and correspondence is thus established in the vicinity of the filament. The potential P is obtained by replacing $\beta^2 r$ by $(r-a)^2/a$. With r close to a:

$$\beta^2\approx(r-a)^2/ar. \quad (31)$$

Comparison with (30) shows that this approximation is valid only up to $r=1.25a$; that is, in the immediate vicinity of the filament. This indication gives the limit of validity of (28), which should be replaced by Langmuir's Eq. (30) when distances beyond the immediate vicinity of the cathode are involved.

Figure 4 shows the shapes of the various curves.

[2] I. Langmuir, Phys. Rev. **2**, 458 (1913). An approximate solution may be found:

$$P=\tfrac{1}{2}(9I)^{2/3}\left(-\frac{m}{e}\right)^{1/3}\frac{(r-a)^{4/3}}{r^{2/3}}$$

it yields correct results for very small or very large r, the error is about 20 percent near $r=4a$.

B. Distant from filament

At a considerable distance from the filament the Langmuir equation gives low values for the term $(\partial/\partial r)[r(\partial P/\partial r)]$. Contrariwise, the magnetic term within the brackets of (27) increases indefinitely. A moment therefore arrives when this first term becomes preponderant. At the limit, the second term may be neglected and the first term kept. An asymptotic solution P_ω, valid at a great distance, is:

$$r \gg a,$$

$$P_\omega(r) = -I^2 \frac{e}{2m\omega_H{}^4}\left(r + \frac{a^4}{r^3}\right)^{-2} \approx -I^2 \frac{e}{2m\omega_H{}^4 r^2}. \quad (32)$$

It is now necessary to discuss how to join these two extreme cases with a view to ascertaining the corresponding values of r and the form of function $P(r)$ obtained.

By comparing the equations, an expression $-eI/m\omega_H{}^3$ appears, homogeneous to a certain length L squared. The following gives an indication of the order of magnitude:

$-e/m = 5.3 \times 10^{17}$ (e.s.c.g.s.),
$I = 3 \times 10^6 J$ (J in milliamperes; I in e.s.c.g.s. units).

If we calculate ω_H according to (5) and express H in gauss:

$$L^2 = -eI/m\omega_H{}^3 = 2.3 \times 10^3 J/H^3. \quad (33)$$

The role played by the characteristic length L is very important and has now to be discussed.

First, the condition stated at the beginning of Section 3 must be satisfied in order to justify use of the solution (32); this necessitates:

$$\frac{\partial}{\partial r}\left(r\frac{\partial P_\omega}{\partial r}\right) = -\frac{2I^2 e}{m\omega_H{}^4 r^3} \ll -\frac{2m}{e}\omega_H{}^2 r.$$

By neglecting a, this condition reduces to

$$r \gg L.$$

For weak fields, which scarcely disturb the diode, the length L is much greater than the dimensions of the bulb; if the field increases, however, the length L decreases considerably to the order of magnitude of the interior dimension of the bulb and may even drop to very low values; for some hundreds of gauss, L is of the order of some tenths of a mm.

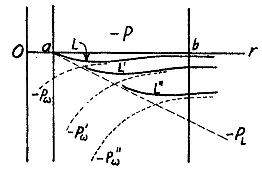

FIG. 5. Value of the apparent potential P_ω for various values of L.

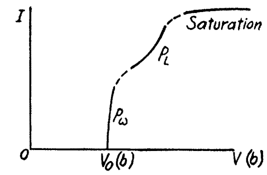

FIG. 6. Magnetron characteristic.

The length L then gives the magnitude of the distance where the Langmuir solution P_L (always valid in the immediate vicinity of the filament) has to rejoin with the solution P_ω which is valid at a great distance.

As a rough criterion of the distance of coincidence, we take the condition:

$$P_L(r) = P_\omega(r),$$

then

$$\tfrac{1}{2}(9I\beta^2 r)^{\frac{2}{3}}(-m/e)^{\frac{1}{3}} = -I^2 e/2m\omega_H{}^4 r^2; \quad (34)$$

hence

$$9\beta^2 r^4 = L^4.$$

The curve representing the apparent potential $P(r)$ is consequently known by the expressions P_L (for $r \ll L$) and P_ω (for $r \gg L$); it can, in any case, be plotted approximately. Figure 5 shows curves for various values of L between the cathode ($r = a$) and the anode ($r = b$).

