# On the Fermi Theory of β-Radioactivity

## II. The "Forbidden" Spectra

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Fermi's theory of the energy distribution of β-particles is extended to first and second forbidden transitions for arbitrarily charged nuclei. The calculations are done not only for Fermi's original "polar vector" form of the theory but also for the scalar, tensor, axial vector and pseudo-scalar forms. Selection rules appropriate for these are given in Table I. The final results are given in the form of a "correction factor" $C$, by which the allowed distribution must be multiplied to give a forbidden spectrum. They are listed in §4. The energy dependence of the correction factors was found to be completely independent of knowledge concerning the details of the nuclear states only for the scalar and pseudo-scalar interactions (which give identical results) and for certain special selection rules in the other interactions. Comparison with experimental data on $Na^{24}$, $P^{32}$ and RaE seems to eliminate the scalar, pseudo-scalar and axial vector possibilities, all of which yield results independent of detailed knowledge concerning the nuclear states involved in these cases. The polar vector and tensor results depend on the unknown ratio of the magnitudes of certain nuclear matrix elements. Arbitrary adjustment of the unknown ratios allows fairly good fitting of the data. Especially striking is the reproduction of a "K-U type" shape for the RaE spectrum. Although the tensor and polar vector theories are equally favored by the evidence of the energy distributions, the fact that the tensor theory leads to Gamow-Teller selection rules perhaps make it preferable.

### §1. INTRODUCTION

THE distinction between "allowed" and "forbidden" β-decay was first made by Sargent[1] on an empirical basis. By plotting the decay constants of the naturally β-radioactive elements as a function of the maximum energy of the β-rays, Sargent showed that there is no one-to-one correspondence between the two quantities, but that for a given energy release the decay constant may have any one of a discrete set of order of magnitudes. Fermi's[2] theory attributes these differences between equally energetic transitions to differences in the change of angular momentum and parity which occur during each, in analogy to the emission of dipole, quadrupole, etc. radiation in atomic spectra. The class of β-emitters with the largest decay constants as a function of the energy release are said to undergo "allowed" transitions; the transitions with a decay rate a distinct order of magnitude smaller (it turns out to be ~100) are "first forbidden"; the next class is "second forbidden," and so on. When the Sargent plot is extended to lighter, artificially radioactive atoms, the distinctions between the classes become obscured, although certain groups can still be distinguished if care is taken to compare only atoms of comparable size.[3] This happens because the decay constants depend also on the nuclear charge and on the complexity of the nucleus.

A formula for the energy distribution $P(W)dW$ of the β-particles emitted in "allowed" transitions was given by Fermi.[4] He obtained:

$$P(W)dW=(G^2/2\pi^3)C_0F(Z,W)pW\times(W_0-W)^2dW, \quad (1)$$

in which $W$ is the electron energy, $p=(W^2-1)^{\frac{1}{2}}$ is its momentum, $W_0$ is the maximum energy of the β-spectrum, and $G$ is the "Fermi constant" having the order of magnitude $5\times10^{-12}$. The function

$$F(Z,W)=\frac{4}{\Gamma^2(1+2s)}(2p\rho)^{2s-2}e^{\pi\alpha ZW/p}\times|\Gamma(s+i\alpha ZW/p)|^2 \quad (2)$$

represents the influence of the nuclear charge $Z$; $\rho$ is the nuclear radius and $s=(1-\alpha^2Z^2)^{\frac{1}{2}}$;

[1] B. W. Sargent, Proc. Roy. Soc. **139**, 659 (1933).
[2] E. Fermi, Zeits. f. Physik **88**, 161 (1934).
[3] L. W. Nordheim and F. L. Yost, Phys. Rev. **51**, 942 (1937); compare also E. J. Konopinski, *Advances in Nuclear Physics* (Interscience Publishers, Inc., in press).
[4] The usual relativistic units will be used throughout; unit of energy is $mc^2$, of time $\hbar/mc^2$ and of length $\hbar/mc$. All formulas will be written for electron emission; for positron emitters one has to change the sign of the nuclear charge $Z$.

$F(0, W)=1$. The quantity $C_0$ (for which exact expressions will be given later) contains a matrix element of the initial and final nuclear states and is independent of the energy $W$. The formula (1) seems to be adequately confirmed by the best experimental measurements of various $\beta$-spectra. Most notable are Lawson and Cork's[5] $In^{114}$ spectrum, Tyler's[6] spectra of the $Cu^{64}$ positrons and electrons, and the measurements[7] on $N^{13}$ and $C^{11}$.

The authors[8] made a criticism of Fermi's formula (1) on the basis of a comparison with older experimental data and advanced a modification of the Fermi theory which seemed to represent those data better. The technical improvements in the most recent measurements, particularly in eliminating scattering, have withdrawn the basis for the criticism. Further, these measurements have confirmed the values of the maximum energy $W_0$ as derived from the nuclear masses. The so-called K-U modification had led to values of $W_0$ which were distinctly too large.[9]

Fermi's formula (1) however still does not represent a great number of observed $\beta$-spectra. Many of these disagreements are undoubtedly due to the superposition of several spectra, as has lately again been emphasized by Bethe, Hoyle and Peierls.[10] Nevertheless, all the disagreements cannot be explained in this way. The very well investigated[11] spectra of RaE and $P^{32}$ show definite deviations from Fermi's formula. No evidence of the $\gamma$-radiation to be expected if these spectra were complex has been found. The feeble $\gamma$-radiation which has been observed can be completely accounted for as internal bremsstrahlung.[12] Now, according to the Sargent relation both $P^{32}$ and RaE seem to belong to the "second forbidden" class. There is, therefore, no *a priori* reason to expect them to obey the "allowed" formula (1). Accordingly, it will be of interest to see whether an extension of Fermi's calculations to forbidden transitions can account for such cases. Attempts in this direction have been made already by Hoyle[13] and by Berestetzky.[14] These authors however have confined themselves to the K-U modification of the theory and moreover have made no systematic examination of the possible forms of the forbidden $\beta$-spectra.

The calculation of the form of forbidden spectra may also have importance for the general theory of $\beta$-decay. As is well known, the Fermi theory can be formulated in several different ways, all of which lead to the same shape for allowed spectra, but can be expected to yield distinct results for forbidden transitions. These various ways can be expressed in terms of five independent forms, listed by Bethe and Bacher.[15] We have especially investigated the shape of the forbidden spectra for each of these forms separately. The method of calculation is illustrated in §§2, 3 for the one of these forms originally adopted by Fermi. The results for all the forms are listed in §§4, 5. In §6, a comparison of these results with the limited accurate experimental data available, namely the $P^{32}$ and RaE spectra, has been made. This seems already able to eliminate some of the possible forms.

