Theory of the Superfluidity of Helium II

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It is well known that liquid helium at temperatures below the λ-point possesses a number of peculiar properties, the most important of which is superfluidity discovered by P. L. Kapitza.[1] As to the theoretical interpretation of these phenomena, Tisza's[2] well-known attempt to consider helium II as a degenerate Bose gas cannot be accepted as satisfactory—even putting aside the fact that liquid helium is not an ideal gas, nothing could prevent the atoms in the normal state from colliding with the excited atoms; i.e., when moving through the liquid they would experience friction and there would be no superfluidity at all.

Consider the quantization of an arbitrary system of interacting particles (a liquid) from a general point of view. This can be done by means of introducing operators of the density of the mass ρ, the density of the flow of the mass (momentum density) $\mathbf{j}$ and the velocity $\mathbf{v}$ of the liquid according to:

$$\rho=\sum_\alpha m_\alpha\delta(\mathbf{r}_\alpha-\mathbf{R}),$$

$$\mathbf{j}=\tfrac{1}{2}\sum_\alpha[\mathbf{p}_\alpha\delta(\mathbf{r}_\alpha-\mathbf{R})+\delta(\mathbf{r}_\alpha-\mathbf{R})\mathbf{p}_\alpha], \qquad (1)$$

$$\mathbf{v}=\frac{1}{2}\left(\frac{1}{\rho}\cdot\mathbf{j}+\mathbf{j}\cdot\frac{1}{\rho}\right)\quad\left(\mathbf{p}_\alpha=\frac{\hbar}{i}\nabla_\alpha\right),$$

$\mathbf{R}$ being the radius vector of an arbitrary point, $\mathbf{r}_\alpha$ the radius vector of the particle m_α. (It must be emphasized that such a description is based on a microscopical picture and does not involve any statistical averaging.) The calculation leads to the following commutation rules:

$$\rho_1\rho_2-\rho_2\rho_1=0,\quad \mathbf{v}_1\rho_2-\rho_2\mathbf{v}_1=\frac{\hbar}{i}\nabla\delta(\mathbf{R}_1-\mathbf{R}_2),$$
$$v_{1i}v_{2k}-v_{2k}v_{1i}=\frac{\hbar}{i}\delta(\mathbf{R}_1-\mathbf{R}_2)\frac{1}{\rho_1}(\mathrm{rot}\ \mathbf{v})_{ik} \qquad (2)$$

(here $i, k=x, y, z$; $(\mathrm{rot}\ \mathbf{v})_{ik}=(\partial v_k/\partial x_i)-(\partial v_i/\partial x_k)$ and the indexes 1 and 2 refer to the points $\mathbf{R}_1$ and $\mathbf{R}_2$ in space). By applying the relations (2) to the macroscopic motion of the liquid, we find, as it should be, the usual hydrodynamic equations written in an operational form (in a macroscopic consideration the internal energy of a liquid is considered as a function of its density only).

From (2) it can be seen that $\mathrm{rot}\ \mathbf{v}$ commutes with ρ and $\mathbf{v}$ and, therefore, also with the Hamiltonian in that case only when $\mathrm{rot}\ \mathbf{v}=0$ over the whole volume of the liquid. This means that $\mathrm{rot}\ \mathbf{v}=0$ is conserved, i.e., a quantum liquid always possesses stationary states in which $\mathrm{rot}\ \mathbf{v}=0$ ("potential motion"). States in which $\mathrm{rot}\ \mathbf{v}\neq0$, but is arbitrarily small over the whole volume, do not exist. In other words, there is no continuous transition between the states of the potential ($\mathrm{rot}\ \mathbf{v}=0$) and the vortex ($\mathrm{rot}\ \mathbf{v}\neq0$) motions of a quantum liquid. Therefore, between the lowest energy levels of vortex and potential motion there must be a certain energy interval Δ. (It must be emphasized that we do not here refer to the levels for single helium atoms but to the levels corresponding to the states of the whole liquid.) One may question which of these levels lies lower; apparently both cases are logically possible. The supposition that the normal level of potential motions lies lower than the beginning of the spectrum of vortex motions leads to the phenomenon of superfluidity. Hence we must suppose that this very case exists in liquid helium.