If the anode radius b is appreciably greater than the length L, the formula $P_\omega(R)$ of (32) gives the anode voltage, calculated from the critical voltage $V_0(R)$; if, however, the length L is very great, the magnetic field being weak, Langmuir's curve P_L is used. Figure 6, accordingly, shows the magnetron characteristic; the internal resistance R is zero in the vicinity of the critical point.

Reverting to the curves of Fig. 5, it should be noted that the electrons, after leaving the filament ($r = a$), first reach the Langmuir region and then travel toward $r = L$ through a region of low apparent potential energy, thereafter rebounding (approaching the anode) toward higher potential energies only slightly below the potential energy of the cathode. The expression $-P(r)$ represents, but for the factor $\epsilon = |e|$, the apparent potential energy of the electrons, controlling their radial movements (Eq. (20)). The speed of rotation around the filament is defined by (15). The curves $-P(r)$ show that the electrons are accelerated from a to L and retarded from L to b. The space charge thus is increased in the second region; close to the filament, Langmuir's space charge applies, and, at a great distance, the constant space charge of (25) applies approximately

$$r \gg L, \quad \rho \approx m\omega_H^2/2\pi e. \tag{35}$$

This space charge density far away from the filament is just the density we have found in Eq. (14) as being the greatest possible density of electrons in a given magnetic field. A more detailed study would require calculation of the junction of the curves in the region L. The following, however, is pertinent:

The electronic paths start radially from the filament; they are slightly curved in the region $r=L$ and then turn around the filament for $r\gg L$, as shown in Fig. 7. This can be understood readily from the following considerations: Let V_r and V_θ be the radial and rotational components of electronic velocity; Eqs. (15) and (21) yield:

$$\frac{V_\theta}{V_r}=\frac{r\dot{\theta}}{I}2\pi r\rho=\frac{2\pi r^2\rho}{I}\omega_H\left(1-\frac{a^2}{r^2}\right).$$

Instead of computing the ratio V_θ/V_r, let us calculate

$$\frac{\rho V_r}{\rho_0 V_\theta}=\frac{2\pi r^2\rho_0}{I}=\frac{r^2 m\omega_H{}^3}{eI}\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{a^4}{r^4}\right), \tag{36}$$

where ρ_0 is the space charge density (25) in the region $r\gg L$. If the radius a of the cathode is neglected, as a first approximation, it is seen that the ratio $\rho V_r/\rho_0 V_\theta$ equals -1 when $r=L$, which proves that V_r and V_θ are of the same order of magnitude at a distance L from the filament. It appears from this discussion that the junction between the two asymptotic solutions A (near the filament) and B (far away from it) has to take place about the distance L from the filament. Series of approximations starting from both sides can be obtained and have already been published in a former paper.[3]

[3] L. Brillouin, "Theory of the magnetron," Elec. Commun. **20**, No. 1 (1941).

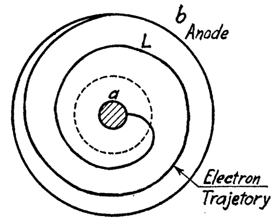

FIG. 7. Electronic paths from filament to anode.

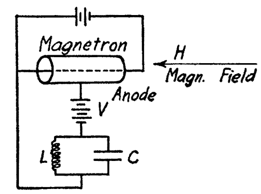

FIG. 8. Oscillator circuit.

5. Oscillations in a Magnetron with Cylindrical Anode: Some General Remarks

Different types of magnetrons are practically used for generating electromagnetic oscillations; some of them possess one anode, of cylindrical shape. Oscillations of large amplitude are often obtained from these devices, but large oscillations are rather difficult to discuss theoretically and the calculations will be given only for the case of small oscillations, which do not introduce too great a perturbation in the original charge density distribution.

The theoretical discussion will be conducted with the following approximations: The formula giving the electric current has to be completed with an additional term, taking account of Maxwell's displacement current. In more general problems, one should also take account of the magnetic fields induced by the current flowing inside the magnetron, but we shall assume the cylindrical symmetry to be obeyed for all quantities, in which case there is no induced magnetic field for a magnetron of infinite length. Magnetic fields would appear only outside of a magnetron of finite length and result in radiation of electromagnetic waves from the magnetron. This is, of course, a very small and secondary effect, causing some damping to the oscillations in the magnetron.