## §2. The Forbidden Spectra for the Fermi Interaction ($Z=0$)

The result of Fermi's formulation for the number of electrons $P(W)dW$ emitted into the energy range $dW$ at $W$ may be written:

$$P(W)dW=G^2/(2\pi)^5\int d\omega_e\int d\omega_n\left|\int d\tau\sum_k\{(V^*Q_kU)(\psi^*\varphi)_k-(V^*\boldsymbol{\alpha}_kQ_kU)\cdot(\psi^*\boldsymbol{\alpha}\varphi)_k\}\right|^2\cdot q^2pWdW. \quad (3)$$

[5] J. L. Lawson and J. M. Cork, Phys. Rev. **57**, 982 (1940).
[6] A. W. Tyler, Phys. Rev. **56**, 125 (1939).
[7] E. M. Lyman, Phys. Rev. **55**, 1123A (1939); Y. Watase and J. Itoh, Proc. Phys. Math. Soc. Japan **22**, 639 (1940); A. A. Townsend, Proc. Roy. Soc. **177**, 357 (1941).
[8] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **48**, 7 (1935). (To be referred to as I.)
[9] Kurie, Richardson, and Paxton, Phys. Rev. **49**, 368 (1936); Bonner, Delsasso, Fowler, and Lauritsen, Phys. Rev. **49**, 203 (1936); J. D. Cockcroft and W. B. Lewis, Proc. Roy. Soc. **154**, 261 (1936).
[10] Bethe, Hoyle and Peierls, Nature **143**, 200 (1939).
[11] J. L. Lawson, Phys. Rev. **56**, 131 (1939); A. Flammersfeld, Zeits. f. Physik **112**, 727 (1939); G. J. Neary, Proc. Roy. Soc. **175**, 71 (1940); L. M. Langer and M. D. Whitaker, Phys. Rev. **51**, 713 (1937); Alichanian and Nikitin, J. Phys. Acad. Sci. USSR **3**, 243 (1940).
[12] E. Stahel and J. Guillissen, J. de phys. et rad. **1**, 12 (1940); Sizoo, Eykman and Groen, Physica **6**, 1057 (1939); Chien-Shiung Wu, Phys. Rev. **59**, 481 (1941).
[13] F. Hoyle, Proc. Roy. Soc. **166**, 249 (1938).
[14] Berestetzky, Comptes rendus Acad. Sci. USSR **23**, 450 (1939).
[15] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936).

The integral over the coordinates of all the nucleons, $\int d\tau \cdots$ represents the matrix element of the interaction between the nucleons and the electron-neutrino field. $U$ and $V$ are the wave functions of the initial and final nuclei; $Q_k$ is an operator which by definition replaces $U$ with a wave function describing a nucleus in which the $k$th neutron is replaced by a proton; $\psi$ and $\varphi$ are the wave functions of the emitted electron and anti-neutrino, respectively, normalized to one particle per unit volume and evaluated at the position of the transforming nucleon. All the wave functions are assumed to be of the four-component Dirac type and the $\boldsymbol{\alpha}$'s are the usual Dirac matrix operators; $q$ is the neutrino momentum, in our units numerically equal to the neutrino energy $K$, since, at the outset, we assume the neutrino mass to be zero. The integrals over $\omega_e$ and $\omega_n$ represent integrations over the directions of the electron and the neutrino momenta and a summation over their spin directions. Accordingly, the formula (3) is written appropriately for the case in which the forces on the electron and the neutrino are neglected, so that their wave functions can be taken as plane waves:

$$\psi = A \exp[i(\mathbf{p}\cdot\mathbf{r})], \quad \varphi = B \exp[-i(\mathbf{q}\cdot\mathbf{r})].$$

Since the de Broglie wave-lengths of the electron and the neutrino are, in general, large compared to nuclear dimensions, one can evaluate the matrix element in (3) in successive approximations, corresponding to an expansion of $\psi$ and $\varphi$ in successive powers of $(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}$:

$$(\psi^*\varphi)_k = (A^*B)[1 - i(\mathbf{p}+\mathbf{q}\cdot\mathbf{r}_k) - \tfrac{1}{2}(\mathbf{p}+\mathbf{q}\cdot\mathbf{r}_k)^2\cdots], \quad (4a)$$

$$(\psi^*\boldsymbol{\alpha}\varphi)_k = (A^*\boldsymbol{\alpha}B)[1 - i(\mathbf{p}+\mathbf{q}\cdot\mathbf{r}_k)\cdots]. \quad (4b)$$

If only the first term of (4a) is kept one is led to a nuclear matrix element in (3), which we shall represent by the symbol $\int 1$:

$$\int 1 = \int d\tau \sum_k (V^* Q_k U). \quad (5)$$

Since this is the matrix element of a scalar quantity it can be different from zero only when the initial and final states have the same angular momentum $J$ and parity. When this is the case we shall call the transition "allowed." By carrying out the further operations in (3) we are then led to Eq. (1) with $Z=0$ and

$$C_{0V}(Z=0) = \left|\int 1\right|^2. \quad (6)$$

The index $V$ is added to $C_0$ to distinguish the Fermi interaction from the other four forms to be discussed later.

The first term in (4b) leads to a nuclear matrix element in (3):

$$\int \boldsymbol{\alpha} = \int d\tau \sum_k (V^* \boldsymbol{\alpha}_k Q_k U), \quad (7)$$

which has the selection rules appropriate for a polar vector:

$$\Delta J = 0, \pm 1 (\text{no } 0\to 0); \text{ yes} \quad (8)$$

where the "yes" signifies that the parity must change in the transition. These same selection rules are likewise characteristic of the nuclear matrix element which arises from the second term of (4a):

$$\int \mathbf{r} = \int d\tau \sum_k (V^* \mathbf{r}_k Q_k U). \quad (9)$$

We must, therefore, take the second term of (4a) and the first term of (4b) together. Whether the matrix elements $\int \boldsymbol{\alpha}$ and $\int \mathbf{r}$ are also of the same order of magnitude will be discussed in §5. Together they lead to the formulas for what we shall call the "first forbidden" transitions. The result can be expressed in the same form as (1) except that $C_0$ is now replaced by the "correction factor":

$$C_{1V}(Z=0) = \left|\int \mathbf{r}\right|^2 (a+b) + \left|\int \boldsymbol{\alpha}\right|^2 + i\left\{\left(\int \boldsymbol{\alpha}\right)\cdot\left(\int \mathbf{r}\right)^* - \text{c.c.}\right\} c, \quad (10)$$

in which $a$, $b$, $c$ are abbreviations for:

$$a = \tfrac{1}{3}(p^2+q^2), \quad b = -\frac{2}{9}\frac{p^2q^2}{WK}, \quad c = -\frac{1}{3}\left(\frac{p^2}{W}+\frac{q^2}{K}\right). \quad (11)$$

To obtain the formulas for the "second forbidden" transitions, we proceed in an analogous fashion. We consider together the third

term of (4a) and the second term of (4b), since each will yield nuclear matrix elements containing products of two polar vectors. These tensors can always be written as combinations of the scalars

$$\int r^2 \quad \text{and} \quad \int (\boldsymbol{\alpha}\cdot\mathbf{r}), \tag{12}$$

the axial vector

$$\int \boldsymbol{\alpha}\times\mathbf{r}, \tag{13}$$

and the symmetrical tensors with spur zero:

$$R_{ij}=\int (x_i x_j-\tfrac{1}{3}\delta_{ij}r^2), \tag{14}$$

$$A_{ij}=\int (\alpha_i x_j+\alpha_j x_i-\tfrac{2}{3}\delta_{ij}(\boldsymbol{\alpha}\cdot\mathbf{r})). \tag{15}$$

Each of these matrix elements has characteristic selection rules. For the scalars (12), these will be the same as for the "allowed" transition (i.e., $\Delta J=0$, no) and their contribution will, therefore, constitute only a presumably small correction to the "allowed" formula. In addition to these (except for $0\to 0$ and $\frac{1}{2}\to\frac{1}{2}$) the axial vector and the tensors give really new selection rules. They are

$$\Delta J=\pm 1;\ \text{no} \tag{16}$$

for the axial vector and

$$\Delta J=\pm 1,\ \pm 2(\text{no } 1\leftrightarrow 0);\ \text{no} \tag{17}$$

for each tensor. Only the contributions of the axial vector and the tensors are, therefore, of interest for the "second forbidden" formula. They give the correction factor,

$$C_{2V}(Z=0)=\sum_{ij}|R_{ij}|^2(\tfrac{1}{6}d+\tfrac{3}{5}ab)+\sum_{ij}|A_{ij}|^2\cdot\tfrac{1}{4}a$$
$$+i(\sum_{ij}A_{ij}R^*_{ij}-\text{c.c.})f+\left|\int \boldsymbol{\alpha}\times\mathbf{r}\right|^2(\tfrac{1}{2}a-b), \tag{18}$$

with the additional abbreviations,

$$d=\tfrac{1}{5}(p^4+q^4)+\tfrac{2}{3}p^2q^2,$$
$$f=\frac{1}{30}\left(\frac{p^4}{W}+\frac{q^4}{K}\right)+\frac{1}{4}W_0 b. \tag{19}$$