Every weakly excited state must be considered as a combination of simple "elementary excitations." In the case of potential internal motions, these excitations are quanta of longitudinal (sound) waves, i.e., phonons. The energy ϵ of the phonons, is, as is well known, a linear function of their momentum: $\epsilon=cp$ (c being the velocity of sound). The "elementary excitations" of the vortex spectrum can be called "rotons." Their energy is a quadratic function of the momentum:

[1] P. L. Kapitza, Nature **141**, 74 (1937); Comptes rendus Acad. Sci. USSR **18**, 28 (1938).
[2] L. Tisza, Nature **141**, 913 (1938).

$\epsilon = \Delta + p^2/2\mu$ (μ being the "effective mass" of a roton and the energy is measured from the normal state of the liquid). If the number of rotons and phonons per unit volume is not too large (sufficiently low temperatures) their aggregate can be considered as a mixture of two ideal gases. The phonon gas obeys Bose statistics. For the rotons we can apply (for $kT \ll \Delta$) the Boltzmann distribution independently of their statistics (probably Bose) owing to the assumed large term Δ (as compared with kT) in their energy.

From these properties of the energy spectrum the heat capacity of helium II must consist of two parts: the "phonon part," i.e., the normal Debye heat capacity proportional to T^4, and the "roton part," depending on the temperature exponentially ($\sim e^{-\Delta/kT}$). A comparison of the calculated phonon heat capacity with that measured experimentally[3] shows that right down to the lowest temperatures ($<1°$K) the roton part plays the dominant role in the heat capacity. If it is expressed through μ and Δ and compared with the experimental values, we get $\Delta/k = 8$–$9°$, $\mu = 7$–8 masses of a helium atom.

At absolute zero, helium is in its normal unexcited state. If such a liquid is considered when flowing as a whole along a capillary, it can be easily shown that the interaction between it and the walls of the capillary cannot lead (when the velocity of the flow is not too great) to an excitation of internal motion, i.e., to an energy dissipation; in other words, the liquid will disclose no viscosity. Owing to the presence of the energy gap in the spectrum the rotons can be excited only at velocities $V > (2\Delta/\mu)^{\frac{1}{2}}$, and the phonons—because of the linear dependence of their energy on the momentum—only at $V > c$.

At temperatures higher than absolute zero there are a certain number of phonons and rotons in helium II. If we consider helium II in a rotating vessel, a statistical investigation leads to the result that a statistical equilibrium must be established in the vessel which is distinguished from the equilibrium in a vessel at rest in that the gas of rotons and phonons rotates with the vessel as if it were carried along by the walls. If the angular momentum of the helium in the rotating vessel is calculated from the corresponding statistical distribution, at absolute zero, i.e., in the entire absence of rotons and phonons, we would get simply zero. At higher temperatures the angular momentum will be non-zero, but the moment of inertia will be, at sufficiently low temperatures, much lower than the usual one (which corresponds to the usual rotation of the whole liquid together with the vessel).

Thus, when the walls of the vessel are in motion, only a part of the mass of liquid helium is carried along by them, and the other part "remains stationary." Therefore we might regard liquid helium as if it consisted of a "mixture" of two liquids—one is "superfluid" without viscosity and not carried along by the walls of the vessel, and the other is "normal." When these two "liquids" move through each other there is "no friction" between them, i.e., there is no transfer of momentum from one to the other. It must be emphasized that when we talk about helium as being a "mixture" of two liquids it is no more than a means of expression convenient for describing the phenomena in helium II. Actually, it should be said that two motions can exist simultaneously in a quantum liquid, each of which is connected with its own effective mass. One of these motions is "normal" and the other is "superfluid." It must be particularly stressed that we have here no real division of the particles of the liquid into "superfluid" and "normal" ones—in a definite sense one can speak only of the "superfluid" and "normal" parts of the mass of the liquid as of masses connected with two simultaneously possible motions, but this does not mean that the liquid can be really divided into two parts.

At every temperature liquid helium is characterized by a definite value for the ratio of the densities ρ_n and ρ_s of the "normal" and "superfluid" liquids ($\rho = \rho_n + \rho_s$ is the true density of helium). At $T = 0$ the ratio $\rho_n/\rho = 0$; if the temperature is raised it increases. The temperature at which ρ_n/ρ becomes unity, i.e., the "superfluid" part vanishes, is the λ-point of helium. The ratio ρ_n/ρ can be measured experimentally in a direct way by measuring the moment of inertia of the rotating vessel filled with helium II.