Equations of Sections 1 and 2 will thus be modified in the following way: The charge density ρ and the radial electric field E will be considered as functions of r and t, an assumption which secures cylindrical symmetry at any time. Equation (11) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rE)=4\pi\rho, \quad E=-\frac{\partial V}{\partial r}. \tag{37}$$

The current I, including displacement current, will be written

$$I=2\pi r\left(\rho v_r+\frac{1}{4\pi}\frac{\partial E}{\partial t}\right)=2\pi\rho r\dot{r}+\frac{r}{2}\frac{\partial E}{\partial t}. \quad (38)$$

The dielectric constant ϵ_0 in vacuum, has been taken 1 in e.s.c.g.s. units. As is well known from general proofs, introduction of the displacement term secures the constancy of I along the whole circuit, which means here that I is independent of the distance r, and a function only of t. All other formulae from Section 2 still apply, and especially Eqs. (15), (19), and (20). Time derivatives may be taken either at a given place, or following the motion of an electron; these two definitions will be distinguished this way:

$\partial/\partial t$	derivative at r constant or ordinary partial derivative,	
		(39)
$d/dt=\partial/\partial t+v_r\partial/\partial r$	following the motion of an electron moving with the velocity v_r, v_θ.	

All quantities being independent of θ, because of cylindrical symmetry, a term in $v_\theta\partial/\partial\theta$ has been omitted in formula (39). Equation of motion (20), for instance, has to be written

$$d^2r/dt^2=-e\partial P/\partial r. \quad (40)$$

A well-known integration method, which has proved very useful in most electronic problems, was omitted on purpose in the preceding sections for the reason that it leads to some difficulties which will be discussed now. The method consists in taking the derivative d/dt of the product rE, following the motion of an electron:

$$\frac{d}{dt}(rE)=\frac{\partial}{\partial t}(rE)+\dot{r}\frac{\partial}{\partial r}(rE)$$
$$=r\frac{\partial E}{\partial t}+\dot{r}4\pi\rho r=2I \quad (41)$$

and in using Eqs. (37) and (38). As noticed before, $I(t)$ is independent of r, which allows easy integration:

$$rE(r,t)=2\int_{t_0}^{t}I(t)dt. \quad (42)$$

t_0 is the time at which one given electron has left the filament, and t the instant when it reaches the distance r.

Let us now show the difficulties involved in the use of this formula, by applying it to the static case. E is independent of t; hence

$$E(r)=2I\tau/r, \quad (43)$$

where $\tau=t-t_0$ is the transit time for an electron going from the filament to the distance r. Suppose the magnetic field to be such as to cut the anodic current, according to condition (18); I is naught, but electrons in this case are moving in circular orbits about the filament, with no radial velocity at all; hence the transit time τ is infinite, while E keeps a finite value. Many theoreticians attempting to use (43) without a former knowledge of the solution were led to entirely wrong conclusions.

6. Small Oscillations in a Magnetron with Cylindrical Anode

After the magnetron has been studied in its static condition, without oscillations, all we need do now is to superimpose small oscillations on the static quantities. The amplitude of the oscillations will be specified by a factor ϵ; static continuous quantities are characterized by a subscript c, while alternating quantities will be given a subscript a.

$V=V_c(r)+\epsilon V_a(r,t)$	electric potential,	
$I=I_c+\epsilon I_a(t)$	electric current independent of r,	(44)
$r=r_c(t)+\epsilon r_a(t)$	position of the electron at the time t.	

Because of the supposed smallness of ϵ, all terms in ϵ^2, $\epsilon^3\cdots$ will be neglected. Let us now rewrite the most important formulae, starting with (20):

$$\frac{d^2r_c}{dt^2}+\epsilon\frac{d^2r_a}{dt^2}=\frac{e}{m}\left(\frac{\partial V_0}{\partial r}-\frac{\partial V_c}{\partial r}\right)-\epsilon\frac{e}{m}\frac{\partial V_a}{\partial r}. \quad (45)$$