The derivation of (18) is greatly facilitated by taking advantage of the fact that the nuclear matrix elements can only occur in the invariant combinations indicated in (18). Just as in deriving Eq. (10) it was sufficient to compute only the coefficient of one of the components in each of the expressions $|\int\mathbf{r}|^2$, $|\int\boldsymbol{\alpha}|^2$ and $(\int\mathbf{r})\cdot(\int\boldsymbol{\alpha})^*$, so in deriving (18) it is enough to consider only certain representative terms. The only complication arises from the fact that in the terms with $A_{ij}$, $\int\boldsymbol{\alpha}\times\mathbf{r}$ and $\int\boldsymbol{\alpha}\cdot\mathbf{r}$ the same combinations of the $\boldsymbol{\alpha}$ and $\mathbf{r}$ components occur. One must, therefore, compute the coefficients of more than one representative term in each of the invariant expressions, in order to obtain enough simultaneous linear equations to determine the coefficients with which each invariant occurs.

## §3. The Forbidden Spectra for the Fermi Interaction ($Z\neq 0$)

When the influence of the Coulomb field on the wave function of the electron is taken into account, it becomes necessary to use spherical coordinates, at least for the electron. The formula (3) is now replaced by:

$$P(W)dW=G^2/4\pi^3\int d\omega_n\sum_{j,l,m}\left|\int d\tau\sum_k\{(V^*Q_kU)(\psi^*\varphi)_k-(V^*\boldsymbol{\alpha}_kQ_kU)\cdot(\psi^*\boldsymbol{\alpha}\varphi)_k\}\right|^2\cdot(q^2W/p)dW. \tag{20}$$

$\psi$ is now the solution of the Dirac equations in the Coulomb field,[16] corresponding to the energy $W$ and the angular momentum quantum numbers $j$, $l$, $m$, normalized to one particle in a sphere of unit radius. For $\varphi$ we still keep the plane wave solution in Cartesian coordinates.

We try now to divide the matrix element occurring in (20) into parts with characteristic selection rules, as in the preceding section. To do this we write the spherical harmonics occurring in $\psi$ as polynomials in $x/r$, $y/r$, $z/r$. When the plane wave $\varphi$ is also developed in powers of $x$, $y$, $z$, for each

[16] Compare M. E. Rose, Phys. Rev. **51**, 484 (1937). His notation will be followed in this paper.

set of values of $j$, $l$, $m$ one obtains formulas for $(\psi^*\varphi)$ and $(\psi^*\boldsymbol{\alpha}\varphi)$ analogous to (4a) and (4b), except that the coefficients still depend on the coordinate $r$. One should evaluate these coefficients at the place of the transforming nucleon. The difficulty then arises that they contain the radial wave functions of the electron, which are singular at the origin. However, this is without physical significance since the Coulomb field undoubtedly breaks down within the nucleus. Following Fermi, we avoid the difficulty by evaluating the radial functions at a value $\rho$ of $r$, which has the order of magnitude of the nuclear radius. It is now possible to select the terms appropriate for the allowed, the first forbidden, etc., transitions, just as in the preceding section.

The result for the allowed spectrum becomes

$$P(W)dW=(G^2/\pi^2)\left|\int 1\right|^2((g_0^2+f_{-2}^2)/4\pi)(q^2W/p)dW, \tag{21}$$

where $g_0$ and $f_{-2}$ are radial wave functions in the notation of reference 16. Evaluating these functions by keeping only the largest terms in their series expansions in powers of $\rho$, one obtains:

$$(g_0^2+f_{-2}^2)/4\pi=(p^2/2\pi)F(Z,\,W)(1+s)/2. \tag{22}$$

This leads to formula (1) with[17]

$$C_{0V}=\left|\int 1\right|^2(1+s)/2. \tag{23}$$

For the first forbidden transitions one obtains analogously:

$$P(W)dW=dW\frac{G^2}{\pi^2}\frac{q^2W}{p}\left[\left|\int \mathbf{r}\right|^2\left\{\frac{q^2}{3}\frac{g_0^2+f_{-2}^2}{4\pi}+\frac{f_0^2+g_{-2}^2}{4\pi\rho^2}+\frac{g_1^2+f_{-3}^2}{2\pi\rho^2}-\frac{q^2}{3K}\frac{g_0f_0-g_{-2}f_{-2}}{2\pi\rho}\right\}\right.$$
$$\left.+\left|\int\boldsymbol{\alpha}\right|^2\frac{g_0^2+f_{-2}^2}{4\pi}+i\left\{\left(\int\boldsymbol{\alpha}\right)\cdot\left(\int\mathbf{r}\right)^*-\text{c.c.}\right\}\left\{\frac{q^2}{3K}\frac{g_0^2+f_{-2}^2}{4\pi}-\frac{g_0f_0-g_{-2}f_{-2}}{4\pi\rho}\right\}\right]. \tag{24}$$

Here new radial wave functions appear which again are to be evaluated from their series expansions. The expansions will always be carried to high enough powers in $\rho$ to give non-vanishing results in the limit of $Z\to 0$, which can then be checked against the formulas of §2. The explicit results are listed in §5, together with those for the second forbidden transitions and for other forms of interaction.

## §4. The Forbidden Spectra for Other Forms of Interaction

In constructing the matrix elements in (3) and (20), Fermi used a combination of $\psi$ and $\varphi$ which behaves like a polar four-vector in Lorentz transformations. The natural generalization of this special form is to include in the consideration the four additional combinations which transform, respectively, like a scalar, a tensor, an axial vector and a pseudo-scalar. For each of these the curly brackets in the integrands of expressions (3) and (20) are replaced by[15]

$$\text{Scalar:}\qquad \{(V^*\beta_k Q_k U)(\psi^*\beta\varphi)_k\}, \tag{25a}$$

$$\text{Tensor:}\qquad \{(V^*\beta_k\boldsymbol{\sigma}_k Q_k U)\cdot(\psi^*\beta\boldsymbol{\sigma}\varphi)_k+(V^*\beta_k\boldsymbol{\alpha}_k Q_k U)\cdot(\psi^*\beta\boldsymbol{\alpha}\varphi)_k\}, \tag{25b}$$

$$\text{Axial V.:}\qquad \{(V^*\boldsymbol{\sigma}_k Q_k U)\cdot(\psi^*\boldsymbol{\sigma}\varphi)_k-(V^*\gamma_{5k}Q_kU)(\psi^*\gamma_5\varphi)_k\}, \tag{25c}$$

$$\text{Pseudo-S.:}\qquad \{(V^*\beta_k\gamma_{5k}Q_kU)(\psi^*\beta\gamma_5\varphi)_k\}. \tag{25d}$$

In these expressions the $\beta$'s are the usual Dirac matrices, $\gamma_5=-i\alpha_x\alpha_y\alpha_z$ and the $\boldsymbol{\sigma}$'s are the familiar Pauli matrices doubled to four rows and four columns; $\boldsymbol{\sigma}=\gamma_5\boldsymbol{\alpha}$.

[17] This differs from Fermi's original result by the factor $(1+s)/2$, which has no effect on the energy distribution.