The ratio ρ_n/ρ can be calculated for low temperatures when the aggregate of phonons and

[3] W. H. Keesom and A. P. Keesom, Physica 2, 557 (1935).

rotons can be regarded as an ideal gas; one gets:

$$\frac{\rho_n}{\rho}=\frac{4}{3}\frac{E^{(\mathrm{ph})}}{c^2}+\mu N^{(\mathrm{rot})} \tag{3}$$

($E^{(\mathrm{ph})}$ is the energy of the phonon gas per 1 g of helium, $N^{(\mathrm{rot})}$ is the number of rotons in 1 g of helium; $E^{(\mathrm{ph})}\sim T^4$, $N^{(\mathrm{rot})}\sim e^{-\Delta/kT}$). In view of a very rapid exponential increase of $N^{(\mathrm{rot})}$, this formula can be approximately applied to the calculation of the temperature of the λ-point; with the values of μ and Δ given above we get 2.3°K for the λ-point, which is in sufficiently good agreement with the known value 2.19°K.

It can be shown that the motion of the "superfluid" liquid is always "potential." Besides this, the motion of the "superfluid" part of helium II does not carry heat. Therefore, a motion of helium II in which only the superfluid part takes part is thermodynamically reversible. When helium II is flowing through a narrow slit it is just the superfluid part which flows through without disclosing any friction. The outflowing helium ought to be at a lower temperature than the helium II in the initial vessel—in the ideal case at absolute zero.

For the heating of the liquid in the vessel when helium flows out through a narrow capillary, and for the temperature gradient of the pressure dp/dT, the formulae are obtained which were given by H. London[4] starting from Tisza's ideas, the verbal formulation of which coincides at this point with the theory here advanced. These formulae are fully confirmed by P. L. Kapitza's experiments.[5]

A temperature gradient along the capillary gives rise to two currents—the current of the "normal" liquid which carries heat from the hot to the cold end and the oppositely directed current of "superfluid" liquid; this mechanism leads to a very large heat transfer.

A complete system of hydrodynamic equations can be advanced describing the macroscopical motion of helium II. At every point the motion is described by two velocities—the "superfluid" $\mathbf{v}_s$ (for which rot $\mathbf{v}_s=0$) and the "normal" $\mathbf{v}_n$; on a hard surface $\mathbf{v}_s$ must fulfill the boundary conditions of an ideal liquid and $\mathbf{v}_n$ that of a viscous one. If the values of $\mathbf{v}_s$ and $\mathbf{v}_n$ are not too large, the equations are found to be of the form:

$$\rho=\rho_s+\rho_n, \quad \mathbf{j}=\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n, \tag{4}$$

$$(\partial\rho/\partial t)+\mathrm{div}\,\mathbf{j}=0, \tag{5}$$

$$\frac{\partial j_i}{\partial t}+\sum_k\frac{\partial \Pi_{ik}}{\partial x_k}=0,$$

$$\Pi_{ik}=p\delta_{ik}+\rho_n v_i^{(n)}v_k^{(n)}+\rho_s v_i^{(s)}v_k^{(s)}, \tag{6}$$

$$\frac{\partial\mathbf{v}_s}{\partial t}=-\mathrm{grad}\left\{\varphi+\frac{v_s^2}{2}-\frac{\rho_n}{2\rho}(\mathbf{v}_n-\mathbf{v}_s)^2\right\}, \tag{7}$$

$$(\partial\rho S/\partial t)+\mathrm{div}(\rho S\mathbf{v}_n)=0 \tag{8}$$

(s and φ being the entropy and thermodynamic potential per 1 g of helium, p the pressure); in (6) and (8) the terms connected with the viscosity of the normal liquid are left out.

The application of these equations to the propagation of sound leads to the result that two velocities of sound must exist in helium II which are approximately equal to

$$u_1=(\partial p/\partial\rho)^{\frac{1}{2}}, \qquad u_2=(Ts^2\rho_s/C\rho_n)^{\frac{1}{2}} \tag{9}$$

(C is the heat capacity). At the λ-point u_2 becomes zero. At $T\rightarrow 0$ these velocities tend to the limits $u_1=c$, $u_2=c/\sqrt{3}$.

A detailed paper will be published in one of the issues of the *Journal of Physics USSR*.

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[4] H. London, Proc. Roy. Soc. **A171**, 484 (1939).
[5] P. L. Kapitza, Phys. Rev. **60**, 354 (1941) (this issue).