Potentials V_0 and V_c have to be taken at the point $r_c+\epsilon r_a$ where the electron is located; hence the expansion $V_c(r)=V_c(r_c)+\epsilon r_a[\partial V_c(r_c)/\partial r]$. Constant terms and oscillating terms are easily separated, and yield the following relations:

$$\frac{d^2r_c}{dt^2}=\frac{e}{m}\left(\frac{\partial V_0(r_c)}{\partial r}-\frac{\partial V_c(r_c)}{\partial r}\right), \tag{46}$$

$$\frac{d^2r_a}{dt^2}=\frac{e}{m}r_a\left(\frac{\partial^2 V_0}{\partial r^2}-\frac{\partial^2 V_c}{\partial r^2}\right)-\frac{e}{m}\frac{\partial V_a}{\partial r}. \tag{47}$$

Treating Eq. (42) in the same way, one gets

$$rE=(r_c+\epsilon r_a)\left(-\frac{\partial V_c}{\partial r}-\epsilon r_a\frac{\partial^2 V_c}{\partial r^2}-\epsilon\frac{\partial V_a}{\partial r}\right)$$
$$=2\int_{t_0}^{t}(I_c+\epsilon I_a)dt;$$

hence two relations are derived

$$r_c\frac{\partial V_c}{\partial r}=-2I_c(t-t_0), \tag{43}$$

$$r_a\frac{\partial V_c}{\partial r}+r_cr_a\frac{\partial^2 V_c}{\partial r^2}+r_c\frac{\partial V_a}{\partial r}=-2\int_{t_0}^{t}I_adt. \tag{48}$$

Equations (46) and (43) between static quantities have already been obtained in the preceding sections and bring nothing new; this is the result of our hypothesis of infinitely small oscillations. If oscillations of greater amplitude should be considered, terms in ϵ^2, $\epsilon^3\cdots$ could not be neglected any more, and would appear in the static formulae (corresponding to detection phenomena) thus perturbing the static regimen.

Equations (47) and (48) rule the magnetronic oscillations and are linear with respect to the unknown quantities r_a and V_a. Eliminating V_a one easily gets

$$\frac{d^2r_a}{dt^2}-\frac{e}{m}r_a\left[\frac{\partial^2 V_0}{\partial r^2}+\frac{1}{r_c}\frac{\partial V_c}{\partial r}\right]=\frac{2e}{mr_c}\int_{t_0}^{t}I_adt, \tag{49}$$

where V_0 and V_c result from the study of the static case.

In order to know how the magnetron is able to react on the oscillating circuit connected to it, we need to calculate the internal impedance of the magnetron; we thus want to find the relation between the alternating current $I_a(t)$ flowing through the magnetron, and the alternating potential V_a on its anode. Let us suppose $I_a(t)$ to be given, then Eq. (49) is a linear differential equation with a right-hand term. As is the rule for such equations, we have to study first the linear differential equation without the second member; it is of the ordinary oscillating type, and indicates a proper frequency $\omega(r)$ as a function of r.

$$\omega^2(r)=-\frac{e}{m}\left(\frac{\partial^2 V_0}{\partial r^2}+\frac{1}{r}\frac{\partial V_c}{\partial r}\right). \tag{50}$$

Let us study first the order of magnitudes of these internal proper frequencies. In a magnetron on critical conditions, the potential repartition is given by Eq. (23)

$$V_0=V_c=-(m/2e)\omega_H^2(r-a^2/r)^2; \tag{51}$$

hence

$$\omega^2(r)=2\omega_H^2(1+a^4/r^4),\quad \omega(r)\approx\sqrt{2}\omega_H. \tag{52}$$

Such critical conditions, however, correspond to no anodic current ($I_c=0$) and infinite transit time τ, and cannot be used for sustaining oscillations. We may, however, suppose the magnetron to be very near this critical state, the continuous current I_c keeping a small value, so that the average potential $V_c(r)$ will differ only slightly from $V_0(r)$. The proper frequencies $\omega(r)$ will then keep values very near $\sqrt{2}\omega_H$.

Returning to Eq. (49) we notice its similarity to the equation of a harmonic oscillator acted upon by an external force represented by the right-hand term. A remarkable feature is that this equation contains no damping term, but this is only the result of the initial assumptions outlined in Section 5. A complete theory, including magnetic oscillating fields, should yield equations with a damping term due to outside radiation, as is immediately realized for physical reasons.