TABLE I. *Selection rules.*

| | | ALLOWED | FIRST FORBIDDEN | | | SECOND FORBIDDEN | | |
|---|---|---|---|---|---|---|---|---|
| Scalar | Matrix | $\int 1$ | $\int \mathbf{r}$ | | | $R_{ij}$ | | |
| | $\Delta J$ | 0 | 0, $\pm 1$ (no $0\to 0$) | | | $\pm 1$, $\pm 2$ (no $1 \leftrightarrow 0$) | | |
| | Parity Change | no | yes | | | no | | |
| Polar V. | Matrix | $\int 1$ | $\int \mathbf{r}$, $\int \boldsymbol{\alpha}$ | | | $R_{ij}$, $A_{ij}$ | | $\int \boldsymbol{\alpha}\times\mathbf{r}$ |
| | $\Delta J$ | 0 | 0, $\pm 1$ (no $0\to 0$) | | | $\pm 1$, $\pm 2$ (no $1 \leftrightarrow 0$) | | $\pm 1$ |
| | Parity Change | no | yes | | | no | | no |
| Tensor | Matrix | $\int \boldsymbol{\sigma}$ | $\int \boldsymbol{\sigma}\cdot\mathbf{r}$ | $\int \boldsymbol{\sigma}\times\mathbf{r}$, $\int \boldsymbol{\alpha}$ | $B_{ij}$ | $\int \boldsymbol{\alpha}\cdot\mathbf{r}$ | $T_{ij}$, $A_{ij}$ | $S_{ijk}$ |
| | $\Delta J$ | 0, $\pm 1$ (no $0\to 0$) | 0 | 0, $\pm 1$ (no $0\to 0$) | 0, $\pm 1$, $\pm 2$ (no $0\to 0$, $\frac{1}{2}\to\frac{1}{2}$, $1 \leftrightarrow 0$) | $0\to 0$ | $\pm 2$ | $\pm 2$, $\pm 3$ (no $2 \leftrightarrow 0$) |
| | Parity Change | no | yes | yes | yes | no | no | no |
| Axial V. | Matrix | $\int \boldsymbol{\sigma}$ | $\int \boldsymbol{\sigma}\cdot\mathbf{r}$, $\int \gamma_5$ | $\int \boldsymbol{\sigma}\times\mathbf{r}$ | $B_{ij}$ | $T_{ij}$ | | $S_{ijk}$ |
| | $\Delta J$ | 0, $\pm 1$ (no $0\to 0$) | 0 | 0, $\pm 1$ (no $0\to 0$) | 0, $\pm 1$, $\pm 2$ (no $0\to 0$, $\frac{1}{2}\to\frac{1}{2}$, $1 \leftrightarrow 0$) | $\pm 2$ | | $\pm 2$, $\pm 3$ (no $2 \leftrightarrow 0$) |
| | Parity Change | no | yes | yes | yes | no | | no |
| Pseudo-S. | Matrix | $\int \gamma_5$ | $\int \gamma_5\mathbf{r}$ | | | $R_{ij}^{\gamma}$ | | |
| | $\Delta J$ | 0 | 0, $\pm 1$ (no $0\to 0$) | | | $\pm 1$, $\pm 2$ (no $1 \leftrightarrow 0$) | | |
| | Parity Change | yes | no | | | yes | | |

All the forms (25) lead to exactly the same formula (1) for allowed spectra except for differences in the nuclear matrix elements, and, therefore, in the selection rules. The scalar leads to the same "allowed" selection rules as Fermi's polar vector because $\beta_k$ is a scalar operator just as the unity in $\int 1$. Since the $\beta_k$ is a mere scalar, it is ignored in all the further considerations. For the pseudo-scalar interaction the nuclear matrix element $\int \gamma_5$ replaces Fermi's $\int 1$ in allowed transitions, and, therefore, its selection rules differ only in that parity is required to change. The axial vector and the tensor interactions each consist of a pair of terms, as does the Fermi interaction. It can be made plausible (for further discussion see §5), that again only the first terms play a role in allowed transitions. However, they lead to completely different selection rules because they involve the matrix element $\int \boldsymbol{\sigma}$ which is non-vanishing only for

$$\Delta J = 0, \pm 1 \text{ (no } 0\to 0); \text{ no.} \tag{26}$$

The difference from (8) is, of course, due to the fact that $\boldsymbol{\sigma}$ is an axial whereas $\boldsymbol{\alpha}$ is a polar vector. The rules (26) for allowed transitions are referred to as the Gamow-Teller rules because evidence for their correctness was first adduced by these authors.[18]

Still further types of nuclear matrix elements must be introduced when the results of the interactions (25) for forbidden transitions are investigated. The scalar interaction brings in elements which differ from $\int \mathbf{r}$ and $R_{ij}$ only in that the operator $\beta_k$ also appears in the integrands; as already mentioned, this distinction is unimportant and will be ignored. The effect of $\gamma_{5k}$ in the matrix elements needed with the pseudo-scalar interaction,

$$\int \gamma_5\mathbf{r} \quad \text{and} \quad R_{ij}^{\gamma} = \int \gamma_5(x_i x_j - \tfrac{1}{3}\delta_{ij} r^2), \tag{27}$$

is to require parity changes opposite to those required by $\int \mathbf{r}$ and $R_{ij}$. The axial vector interaction

[18] G. Gamow and E. Teller, Phys. Rev. **49**, 895 (1936).

will, for first forbidden transitions, introduce $\int\gamma_5$ and combinations of $\boldsymbol{\sigma}$ with $\mathbf{r}$:

$$\int\boldsymbol{\sigma}\cdot\mathbf{r},\quad \int\boldsymbol{\sigma}\times\mathbf{r},\quad B_{ij}=\int(\sigma_i x_j+\sigma_j x_i-\tfrac{2}{3}\delta_{ij}(\boldsymbol{\sigma}\cdot\mathbf{r})). \tag{28}$$

For the second forbidden transitions $\int\gamma_5\mathbf{r}$ and combinations of $\boldsymbol{\sigma}$, $\mathbf{r}$, and $\mathbf{r}$ are needed. The irreducible forms consist of two vectors $\int r^2\boldsymbol{\sigma}$ and $\int(\boldsymbol{\sigma}\cdot\mathbf{r})\mathbf{r}$, one symmetrical tensor of the second rank and with zero spur

$$T_{ij}=\int([\boldsymbol{\sigma}\times\mathbf{r}]_i x_j+[\boldsymbol{\sigma}\times\mathbf{r}]_j x_i), \tag{29}$$

and finally a completely symmetrical tensor of the third rank with diagonal "spurs" zero,

$$S_{ijk}=\int\{\sigma_{(i}x_j x_{k)}-\tfrac{1}{5}\delta_{(ij}[\sigma_{k)}r^2+2k_{k)}(\boldsymbol{\sigma}\cdot\mathbf{r})]\}, \tag{30}$$

where enclosing indices in parenthesis signifies a sum of terms for all permutations of the indices. Except for the trivial $\beta_k$, it is easy to see that the tensor interaction requires all the matrix elements already introduced which contain $\boldsymbol{\sigma}$ or $\boldsymbol{\alpha}$.

The selection rules characteristic of the various matrix elements are presented in Table I. Just as in §2, all matrix elements introduced in the second forbidden approximation which only repeat the allowed selection rules are left out; also, for the matrix elements which are admitted, only those rules are quoted which are not also "allowed."

It can be seen from Table I that 0→0 transitions possess a special property. The scalar and polar vector interaction do not allow 0→0 transitions with parity change in any approximation. The axial vector and pseudo-scalar interaction similarly forbid completely 0→0 transitions with no parity change. The table shows this only up to the second forbidden approximation, but it is easy to convince oneself that this property extends to all approximations.

The calculations proceed as illustrated in §§2, 3. The final results for the energy distributions are always expressible in the form (1), with new correction factors taking the place of $C_0$. The subscripts $S$, $V$, $T$, $A$ will refer to the scalar, vector, tensor and axial vector interactions, respectively. The "1" and "2" will distinguish the first forbidden and second forbidden formulas. The pseudo-scalar gives exactly the same results as the scalar, except for the appearance of $\gamma_5$ in the matrix element itself. The calculations were always done first for $Z=0$ and then for arbitrary $Z$, so that a check was available by going to the limit $Z\to 0$. The results for the first forbidden transitions are:

$$C_{1S}=\left|\int\mathbf{r}\right|^2(\tfrac{1}{3}K^2L_0+2L_1+M_0+\tfrac{2}{3}KN_0),$$

$$C_{1V}=\left|\int\mathbf{r}\right|^2(\tfrac{1}{3}K^2L_0+2L_1+M_0-\tfrac{2}{3}KN_0)+\left|\int\boldsymbol{\alpha}\right|^2L_0+i\left\{\left(\int\boldsymbol{\alpha}\right)\cdot\left(\int\mathbf{r}\right)^*-\text{c.c.}\right\}(\tfrac{1}{3}KL_0-N_0),$$

$$C_{1T}=\left|\int(\boldsymbol{\sigma}\cdot\mathbf{r})\right|^2(\tfrac{1}{9}K^2L_0+M_0+\tfrac{2}{3}KN_0)+\left|\int\boldsymbol{\alpha}\right|^2L_0+\left|\int\boldsymbol{\sigma}\times\mathbf{r}\right|^2(\tfrac{1}{6}K^2L_0+\tfrac{1}{2}L_1+M_0-\tfrac{2}{3}KN_0)$$
$$-\left\{\left(\int\boldsymbol{\sigma}\times\mathbf{r}\right)\cdot\left(\int\boldsymbol{\alpha}\right)^*+\text{c.c.}\right\}(\tfrac{1}{3}KL_0-N_0)+\sum_{ij}|B_{ij}|^2(\tfrac{1}{12}K^2L_0+\tfrac{3}{4}L_1),$$

$$C_{1A}=\left|\int\boldsymbol{\sigma}\cdot\mathbf{r}\right|^2(\tfrac{1}{9}K^2L_0+M_0-\tfrac{2}{3}KN_0)+\left|\int\gamma_5\right|^2L_0-i\left\{\left(\int\boldsymbol{\sigma}\cdot\mathbf{r}\right)\left(\int\gamma_5\right)^*-\text{c.c.}\right\}(\tfrac{1}{3}KL_0-N_0)$$
$$+\left|\int\boldsymbol{\sigma}\times\mathbf{r}\right|^2(\tfrac{1}{6}K^2L_0+\tfrac{1}{2}L_1+M_0+\tfrac{2}{3}KN_0)+\sum_{ij}|B_{ij}|^2(\tfrac{1}{12}K^2L_0+\tfrac{3}{4}L_1).$$

For the second forbidden transitions one obtains:

$$C_{2S}=\sum_{ij}|R_{ij}|^2\{\tfrac{1}{30}K^4L_0+\tfrac{2}{15}K^3N_0+\tfrac{1}{3}K^2(2L_1+M_0)+2KN_1+\tfrac{9}{2}L_2+3M_1\},$$

$$C_{2V}=\sum_{ij}|R_{ij}|^2\{\tfrac{1}{30}K^4L_0-\tfrac{2}{15}K^3N_0+\tfrac{1}{3}K^3(2L_1+M_0)-2KN_1+\tfrac{9}{2}L_2+3M_1\}$$
$$+\sum_{ij}|A_{ij}|^2(\tfrac{1}{12}K^2L_0+\tfrac{3}{4}L_1)-i\{\sum_{ij}R_{ij}A^*_{ij}-\text{c.c.}\}(\tfrac{1}{30}K^3L_0-\tfrac{1}{6}K^2N_0+\tfrac{1}{2}KL_1-\tfrac{3}{2}N_1)$$
$$+\left|\int\boldsymbol{\alpha}\times\mathbf{r}\right|^2(\tfrac{1}{6}K^2L_0+\tfrac{2}{3}KN_0+\tfrac{1}{2}L_1+M_0),$$

$$C_{2T}=\sum_{ijk}|S_{ijk}|^2\cdot\tfrac{1}{12}(\tfrac{1}{15}K^4L_0+2K^2L_1+15L_2)+\sum_{ij}|T_{ij}|^2\cdot\tfrac{1}{12}\{\tfrac{1}{15}K^4L_0-\tfrac{2}{5}K^3N_0+K^2(L_1+M_0)$$
$$-6KN_1+6L_2+9M_1\}+\sum_{ij}|A_{ij}|^2(\tfrac{1}{12}K^2L_0+\tfrac{3}{4}L_1)-\{\sum_{ij}T_{ij}A^*_{ij}+\text{c.c.}\}$$
$$\times\tfrac{1}{4}(\tfrac{1}{15}K^3L_0-\tfrac{1}{3}K^2N_0+KL_1-3N_1)+\left|\int\boldsymbol{\alpha}\cdot\mathbf{r}\right|^2(\tfrac{1}{9}K^2L_0+\tfrac{2}{3}KN_0+M_0),$$

$$C_{2A}=\sum_{ijk}|S_{ijk}|^2\cdot\tfrac{1}{12}(\tfrac{1}{15}K^4L_0+2K^2L_1+15L_2)+\sum_{ij}|T_{ij}|^2$$
$$\cdot\tfrac{1}{12}\{\tfrac{1}{15}K^4L_0+\tfrac{2}{5}K^3N_0+K^2(L_1+M_0)+6KN_1+6L_2+9M_1\}.$$

Here the quantities $L$, $M$, and $N$ represent combinations of radial wave functions. They are listed below, together with their values, computed as explained in §3. Following the arrows are given expressions which are good approximations for the case $\alpha Z\ll 1$.

$$L_0=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{g_0^2+f_{-2}^2}{4\pi}=\frac{1+s}{2}\qquad\rightarrow 1,$$

$$L_1=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{g_1^2+f_{-3}^2}{4\pi\rho^2}=\frac{F_1}{F}\frac{p^2}{9}\frac{2+s_1}{4}\rightarrow\frac{p^2}{9},$$

$$L_2=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{g_2^2+f_{-4}^2}{4\pi\rho^4}=\frac{F_2}{F}\frac{p^4}{225}\frac{3+s_2}{6}\rightarrow\frac{p^4}{225},$$

$$M_0=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_0^2+g_{-2}^2}{4\pi\rho^2}=\frac{1-s}{2\rho^2}+\frac{\alpha Z}{\rho}\frac{1}{2s+1}\left\{(2s-1)W-\frac{s}{W}\right\}+\frac{1}{(2s+1)^2}\{\alpha^2Z^2+(4s^2-3)sp^2\}$$
$$\rightarrow\frac{1}{9}p^2+\frac{\alpha^2Z^2}{4\rho^2}+\frac{\alpha Z}{3\rho}\left(W-\frac{1}{W}\right),$$

$$M_1=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_1^2+g_{-3}^2}{4\pi\rho^4}=\frac{F_1}{F}\frac{p^2}{36}\left[\frac{2-s_1}{\rho^2}+\frac{\alpha Z}{\rho}\frac{1}{2s_1+1}\left\{2W(2s_1-3)-\frac{s_1}{W}\right\}+\frac{1}{(s_1+1)(2s_1+1)^2}\{2s_1(s_1^2-1)p^2\right.$$
$$\left.+\alpha^2Z^2(2s_1+1)^2+2\alpha^2Z^2W^2(3+s_1-4s_1^2)\}\right]\rightarrow\frac{p^4}{225}+\frac{p^2}{36}\left\{\frac{\alpha^2Z^2}{4\rho^2}+\frac{2\alpha Z}{5\rho}\left(W-\frac{1}{W}\right)\right\},$$

$$N_0=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_0g_0-f_{-2}g_{-2}}{4\pi\rho}=-\frac{\alpha Z}{2\rho}-\frac{1}{2s+1}\left(\frac{sp^2}{W}-2\alpha^2Z^2W\right)\qquad\rightarrow-\frac{p^2}{3W}-\frac{\alpha Z}{2\rho},$$

$$N_1=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_1g_1-f_{-3}g_{-3}}{4\pi\rho^3}=\frac{F_1}{F}\frac{p^2}{18}\left[-\frac{\alpha Z}{2\rho}-\frac{1}{2s_1+1}\left(\frac{s_1p^2}{W}-2\alpha^2Z^2W\right)\right]\rightarrow-\frac{p^4}{45W}-\frac{p^2}{18}\frac{\alpha Z}{2\rho}.$$