We are thus allowed to proceed as though such a damping were actually found in the equations, and search only for the forced oscillations, neglecting free oscillations [solutions of Eq. (49) without right-hand term]. Let us thus suppose conditions very near the critical state and call

$$\tau(r)=t-t_0 \tag{53}$$

the transit time for an electron running from the filament ($r=a$) to the distance r. Supposing

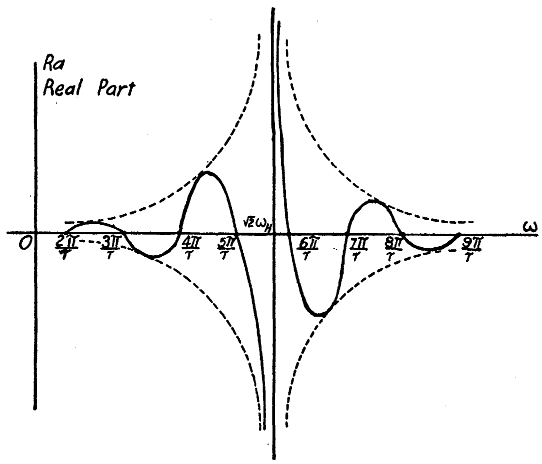

FIG. 9. Approximate values of R_a.

sinusoidal oscillations, we write down

$$I_a = J_a e^{i\omega t}, \quad J_a > 0 \text{ real},$$

$$\int_{t_0}^{t} I_a dt = \frac{J_a}{i\omega}(e^{i\omega t} - e^{i\omega t_0}) = 2\frac{J_a}{\omega}\sin\frac{\omega\tau}{2}e^{i\omega t + i\varphi}. \quad (54)$$

$$\varphi = -\omega\tau/2.$$

Equation (49) takes the following form and defines, for each distance r_c, the amplitude r_a of electronic oscillations.

$$r_a(\omega^2(r) - \omega^2) = \frac{4eJ_a}{mr_c\omega}\sin\frac{\omega\tau}{2}e^{i\omega t + i\varphi}. \quad (55)$$

Phase φ of these oscillations is directly related to the transit time τ.

From the r_a values, one easily calculates V_a using Eq. (47).

$$\frac{\partial V_a}{\partial r} = r_a\left[\frac{\partial^2 V_0}{\partial r^2} - \frac{\partial^2 V_c}{\partial r^2} + \frac{m}{e}\omega^2\right]. \quad (56)$$

V_0 and V_c are very nearly equal to each other (almost critical state) and formula (56) approximately yields

$$\frac{\partial V_a}{\partial r} \approx \frac{m}{e}\omega^2 r_a \approx \frac{4J_a\omega}{r[\omega^2(r) - \omega^2]}\sin\frac{\omega\tau}{2}e^{i\omega t + i\varphi}, \quad (57)$$

$$V_a \approx \frac{4J_a\omega}{\omega^2(r) - \omega^2}\left[\frac{1}{a^2} - \frac{1}{r^2}\right]\sin\frac{\omega\tau}{2}e^{i\omega t + i\varphi}. \quad (58)$$

One must, of course, suppose V_a to be zero on the filament $r = a$. These simpler formulae result from the assumption of a magnetron working very near its critical point. On a more general case, formulae would be of a more complicated type, but one very important point would remain unchanged, namely, the relation between phase angle φ and transit time τ.

Having obtained expressions for alternating current J_a and potential V_a, one immediately finds the internal impedance of the magnetron

$$R_a = \frac{V_a}{J_a} \approx \frac{4\omega}{\omega^2(r) - \omega^2}\left(\frac{1}{a^2} - \frac{1}{r^2}\right)\sin\frac{\omega\tau}{2}e^{-i\omega\tau/2}, \quad (59)$$

$$\varphi = -\omega\tau/2.$$

The question is now to discuss this expression and to see under which conditions it may yield a negative real part, indicating a negative resistance term in the magnetron. It is well known that, in order to be able to sustain oscillations in an electric circuit with positive resistance, an electronic tube must play the role of a negative resistance. Depending on secondary effects in the electronic tube, one may connect this negative resistance in series with the electric circuit or in parallel with it. Magnetrons are actually used with the parallel connection, according to Fig. 8.