Above are used the abbreviations:

$$s_1=(4-\alpha^2Z^2)^{\frac{1}{2}},\quad s_2=(9-\alpha^2Z^2)^{\frac{1}{2}},$$

$$F_1(Z,W)=\frac{(4!)^2}{\Gamma^2(2s_1+1)}(2p\rho)^{2s_1-4}e^{\pi\alpha ZW/p}\,|\,\Gamma(s_1+i\alpha ZW/p)\,|^2,$$

$$F_2(Z,W)=\frac{(6!)^2}{4\Gamma^2(2s_2+1)}(2p\rho)^{2s_2-6}e^{\pi\alpha ZW/p}\,|\,\Gamma(s_2+i\alpha ZW/p)\,|^2.$$

## §5. General Discussion of the Formulas

The characteristic features of the formulas can be seen completely by using the $\alpha Z\ll 1$ approximations for $L$, $M$, and $N$. For example, one then obtains:

$$C_{1V}=\left|\int\mathbf{r}\right|^2\left\{a+b+\frac{\alpha^2Z^2}{4\rho^2}+\frac{\alpha Z}{3\rho}\left(W_0-\frac{1}{W}\right)\right\}+\left|\int\boldsymbol{\alpha}\right|^2+i\left\{\left(\int\boldsymbol{\alpha}\right)\cdot\left(\int\mathbf{r}\right)^*-\text{c.c.}\right\}\left(c+\frac{\alpha Z}{2\rho}\right),$$

$$C_{2V}=\sum_{ji}|R_{ij}|^2\left[\tfrac{1}{6}d+\tfrac{3}{5}ab+\frac{\alpha^2Z^2}{12\rho^2}(q^2+\tfrac{1}{4}p^2)+\frac{\alpha Z}{3\rho}\left\{\left(W-\frac{1}{W}\right)(\tfrac{1}{3}q^2+\tfrac{1}{10}p^2)+K(\tfrac{1}{5}q^2+\tfrac{1}{6}p^2)\right\}\right]$$

$$+\sum_{ij}|A_{ij}|^2\cdot\tfrac{1}{4}a+i\left\{\sum_{ij}A_{ij}R^*{}_{ij}-\text{c.c.}\right\}\left\{f+\frac{\alpha Z}{6\rho}(q^2+\tfrac{1}{2}p^2)\right\}+\left|\int\boldsymbol{\alpha}\times\mathbf{r}\right|^2$$

$$\cdot\left\{\tfrac{1}{2}a-b+\frac{\alpha^2Z^2}{4\rho^2}+\frac{\alpha Z}{3\rho}\left(2W-W_0-\frac{1}{W}\right)\right\},$$

where $a$, $b$, $c$, $d$, and $f$ are defined as in §2 (Eqs. (11) and (19)).

The first characteristic of these formulas which stands out immediately is that, in contrast to the allowed formulas, these involve more than one nuclear matrix element. The only exceptions to this are the scalar and pseudo-scalar correction factors, and, for certain special cases of selection rules, also some of the other factors. When more than one matrix element is thus involved, one

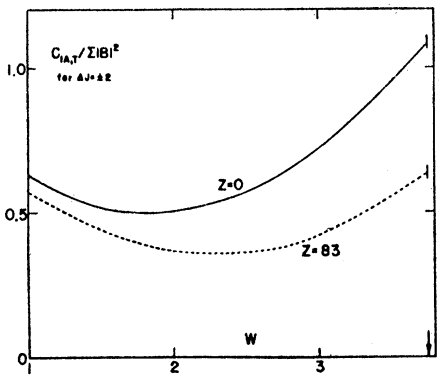


FIG. 1. The first forbidden correction factors $C_{1A}$ and $C_{1T}$ for the axial vector and tensor interactions with $\Delta J=\pm 2$, divided by the nuclear matrix element expression $\Sigma|B_{ij}|^2$, as functions of the electron energy $W$. The arrow indicates the end point used, $W_0=3.75$, which is appropriate for $Na^{24}$. $Z=11$ would be scarcely distinguishable from $Z=0$.

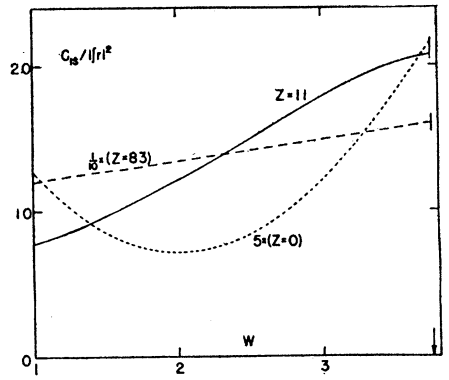


FIG. 2. The first forbidden correction factor for the scalar interaction divided by the square of the matrix element $\int\mathbf{r}$ as a function of the electron energy $W$. The arrow indicates the end point used, $W_0=3.75$, which is appropriate for $Na^{24}$. Expressions like $5\times(Z=0)$ signify that the actual ordinates were multiplied by five in plotting here. The pseudo-scalar interaction gives precisely the same dependence on the energy.

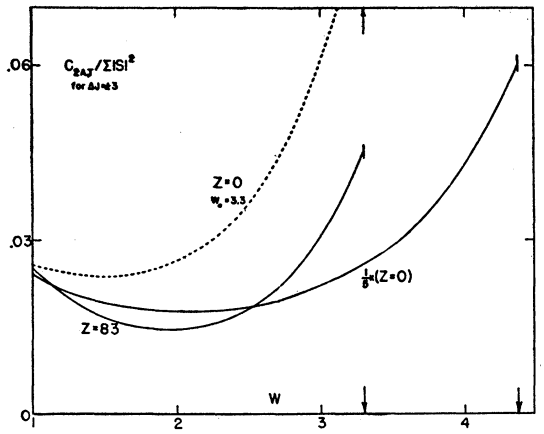


FIG. 3. The second forbidden correction factors $C_{2A}$ and $C_{2T}$, for the axial vector and tensor interactions with $\Delta J=\pm 3$, divided by the nuclear matrix element expression $\Sigma|S_{ijk}|^2$, as functions of the energy $W$. The arrows indicate the end points used: $W_0=3.3$ (appropriate for RaE) for $Z=83$ and one curve with $Z=0$; $W_0=4.37$ (appropriate for $P^{32}$) for the other curve with $Z=0$. $Z=15$ gives results scarcely distinguishable from $Z=0$, here.

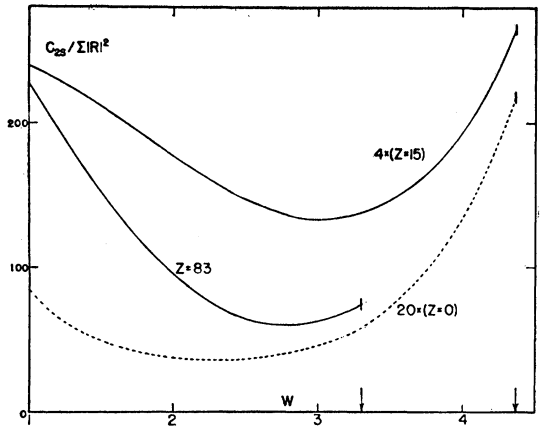


FIG. 4. The second forbidden correction factor $C_{2S}$ for the scalar interaction divided by the nuclear matrix element expression $\Sigma|R_{ij}|^2$ as a function of the electron energy $W$. The arrows indicate the end points used: $W_0=4.37$ (appropriate for $P^{32}$) for $Z=0$ and $Z=15$; $W_0=3.3$ for $Z=83$.

expects it to be necessary to know something about the nuclear states before anything definite concerning the energy dependence of the correction factors can be concluded. However, one can try to make some estimates of the order of magnitude of the matrix elements. Fermi has already mentioned that matrix elements like $\int\boldsymbol{\alpha}$ and $\int\gamma_5$ are of the order $v/c$ smaller than $\int 1$ or $\int\boldsymbol{\sigma}$, $v$ being the velocity of the nucleons. The ratio $v/c$ is thought to be of the order 1/10, which is just about the same as the magnitude of $(p+q)\rho$ in Eq. (4a). This serves as an additional argument for combining the first term of (4b) with the second term in (4a), etc., besides the fact that these terms lead to the same selection rules. One is tempted to make a sharper estimate of such matrix elements as $\int\boldsymbol{\alpha}$, by making use of the symmetrical nuclear Hamiltonian. If one neglects the difference in the Coulomb energy in the initial and final nuclei, one can derive for example relations like

$$\int\boldsymbol{\alpha}=iW_0\int\mathbf{r},\quad A_{ij}=iW_0R_{ij}.$$

This would, for instance, make the energy dependence of the correction factor $C_{1V}$ completely definite. However it seems unjustifiable to neglect the Coulomb energy. We prefer, therefore, to draw only such conclusions which are unaffected by using arbitrary ratios of the nuclear matrix elements.

Another outstanding characteristic of the forbidden formulas in contrast to the allowed is their sensitivity to the position $\rho$ at which the wave functions are evaluated. Except for the very lightest nuclei, the dominant terms are those containing $\alpha Z/\rho$, as long as $\rho$ is of nuclear dimensions. The assumption made by Fermi that $\rho$ is to be set equal to the nuclear radius here becomes quite critical. Exceptions to this can be found in the tensor and axial vector interactions for cases in which $\Delta J$ has the highest possible value; in the only contributing terms $\alpha Z/\rho$ does not appear.

More detailed conclusions which can be made regarding the first forbidden formulas follow:

(1$a$) For $\Delta J=\pm 2$, which is allowed in this approximation only by the axial vector and the tensor interactions, the energy dependence of the correction factors becomes completely definite since only a single matrix element does not vanish. This dependence is shown in Fig. 1 for two values of $Z$. It can be seen that the influence of $Z$ is comparatively small, as is to be expected from the absence of terms in $\alpha Z/\rho$.

(1*b*) Also completely definite are the correction factors for the scalar and pseudo-scalar interactions. These are shown in Fig. 2 for three values of $Z$. This time, due to the presence of $\alpha Z/\rho$, the Coulomb influence is quite large.

(1*c*) All the interactions give the allowed form to first forbidden spectra with $\Delta J=0, \pm 1$ if the nuclear charge is large enough and the energy small enough. In such cases the $\alpha^2Z^2/\rho^2$ term predominates in the correction factors and it is independent of energy. This tendency is displayed by the $C_{1S}$ curve for $Z=83$ in Fig. 2.

(1*d*) It may happen that first forbidden spectra take the allowed form even for light nuclei according to the vector and tensor interactions. If it should happen that the matrix element $|\int \boldsymbol{\alpha}|^2$ in $C_{1V}$ or $C_{1T}$, or $|\int \gamma_5|^2$ in $C_{1A}$, has a dominant magnitude, then the correction factor becomes nearly independent of the energy.

(1*e*) A definite result is also possible for $C_{1T}$ in $0\rightarrow 0$ transitions and for $C_{1A}$ in $1\leftrightarrow 0$ transitions. In each case only a single matrix element does not vanish. The energy dependence in these cases is very similar to that of $C_{1S}$ as illustrated in Fig. 2.

Concerning the second forbidden correction factors one can make the following statements:

(2*a*) A definite energy dependence is obtained for $C_{2A}$ and $C_{2T}$ in case $\Delta J=\pm 3$. It is shown in Fig. 3 for a large and a small value of $Z$. Again the Coulomb influence is comparatively small.

(2*b*) The scalar and pseudo-scalar correction factors are again completely definite and are shown in Fig. 4. As in the first forbidden transitions, the Coulomb influence is considerable.

(2*c*) Excepting $\Delta J=\pm 3$, all the interactions in general give for large $Z$ a correction factor which decreases steeply with energy, approximately as $(W_0-W)^2$. This is illustrated for $C_{2S}$ in Fig. 4.

(2*d*) For $0\rightarrow 0$ transitions, $C_{2T}$ assumes a definite energy dependence much like that of $C_{1S}$. For $2\leftrightarrow 0$ transitions, $C_{2A}$ is definite and very nearly like $C_{2S}$. Finally, $C_{2V}$ becomes definite for $1\leftrightarrow 0$ transitions and gives a distribution much like $C_{1S}$ again.

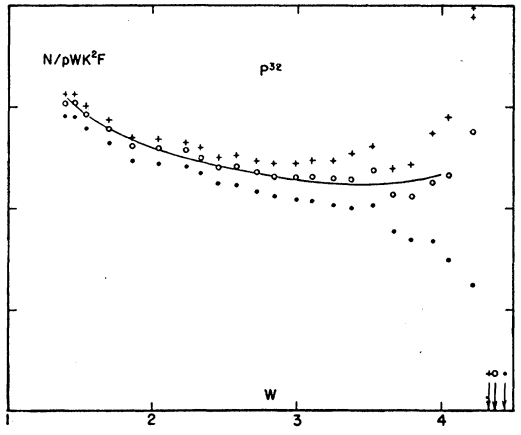


FIG. 5. The experimental number of electrons emitted by $P^{32}$ per unit energy range in ratio to the relative numbers expected according to Fermi's allowed formula, $N\sim pW(W_0-W)^2F$. The data are Lawson's.[11] The blank circles indicate points computed for $W_0=4.37$, the value given by Lawson as most probably correct. The crosses are for $W_0=4.33$ and the solid dots for $W_0=4.44$; this shows the influence of small uncertainties in the upper limit. The curve is drawn to represent an average of the measurements. The ordinates are on an arbitrary scale which is the same for all the points.

## §6. Comparison with Experiments

There seem to be available for comparison with the "forbidden" formulas only three reliable $\beta$-spectrum measurements, namely those of $Na^{24}$, $P^{32}$ and RaE. By dividing the experimental number of electrons per unit energy range by $pW(W_0-W)^2F(Z, W)$, one obtains, according to (1), the energy dependence of the correction factor $C$. Since one divides by a quantity which becomes small at both ends of the spectrum, it is clear that great accuracy is needed to obtain a reliable correction curve. Small uncertainties in the value of the upper limit $W_0$ have great effect. All this is illustrated by Fig. 5.

$Na^{24}$ is expected to be first forbidden according to the Sargent law.[19] The experimental correction factor, shown in Fig. 6, must accordingly be compared with the formulas for $C_1$. It seems impossible however for any of the $C_1$'s to represent the data, whatever choice is made for the values of the nuclear matrix elements. This

[19] Compare Konopinski, reference 8.