7. The Magnetron as a Source of Electrical Oscillations

Referring to formula (59) we notice, first, that the denominator is zero for ω^2 equal to $\omega^2(r)$, which seems to indicate the possibility of infinite internal impedance R_a. This, of course, is the result of some simplifying assumptions discussed in Section 5. A complete theory would include direct radiation from the moving electrons, thus giving a damping term s in the equation, and the impedance R_a would have a denominator avoiding infinity like

$$\{[\omega^2(r) - \omega^2]^2 + s^2\}^{\frac{1}{2}}.$$

At any rate, large values of R_a will only be obtained for

$$\omega \approx \omega(r) \approx \sqrt{2}\omega_H. \quad (60)$$

Now we determine the real part of R_a

$$\mathrm{Re}(R_a)=\frac{2\omega}{\omega^2(r)-\omega^2}\left(\frac{1}{a^2}-\frac{1}{r^2}\right)\sin\omega\tau. \quad (61)$$

Figure 9 gives an approximative visualization of such a function. This will be negative in the following cases:

$$\begin{aligned}&\omega^2<\omega^2(r), \quad \sin\omega\tau<0,\\ &\pi<\omega\tau<2\pi,\\ &3\pi<\omega\tau<4\pi, \quad \cdots\end{aligned} \quad (62)$$

and

$$\begin{aligned}&\omega^2>\omega^2(r), \quad \sin\omega\tau>0,\\ &0<\omega\tau<\pi,\\ &2\pi<\omega\tau<3\pi, \quad \cdots.\end{aligned} \quad (63)$$

We recall that transit time τ in a magnetron may reach values much greater than the period of oscillations; the best values of τ in both cases will correspond to $\sin\omega\tau$ equal 1, that is,

$$\omega^2<\omega^2(r), \quad \omega\tau=(2n+1)\pi+\tfrac{1}{2}\pi, \quad (64)$$

$$\omega^2>\omega^2(r), \quad \omega\tau=2n\pi+\tfrac{1}{2}\pi. \quad (65)$$

One thing may appear rather surprising in the preceding results, and that is the role played by frequencies of the order of $\sqrt{2}\omega_H$. If we refer to the electronic motions in the static magnetron, we notice that the electrons move around the filament with angular velocities ranging from 0 (near the filament) up to ω_H at a great distance from the filament [Eq. (15)]. None of these frequencies happens to play a role in the operation of sustaining oscillations, and the $\sqrt{2}\omega_H$ frequency is noticeably higher. This can be understood, as the angular velocities of the electrons cannot be observed in a magnetron whose anode is built as a complete cylinder around the filament. The only motions which may be noticed as giving a change in the anodic current are radial motions. The oscillations result, in this case, in electronic motions, where all electrons located on a certain cylindrical layer (at distance r_c) move together either to or fro, the whole cylindrical layer expanding or narrowing at one time. This is the type of motion corresponding to the frequency $\sqrt{2}\omega_H$, and it has nothing to do with the rotational frequency ω.

Another point to be emphasized is the great variety of conditions under which the magnetron is able to sustain oscillations. Conditions (62) and (64), for $\omega<\omega(r)$, and (63) and (65) for $\omega>\omega(r)$, offer a great number of different solutions, a characteristic almost never found in the discussion of electron tubes.

Where does the energy necessary for sustaining oscillations in the outer circuit come from? All our calculations refer to a magnetron very near its critical conditions ($I=0$). Such a magnetron stores up energy in kinetic form from the electrons rotating about the filament. In the process of sustaining oscillations in an outer circuit, some energy will be taken out of this reserve, thus slowing down the electrons; in order to keep the electron cloud in rotation, a small current I must be allowed to flow between anode and cathode, and the power IV supplied by the battery will restore the necessary energy and keep the cloud in steady rotation.

Our conclusion, that such a magnetron should be operated near $\sqrt{2}\omega_H$, fits with the results obtained by Blewett and Ramo[4] in a recent paper and is checked by experimental facts. Blewett and Ramo, however, failed to get the conditions for negative resistance, a difficulty they themselves emphasized at the end of their paper.

The first part of this paper, including Sections 1–4, was written in January, 1939, but its publication was delayed because of present circumstances. Several copies, however, were circulated; the author has to thank M. M. Blewett and Ramo for their courtesy in making reference to this unpublished paper.

[4] J. P. Blewett and S. Ramo, Phys. Rev. **57**, 635 (1940).