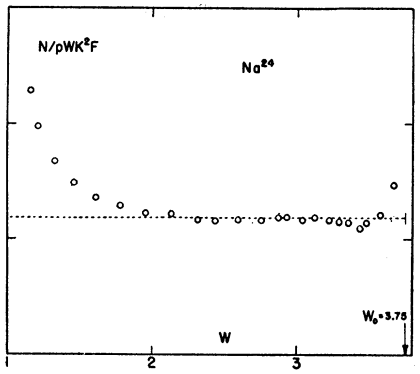


FIG. 6. The ratio of the number of electrons emitted by $Na^{24}$ per unit energy range to the relative number expected according to Fermi's allowed formula. The data are due to Lawson.[11] The circles are the experimental points and the straight line is drawn on the hypothesis that the spectrum is complex with the main component of the spectrum having the allowed form as indicated by the experimental points for energy greater than $W=2$. The ordinates are on an arbitrary scale.

may not at all be a real difficulty, since it seems quite possible for the $Na^{24}$ spectrum to be complex. The sudden rise of $C$ toward low energies must then be interpreted as due to a superposed spectrum with an upper limit of around 1 Mev. Gamma-rays of about the right energy have been observed.[20] This conclusion is contradictory to that drawn by Langer, Mitchell and McDaniel[21] and by Feather and Dunworth[22] from their coincidence experiments. These experiments do not seem to be quite conclusive however, especially since they have not been extended to low enough energies. Accepting the conclusion that the spectrum is complex, one sees from Fig. 5 that, for the main component, the correction factor is constant over a large part of the spectrum. This already excludes the purely scalar and pseudo-scalar theories, as can be seen from Fig. 2. To explain the result with the vector or tensor correction factors, one must assume that the matrix elements $\int\alpha$ or $\int\gamma_5$ which have coefficients independent of the energy are much larger than all the others. This may very well be the case. Of course, the possibility still remains that $Na^{24}$ is not first forbidden but allowed according to our selection rules, especially since the experimental form of the main component seems to be the allowed one. One then must explain the long lifetime by the violation of some further selection rule.

As has already been mentioned, $P^{32}$ and RaE are most likely to be second forbidden and their spectra are almost certainly single. Comparison of Figs. 4, 5 and 7 shows that $C_{2S}$ cannot explain the data. One can also exclude $C_{2A}$. This follows from the fact that $S^{32}$ and RaF are even-even nuclei and so can be taken to have $J=0$ in their normal states. According to the axial vector theory, then, the transitions must be $2\rightarrow0$ or $3\rightarrow0$ in the second forbidden approximation. As mentioned in conclusions (2*a*) and (2*d*), both these cases lead to definite energy dependences for $C_{2A}$. The $2\rightarrow0$ case is excluded for the same reason as $C_{2S}$, while in the $3\rightarrow0$ case $C_{2A}$ cannot explain the data either, as can be seen from Fig. 3. There remain now only the polar vector and the tensor interactions. For the first we can exclude the $1\rightarrow0$ and for the second the $0\rightarrow0$ and $3\rightarrow0$ possibilities, since these are similar either to $C_{1S}$ or to $C_{2A}$, as pointed out in conclusions (2*d*) and (2*a*). There is left now only the possibility of $2\rightarrow0$ transitions for $P^{32}$ and RaE. The energy distribution which $C_{2V}$ and $C_{2T}$ yield for the $2\rightarrow0$ case depends on the

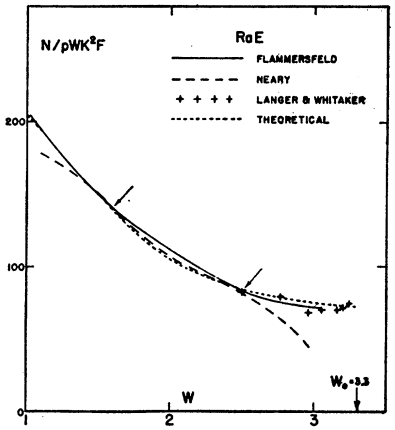


FIG. 7. The ratio of the number of electrons emitted by RaE per unit energy range to the relative number expected according to Fermi's allowed formula. The experimenters responsible for the data are indicated.[11] The dashed line gives $C_{2T}/\Sigma|T_{ij}|^2$ with the ratio of the matrix elements involved in $C_{2T}$ adjusted to fit the experimental curves at the points indicated by the arrows. The adjusted ratio was $A_{ij}/T_{ij}=-5.8$, which does not seem implausible. The ordinates are on an arbitrary scale.

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[21] Langer, Mitchell and McDaniel, Phys. Rev. **56**, 962 (1939).
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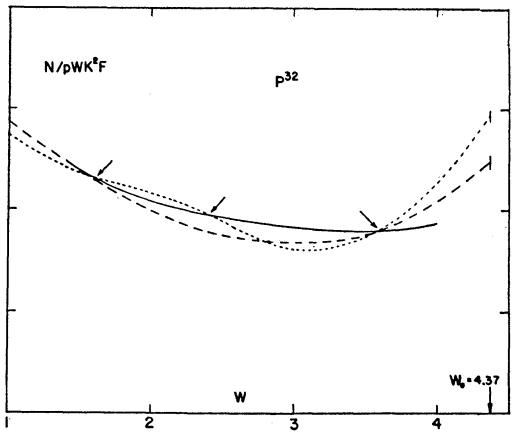


FIG. 8. The solid curve is the same as the curve in Fig. 5, representing an average experimental correction factor for $P^{32}$. The broken lines give $C_{2T}/\Sigma|T_{ij}|^2$, the correction factor according to the tensor theory. The dashed line was fitted to the experimental curve at two points indicated by arrows, leading to $A_{ij}/T_{ij}=-2.2$. The dotted curve was fitted at three points with $A_{ij}/T_{ij}=-1.5\pm i3.0$. Compare with Fig. 5.

ratio of two matrix elements. By adjustment of the ratio, $C_{2V}$ and $C_{2T}$ can be made to give an equally good account of the facts since two of the three terms in each depend on the energy identically and the third terms are very nearly alike. We have done the computations for $C_{2T}$ since the tensor theory is, perhaps, to be favored over the polar vector theory, considering that there is some evidence that the Gamow-Teller selection rules are to be preferred to the original Fermi ones. The fitting of $C_{2T}$ to the RaE and $P^{32}$ experimental data is shown in Figs. 7 and 8.

The one encouraging feature of the application of the theory to the experiments is that the decided deviation of the RaE from the allowed form can be at all explained by the theory. According to statement (2*c*) of §5, the theory gives a correction factor approximately proportional to $(W_0-W)^2$ for an element like RaE. This accounts for the surprising agreements found by the experimenters between their data and the so-called K-U distribution.

The authors wish to express their appreciation for helpful discussions of the problem with Professors E. Wigner and O. Laporte.

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# A Dilatometric Study of the Order-Disorder Transformation in Cu-Au Alloys

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The order-disorder transformation in single crystals of Cu-Au alloys, containing 22, 25, and 30 atomic percent Au, has been studied by dilatometric means. The plots of true coefficient of thermal expansion *versus* temperature for previously well-ordered alloys display a slow rise from −190°C to about +50°C, followed by a flat plateau to about 250°C. From the flat plateau they rise rapidly to a peak at the critical ordering temperature $T_c$, followed by a fall to a second flat plateau which extends to at least 450°C above $T_c$. The lower portion, that portion below the flat plateau, can be well represented by a Grueneisen equation if one assumes additivity for the constants taken from the equation for the pure metals Cu and Au. The presence of the flat plateau above $T_c$ is in disagreement with Bethe-Peierls' theory of the vanishing of short range order. Plots of true coefficient of expansion *versus* temperature, for specimens previously quenched from above $T_c$, display valleys similar to those in the comparable specific heat curves of Sykes and Jones.

THE transformation from a random or disordered state to an ordered state in Cu-Au alloys, containing 22 to 30 atomic percent Au, is accompanied by a decrease in volume. The greater the final degree of order attained, the greater the volume decrease. The contraction in volume on ordering for $Cu_3Au$, as deduced from the contrast between quenched and fully annealed alloys, has been found by x-ray measurements of lattice-constant to be 0.6 percent.[1] The volume change accompanying the redistribution of atoms over the lattice sites depends on the relative

[1] C. H. Johansson and J. O. Linde, Ann. d. Physik **25**, 1 (1936).